Families of vector bundles

families of isocrystals vs. families of vector bundles on FF curve:

moduli space of elliptic curves

$\mathcal{M}_{ell} \;/\mathbb{Z}$ Deligne-Mumford stack (or affine scheme if one puts auxiliary level structure).

In char. p,

$$\mathcal{M}_{ell,\mathbb{F}_p} = \mathcal{M}^{ord}_{ell,\mathbb{F}_p} \cup \mathcal{M}^{ss}_{ell,\mathbb{F}_p}.$$

curve. open closed, finite.

E/k, char $k = p$, is ordinary ($k = \overline{\mathbb{F}}_p$).

$$H^1_{crys}(E/W(k))\left[\tfrac{1}{p}\right] \in \mathrm{Isoc}_{\mathbb{Q}_p}$$

$\phi \circlearrowleft$

is isomorphic to either

$$\left(\breve{\mathbb{Q}}_p^2, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\sigma\right) \quad \text{or} \quad \left(\breve{\mathbb{Q}}_p^2, \begin{pmatrix} & 1 \\ p & \end{pmatrix}\sigma\right).$$

slopes $0, 1$. ordinary

slopes $1/2$. supersingular.

Over $\mathcal{M}_{ell, \mathbb{F}_p}$, have family of isocrystals, degenerating from ordinary to supersingular.

This picture gets reversed when studying vector bundles on the FF curve.

$$\mathcal{M}_{ell, \overline{\mathbb{F}}_p} \subseteq \mathcal{M}_{ell, \hat{\overline{\mathbb{Z}}}_p} \supseteq \mathcal{M}^{ad}_{ell, \mathbb{C}_p}$$

formal scheme $/ \hat{\overline{\mathbb{Z}}}_p = \mathcal{O}_{\mathbb{C}_p}$.

adic generic fibre.

$\mathrm{sp}: |\mathcal{M}^{ad}_{ell,\mathbb{C}_p}| \longrightarrow |\mathcal{M}_{ell,\overline{\mathbb{F}}_p}|$ continuous.

$\rightsquigarrow$ similar stratification of $\mathcal{M}^{ad}_{ell,\mathbb{C}_p}$ by pullback.

$$\mathcal{M}_{ell,\mathbb{C}_p,p^\infty} \sim \varprojlim_m \mathcal{M}^{ad}_{ell,\mathbb{C}_p,p^m}$$

full isom. $E[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2$.

param. isom. $E[p^m] \cong (\mathbb{Z}/p^m)^2$

exists as perfectoid space.

$$\pi_{HT}: \mathcal{M}_{ell,\mathbb{C}_p,p^\infty} \longrightarrow \mathbb{P}^1_{\mathbb{C}_p}$$

$$(E/\mathbb{C}_p,\ E[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2)$$

$$0 \to (\mathrm{Lie}\,E^*)^*_{(1)} \to \underset{\mathbb{C}_p^2}{\rotatebox{90}{$\cong$}}\; T_pE \otimes_{\mathbb{Z}_p} \mathbb{C}_p \longrightarrow \mathrm{Lie}\,E \to 0.$$

natural Hodge-Tate sequence

Prop. E has ordinary reduction

$\iff$ Hodge-Tate filtration is $\mathbb{Q}_p$-rational.

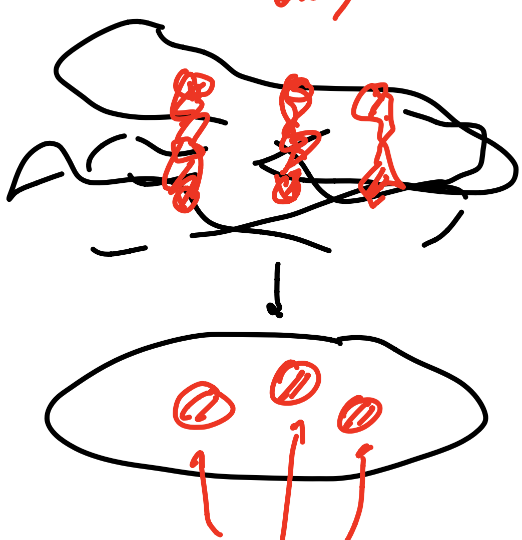
has extra rk 2 points.

essentially,

$$\mathcal{M}^{ord}_{ell,\mathbb{C}_p,p^\infty} \subseteq \pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p)).$$

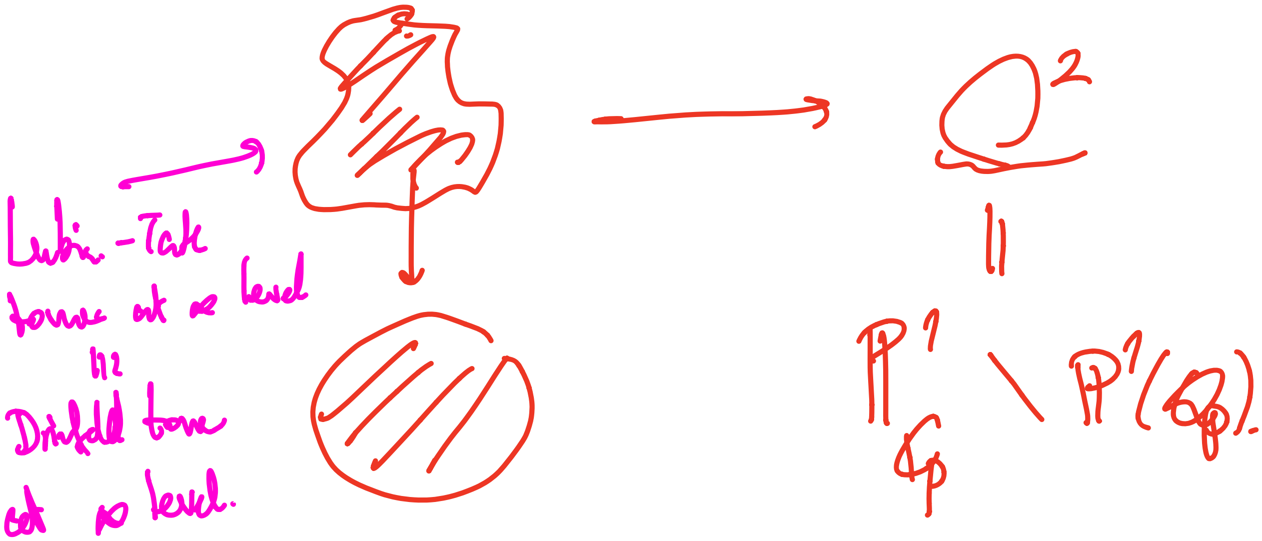
only on rk 1 points.

$$\mathbb{P}^1(\mathbb{Q}_p) \subseteq \mathbb{P}^1_{\mathbb{C}_p} \quad \underline{\text{closed}}.$$



$$\begin{tikzcd} \mathcal{M}_{ell,\mathbb{C}_p,p^\infty} \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{C}_p}. \\ \mathcal{M}_{ell,\mathbb{C}_p}. & \end{tikzcd}$$

discs of supersing reduction.



go to boundary of disc

$\Rightarrow$ go to boundary of Ω^2

$= \mathbb{P}^1(\mathbb{Q}_p)$.

the rk 2- boundary point maps into $\mathbb{P}^1(\mathbb{Q}_p)$.

in terms of vector bundles on FF curve:

$\mathbb{P}^1_{\mathbb{C}_p}$ parametrizes modifications of trivial rk 2 vector bundle.

$\mathcal{O}^2_{X_{\mathbb{C}_p^\flat}}$ / $X_{\mathbb{C}_p^\flat, \mathbb{Q}_p}$. $\hookleftarrow$ $\mathrm{Spa}\,\mathbb{C}_p$. degree 1.

Choosing a varying line at this point, $\mathbb{C}_p^2 \twoheadrightarrow L$.

get varying

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}(L) \arrow[r] & \mathcal{O}^2_{X_{\mathbb{C}_p^\flat}} \arrow[r] \arrow[d, two heads] & L \arrow[r] & 0 \\
 & & \mathbb{C}_p^2 \arrow[ur, "?"'] & &
\end{tikzcd}$$

Over Ω^2, $\mathcal{E}(L) \cong \mathcal{O}(-1/2)$

over $\mathbb{P}^1(\mathbb{Q}_p)$, $\mathcal{E}(L) \cong \mathcal{O}(-1) \oplus \mathcal{O}$.

E nonarch local field

$\cup$

$\mathcal{O}_E \ni \pi$, $\mathbb{F}_q$. $\overline{\mathbb{F}_q}$.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ perfectoid space.

$\rightsquigarrow$ FF curve $X_S = X_{S,E}$.

Let $\mathcal{E}$ vector bundle on X_S.

For each geometric point

$$\bar{s} = \mathrm{Spa}(C, C^+) \longrightarrow S,$$

(rather the strict henselization)

Can consider $\mathcal{E}_{\bar{s}} / X_{\bar{s}}$

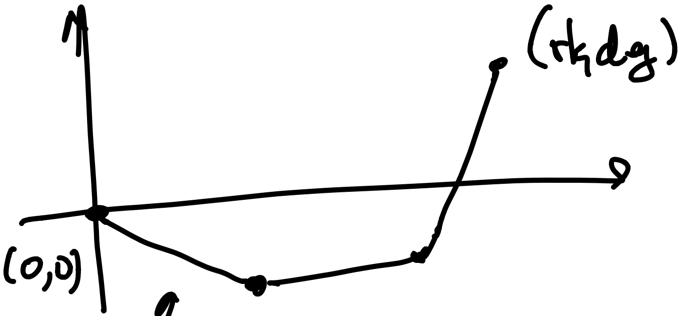
<u>Note:</u> $VB(X_{\bar{s}}) \cong VB(X_{\mathrm{Spa}(C,\mathcal{O}_C)})$, so

can forget about C^+.

Classification of vector bundles.

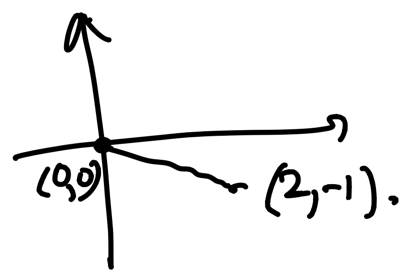
$$\rightsquigarrow \quad \mathcal{E}_{\bar{s}} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_{\bar{s}}}(\lambda)^{n_\lambda(\bar{s})}$$

$\rightsquigarrow$ Newton polygon:

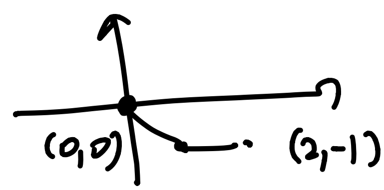


length $\dim n_\lambda(\bar{\mathcal{E}})$. for each slope λ.

<u>Ex.</u> $\mathcal{O}(-1/2)$:



$\mathcal{O}(-1) \oplus \mathcal{O}$:



How does the Newton polygon vary?

Ordering on Newton polygons:

$P \geq P'$ if P lies on or above P'

with same endpoints.

<u>Thm</u> (Kedlaya-Liu '15).

1). The function

$$\bar{s} \longmapsto NP(\mathcal{E}_{\bar{s}}) : |S| \to \left\{\begin{smallmatrix}\text{Newton}\\ \text{polygons}\end{smallmatrix}\right\}.$$

is semicontinuous.

2) If the Newton polygon is constant, then there is a global Harder-Narasimhan filtration

$$\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$$

by $\underset{\text{strict}}{\wedge}$ subvector bundles, and each

$$\mathcal{E}^{\lambda} = \mathcal{E}^{\geq\lambda} \Big/ \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$$

is everywhere semistable slope of λ.

Pro-étale locally on S, there is an isom.

$$\mathcal{E} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_S}(\lambda)^{n_\lambda}.$$

want to explain a new proof, again relying on geometry of diamonds + v-descent.

<u>Key</u>. Projectivized Banach–Colmez spaces are <u>proper</u>.

<u>Separated + Proper maps</u>.

Back to setting of v-sheaves on Perf = {perf'oid spaces of char. p}.

<u>Def'n</u>. Let $f: \mathcal{F} \longrightarrow \mathcal{G}$ map of small v-sheaves.

1). f is a <u>closed immersion</u> if for all strictly totally disconnected X, $X \to \mathcal{G}$, the fibre product

$\mathcal{F} \times_{\mathcal{G}} X$ is repr. by a perfectoid space X',

$X' \longrightarrow X$ is a (Zariski) closed immersion.

<u>equiv.:</u> There is a closed generalizing subset $Z \subseteq |\mathcal{Y}|$ s.th. $\mathcal{F} \subset \mathcal{Y}$ subfunctor of all maps $X \to \mathcal{Y}$ s.th. $|X| \to |\mathcal{Y}|$ factors over Z.

2) f is <u>separated</u> if Δ_f is a closed immersion.

3) f is <u>proper</u> if f separated, quasicompact + universally closed.

<u>Note:</u> No "finite type" assumption!!

There are valuative criteria:

<u>Prop'n</u>. f separated (resp. proper) iff it is quasiseparated (resp. qcqs) and for all diagrams

$$\begin{tikzcd} \mathrm{Spa}(R,R^\circ) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathrm{Spa}(R,R^+) \arrow[r] \arrow[ur, dotted] & \mathcal{G}. \end{tikzcd}$$

then exists at most one (resp. exactly one) dotted arrow.

for all affinoid perfectoid $\mathrm{Spa}(R,R^+)$.

In fact, enough to check for

$$(R,R^+) = (C, C^+).$$

valuation subring.

Complete alg closed.

Definition. $f: \mathcal{F} \to \mathcal{G}$ partially proper if "proper without quasicompact", i.e.:

separated +

$$\begin{tikzcd} \mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathcal{G} \end{tikzcd}$$

Proposition. Let $f: \mathcal{F} \to \mathcal{G}$ be quasicompact. Then f is surjective as a map of v-sheaves iff $|f|: |\mathcal{F}| \to |\mathcal{G}|$ is surjective.

Sketch. reduce to repr. $\mathcal{F}, \mathcal{G}$.
$$\mathcal{F} = X, \quad \mathcal{G} = Y.$$
But then $X \to Y$ is a v-cover. $\square$

Projectivized Banach–Colmez Spaces.

Prop'n. Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{E} \in VB(X_S)$.

Then
$$\mathcal{BC}(\mathcal{E}) : T_{/S} \longmapsto H^0(X_T, \mathcal{E}|_{X_T}).$$
is a locally spatial diamond, partially proper/S.

The projectivized Banach–Colmez space.
$$\left(\mathcal{BC}(\mathcal{E}) \setminus \{0\}\right) / \underline{E}^\times.$$
is a locally spatial diamond, proper/S.

Sketch. $\mathcal{O}(1)$ ample $\rightsquigarrow$ $\exists$ surjection
$$\mathcal{O}_{X_S}(-n)^N \twoheadrightarrow \mathcal{E}. \quad \text{for } n, N \gg 0.$$
$$\rightsquigarrow \quad \mathcal{E} \hookrightarrow \mathcal{O}_{X_S}(n)^N.$$

$\rightsquigarrow$ closed immersion

$$\mathcal{BC}(\mathcal{E}_\epsilon) \hookrightarrow \mathcal{BC}(\mathcal{O}_{X_S}(n)^N).$$

$\rightsquigarrow$ reduce to $\mathcal{E}_\epsilon = \mathcal{O}_{X_S}(n)^N$.

partially proper: Clear, as theory of vector bundles does not depend on $+$- rings.

$\uparrow$.
valuative criterion of.

$\mathcal{BC}(\mathcal{O}(n))$ can be analyzed inductively:

$$0 \to \mathcal{O}_{X_S}(n-1) \to \mathcal{O}_{X_S}(n) \to \mathcal{O}_{S^\sharp} \to 0.$$

$$\rightsquigarrow$$

$$0 \to \mathcal{BC}(\mathcal{O}(n-1)^N) \to \mathcal{BC}(\mathcal{O}(n)^N) \to (\mathbb{A}^N_{S^\sharp})^\diamond \to 0.$$

also.

qs. locally spatial diamond

OK for $n-1=1$,
or by induction.

hard part: $\left(\mathcal{B}\mathcal{C}(\mathcal{E})\setminus\{0\}\right)/\underline{E}^\times$ is quasicompact.

Assume that S is qcqs.

$\mathcal{O}_E^\times$ compact $\Updownarrow$

$\left(\mathcal{B}\mathcal{C}(\mathcal{E})\setminus\{0\}\right)/\pi^{\mathbb{Z}}$ is qc.

$\Updownarrow$

$\left(|\mathcal{B}\mathcal{C}(\mathcal{E})|\setminus\{0\}\right)/\pi^{\mathbb{Z}}$ is qc.

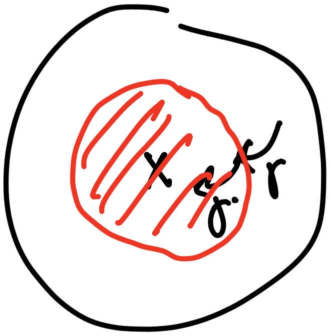
This follows from a general lemma about "contracting" automorphisms of locally spectral spaces:

(action of π $=$) $\gamma \circlearrowright T$. $\left(=|\mathcal{B}\mathcal{C}(\mathcal{E})|\right)$.

closed.

$T_0 = T^\gamma$ ← spectral, i.e. qcqs.

<u>Hypothesis:</u> · "T looks like analytic adic space".



· "for $n \to \infty$, action of γ^n contract towards T_0"

· "for $n \to -\infty$, action of γ^n on $T \setminus T_0$ diverges".

<u>Output:</u> γ acts freely on $T \setminus T_0$, and + discont.

$$(T \setminus T_0) / \gamma^{\mathbb{Z}} \quad \text{spectral, i.e. qcqs.}$$

This gives desired result. $\square$

<u>Back to Thm of Kedlaya-Liu:</u>

$S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{E} \in VB(X_S)$.

1): $NP(\mathcal{E}) : s \mapsto NP(\mathcal{E}_s)$

is semicontinuous.

<u>Note:</u> $NP(\mathcal{E}_s)$ = convex hull of all points (i, d_i) for $i = 0, \dots, \mathrm{rk}\, \mathcal{E}_s$ s.th there exists a nonzero section of

$$(\Lambda^i \mathcal{E}_s)(-d_i).$$

$\rightsquigarrow$ enough to prove: the locus of all points where $\mathcal{E}_s$ has a nonzero section is closed in S.

But this is precisely the image of

$$\underbrace{\left| \left(\mathcal{BC}(\mathcal{E}) \setminus \{0\} \right) / \underline{E}^\times \right|} \longrightarrow S.$$

But proper $/S$, so image closed.

2): want: if $NP(\mathcal{E})$ constant, then

then exists global HN filtration,

and pro-étale locally

$$\mathcal{E}_{\xi} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_S}(\lambda)^{n_\lambda}.$$

<u>Claim.</u> enough to see that v-locally,

$$\mathcal{E}_{\xi} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_S}(\lambda)^{n_\lambda}.$$

Indeed, then HN filtr. exists v-locally, so descends (by v-descent of vector bundles on FF curve)

cf. next time.

isom. $\mathcal{E}_{\xi}^{\lambda} \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$ form a torsor

under $\underline{GL_{n_\lambda}(D_\lambda)}$, thus a pro-étale torsor.

Can after pro-étale loc. find $\mathcal{E}_{\xi}^{\lambda} \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$.

split HN filtr: use

$$H^1(X_S, \mathcal{O}_{X_S}(\lambda)) = 0 \text{ for } \lambda \geq 0,$$
S affinoid.

<u>Proof of Claim:</u> Let λ max'l slope of $\mathcal{E}$.

Want to find a fibrewise nonzero

map $\mathcal{O}_{X_S}(\lambda) \longrightarrow \mathcal{E}$ after a v-cover.

Then

$$0 \to \mathcal{O}_{X_S}(\lambda) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0.$$

NP still constant,

so win by induction.

But let $\mathcal{E}' = \mathcal{H}om(\mathcal{O}_{X_S}(\lambda), \mathcal{E})$

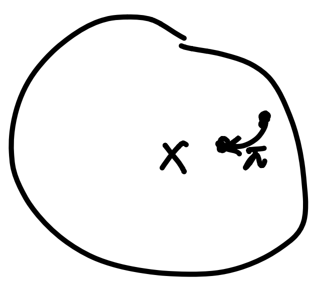
$$\mathcal{BC}(\mathcal{E}') \setminus \{0\} \longrightarrow (\mathcal{BC}(\mathcal{E}') \setminus \{0\}) / \underline{E}^\times \longrightarrow S.$$

surjective as v-sheaves. proper, surj. on here, have $\Longrightarrow$ Surj. as v-sheaves. geometric points.

map $\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$. that is nonzero in each fibre. $\square$

$$\left(\underbrace{\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}}\right) / \underline{E}^\times$$

perfectoid punctured open unit disc.



on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$:

$$\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \cong \left(\mathrm{Spa}\ \breve{E}_\infty^{LT}\right)^\diamond .$$

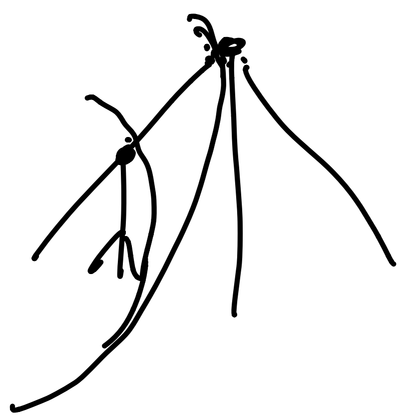
by Lubin-Tate theory.

$$\left(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}\right) / \underline{E}^\times \cong \left(\mathrm{Spa}\ \breve{E}_\infty^{LT}\right)^\diamond / \underline{E}^\times$$

$$0 \to \mathcal{O} \hookrightarrow \mathcal{O}(1) \to \mathcal{O}_D \to 0 \quad \cong (\mathrm{Spa}\,\breve{E})^{\diamond}/\varphi^{\mathbb{Z}}$$

$$= \mathrm{Div}^1_X.$$

$$\left(\mathcal{B}\mathcal{C}(\mathcal{O}(d))\setminus\{0\}\right)/\underline{E}^{\times} \xrightarrow{\sim} \mathrm{Div}^d_X.$$



if X str. tot. disc.

$$\{\text{closed } Z \subset X\} \cong \{\text{closed } S \subset \underbrace{\pi_0 X}_{\text{profinite}}\}.$$

$$H^1(X_S, \mathcal{O}) = H^1_{\text{proét}}(S, \underline{E}).$$

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\#} \to 0.$$

$$\begin{tikzcd}
\tilde{G}^{ad}(R^\#) \arrow[r, "\log_{\tilde{G}}"] \arrow[dr] & R^\# \arrow[r] & H^1(X_S, \mathcal{O}) \arrow[r] & 0 \\
 & G^{ad}(R^\#) \arrow[u, "\log_G"'] & &
\end{tikzcd}$$

pro-étale locally surjective.