KÄHLER HYPERBOLIC SPACES, RIGIDITY VIA HARMONIC MAPS.

In this seminar we will deal with several aspects of Kähler geometry and harmonic maps. It consists of 4 parts which are essentially independent of each other.

The first part of this seminar will deal with 'Kähler Hyperbolic Spaces' and is based on Section 8 in [1]. (The following paragraph is taken from [1]). A question attributed to Chern asks whether the Euler characteristic of a closed Riemannian manifold M of dimension $2m$ satisfies

$$(-1)^m \chi(M) > 0$$

if the sectional curvature K of M is negative. The answer is yes if $m \leq 2$. Dodziuk and Singer remarked that Atiyah?s L^2-index theorem implies $(-1)^m \chi(M) > 0$ if the space of square integrable harmonic forms on the universal covering space of M vanishes in degree $\neq m$ and does not vanish in degree $= m$. In [3], Gromov proves this and more in the case where M is a Kähler manifold.
Following Section 8 in [1], we are concerned with Gromov's arguments and results.

In the second part of this seminar we consider the manuscript [4] by Hernandez in which the author investigates the restrictions coming from the existence of a metric with negative complex sectional curvature. We will consider the following two results:

- Let M be a compact manifold of dimension greater than 2. If M admits a metric with negative complex sectional curvature at every point, then M cannot admit a Kähler metric.
- Let (M, g) be a compact Kähler manifold, and let h be another metric for M. Assume that (M, h) has negative sectional curvature with pointwise pinching at least $1/4$. Then (M, h) is a complex hyperbolic spaceform $\pm$ biholomorphic to (M, g).

Furthermore, we will discuss relations between complex sectional curvature and the study of harmonic maps from Kähler to Riemannian manifolds. These can be found in [4] and [8].

The third part of this seminar will deal with 'twisted harmonic maps'. Here we intend to discuss the contents of [2, 5].
Donaldson [2] investigates twisted harmonic map, i.e., a harmonic section of a flat bundle. The existence theorem for such a map over a Riemann surface X associated to an irreducible representation $\pi_1 X \to PSL(2, \mathbb{C})$ is discussed.
Labourie [5] generalizes a result of Corlette, namely he proves: If M has no flat half-strip, then there exists a ρ-twisted harmonic map if and only if $\text{Im}(\rho)$ is a reductive subgroup of $I(M)$.
Here the following notation is used: let $\tilde{N}$ be the universal covering of a manifold N, and M a simply connected manifold of nonpositive curvature. Furthermore, let ρ denote

a representation of $\pi_1(N)$ into the isometry group of $I(M)$ of M. A ρ-twisted map is a ρ-equivariant map $f : \tilde{N} \to M$.

Finally we want understand a proof of Mostow's rigidity theorem (the geometry of a complete, finite-volume hyperbolic manifold of dimension greater than two is determined by the fundamental group). As reference we use [6] and [7]. The essential ingredient for the proof is a Bochner formula – we recommend consulting [7] for this part.

References

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