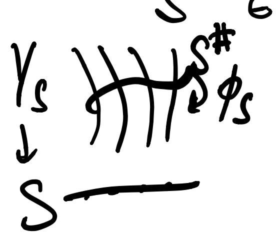
Banach–Colmez Spaces + Classification of Vector Bundles

Last Time: $E \supseteq \mathcal{O}_E \ni \pi$, $\mathbb{F}_q$.

$S \in \mathrm{Perf}_{\mathbb{F}_q} \rightsquigarrow$



$$\begin{array}{ccc} Y_{S,E} & = & Y_S \\ \downarrow & & \downarrow \\ X_{S,E} & = & X_S \end{array}$$

"Sections of $Y_{S,E} \to S$":

Proposition. The following sets are canonically in bijection:

- sections of $Y_{S,E}^\diamond \longrightarrow S$
- maps $S \longrightarrow (\mathrm{Spa}\, E)^\diamond$
- untilts $S^\# / E$ of S
- degree 1 closed Cartier divisors $D \subset Y_{S,E}$. $(\cong S^\#)$

$\rightsquigarrow$ moduli problem $\mathrm{Div}^1_Y = (\mathrm{Spa}\, E)^\diamond$.

variant for X_S:

<u>Proposition.</u> The following sets are canonically in bijection:

- maps $S \longrightarrow (\mathrm{Spa}\ E)^{\diamond}/\varphi^{\mathbb{Z}}$
- degree 1 closed Cartier divisors $D \subset X_S$.

$\rightsquigarrow$ moduli problem $\mathrm{Div}^1_X = (\mathrm{Spa}\ E)^{\diamond}/\varphi^{\mathbb{Z}}$.

often work over $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ instead. Then get

$$\mathrm{Div}^1_Y = (\mathrm{Spa}\ \breve{E})^{\diamond}, \quad \mathrm{Div}^1_X = (\mathrm{Spa}\ \breve{E})^{\diamond}/\varphi^{\mathbb{Z}}.$$

$$= \underbrace{(\mathrm{Spa}\ C)^{\diamond}}_{\mathrm{Spa}\ C^{\flat}} / \underbrace{I_E \times \varphi^{\mathbb{Z}}}_{W_E}.$$

$C = \hat{\breve{E}}$

Weil group of E

$\rightsquigarrow \pi_1(\mathrm{Div}^1_X) \cong W_E$.

$$\begin{tikzcd}
\mathrm{Spa}\ \breve{E}_\infty \arrow[r] \arrow[d, "\underline{E}^\times\text{-torsor}"'] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\
\mathrm{Div}^1_X \arrow[r] & \mathrm{Pic}^1_X \\
D \arrow[r, mapsto] & \mathcal{O}(D)
\end{tikzcd}$$

given by $\mathcal{O}(1)$, is an $\underline{E}^\times$-torsor

Pic^1_X = moduli space of line bundles of degree 1.

$\mathrm{Pic}^1_X \cong [*/\underline{E}^\times]$.

Let $G / O_{\breve{E}}$ Lubin–Tate group with $\circlearrowleft O_E$

$$\log_G(X) = X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \dots$$

$$\breve{E}_\infty = \widehat{\breve{E}(G[\pi^\infty])}.$$

<u>Proposition</u>. The follow. sets are canonically in bijection:

- maps $S \longrightarrow (\mathrm{Spa}\, \breve{E}_\infty)^\diamond = \mathrm{Spa}\, \breve{E}_\infty^{\flat}$

$$\left(\breve{E}_\infty^{\flat} \simeq \overline{\mathbb{F}_q}((X^{1/p^\infty})) \right)$$

- degree 1 closed Cartier divisors

$$D \subset X_S \quad + \quad \text{isom. } \mathcal{O}(D) \simeq \mathcal{O}(1).$$

$$AJ^1 : \mathrm{Div}^1_X \longrightarrow \mathrm{Pic}^1_X = [\ast / \underline{E}^\times].$$

$\leadsto$ map $W_E = \pi_1(\mathrm{Div}^1_X) \longrightarrow \pi_1(\mathrm{Pic}^1_X) = E^\times$.

Artin reciprocity map of local class field theory.

Fargues ("Simple connexité des fibres d'une application d'Abel–Jacobi et corps de classes local").

Using also $AJ^d : \mathrm{Div}^d_X \longrightarrow \mathrm{Pic}^d_X$, can

$d \geq 1$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{[*/\underline{E}^\times]}$

imitate Deligne's proof of geometric class field theory

$\rightsquigarrow$ Artin reciprocity induces isom. $W_E^{ab} \cong E^\times$.

(Any 1-dim'l class. of W_E induces 1-dim'l local system on $\mathrm{Div}^d_X = (\mathrm{Div}^1_X)^d/\Sigma_d$. Fibres of AJ^d are simply connected for $d \geq 2$ (or 3) $\rightsquigarrow$ descends to $\mathrm{Pic}^d_X = [*/\underline{E}^\times]$ $\rightsquigarrow$ get class. of $E^\times$.)

Banach–Colmez Spaces.

Reference: A.-C. & Bras. Coherent sheaves on the FF curve ...

Berkeley Lectures.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\rightsquigarrow X_S$. Let $\mathcal{E}$ vector bundle on X_S.

($=$ locally free $\mathcal{O}_{X_S}$-module of finite rank.)

__Thm__ (Kedlaya-Liu) If $X = \mathrm{Spa}(A, A^+)$ affinoid analytic adic space (so $\mathcal{O}_X$ sheaf),

$$\mathrm{VB}(X) \xleftarrow{\sim} \{\text{fin. proj. } A\text{-modules}\}.$$

$$M \otimes_A \mathcal{O}_X \longleftarrow\!\shortmid \quad M.$$

$+ \quad H^i(X, \mathcal{E}) = 0 \quad$ for $i > 0$, $\mathcal{E} \in \mathrm{VB}(X)$.

(+ also for H^i_{et} if $\mathcal{O}_X$ is an étale sheaf.)

Back to $\mathcal{E} | X_S$.

__Proposition.__ If S affinoid,

$H^i(X_S, \mathcal{E}) = 0 \quad$ for $i \geq 2$

$H^i(Y_S, \mathcal{E}) = 0 \quad$ for $i \geq 1$.

__Sketch.__ Pick ϖ pseudouniformizer

$\rightsquigarrow$ radius $\quad \mathrm{rad}: Y_S \longrightarrow (0, \infty)$.

comparing $|[\varpi]|$, $|\pi|$. $\quad \phi_S \circlearrowleft Y_S$, $\quad (0,\infty) \circlearrowleft \times q$.

for interval $I = [a, b] \quad a, b \in \mathbb{Q} \quad 0 < a \leq b < \infty$.

have rational subset $Y_{S,I} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$.

$$\{|[\varpi]|^b \leq |\pi| \leq |[\varpi]|^a \neq 0\} = Y_{S,I} \subset Y_S \subset$$

$$Y_{S,I} \supseteq \mathrm{rad}^{-1}(I)$$

same rk 1 points.

affinoid, analytic.

Then

$$X_S = Y_{S,[1,q]} \Big/ \left(Y_{S,[1,1]} \underset{\varphi}{\cong} Y_{S,[q,q]} \right).$$

$\rightsquigarrow$ Čech complex

$$R\Gamma(X_S, \mathcal{E}) \cong \left[\mathcal{E}(Y_{S,[1,q]}) \xrightarrow{\varphi - 1} \mathcal{E}(Y_{S,[q,q]}) \right].$$

$\rightsquigarrow$ vanishing in deg ≥ 2.

for Y_S: $\quad Y_S = \bigcup_I Y_{S,I}$, $\quad \mathcal{O}(Y_{S,I'}) \to \mathcal{O}(Y_{S,I})$

aff'd $\qquad$ have dense image.

$$R\Gamma(Y_S, \mathcal{E}) = R\varprojlim_I \mathcal{E}(Y_{S,I}). \quad \text{"}Y_S \text{ Stein"}.$$

$\varprojlim^1$ vanishes by Mittag-Leffler.

$\square$

Proposition. $T \in \mathrm{Perf}_{/S} \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf. In fact, $T \longmapsto R\Gamma(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf of complexes (in $\mathcal{D}(\mathbb{Z})$).
In particular, if $H^0(X_T, \mathcal{E}|_{X_T}) = 0 \quad \forall T \in \mathrm{Perf}_{/S}$,
then $T \mapsto H^1(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf.

Sketch. enough after $-\hat{\otimes}_E E_\infty$. $\quad$ (E_∞ perfectoid.)

$X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ is perfectoid.

v-covers on S induce v-covers.

Now use v-sheaf + acyclicity properties for general perfectoid spaces. $\square$

Def^n. 1). $\mathcal{BC}(\mathcal{E}) : \mathrm{Perf}_{/S} \longrightarrow \mathrm{Sets}$.

$$T \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$$

Banach-Colmez space assoc to $\mathcal{E}$.

2) If $\mathcal{BC}(\mathcal{E}) = 0$, then

$$\mathcal{BC}(\mathcal{E}_c[1]) : T \mapsto H^1(X_T, \mathcal{E}/_{X_T})$$

"negative Banach–Colmez space".

<u>Proposition</u>. 1). $\mathcal{BC}(\mathcal{E}_c)$, $\mathcal{BC}(\mathcal{E}[1])$

locally spatial diamonds$/S$.

2). If $E \cong \mathbb{F}_q((t))$, $\mathcal{BC}(\mathcal{E})$ repr. by perfectoid space.

3). If $\mathcal{E}_c = \mathcal{O}(\lambda)$, $0 < \lambda \leq [E:\mathbb{Q}_p]$ (resp. all $0 < \lambda$ if $E \cong \mathbb{F}_q((t))$)

$\frac{r}{s}$ $(s,r)=1$, $r,s>0$.

then $\mathcal{BC}(\mathcal{E}_c) \cong \widetilde{\mathbb{D}}^r_S$ r-dim'l open perfectoid disc$/S$.

+ for S affinoid

$$H^1(X_S, \mathcal{E}) = 0 .$$

4) $R\Gamma(X_S, \mathcal{O}_{X_S}) \cong R\Gamma_{\text{proet}}(S, \underline{E})$.

In part., if $S = \mathrm{Spa}\, C$, $R\Gamma(X_C, \mathcal{O}_{X_C}) = E[0]$.

<u>Sketch</u>. 3) : similar to identification.

$$\mathcal{BC}(\mathcal{O}(1)) \cong \tilde{\mathbb{D}}_S \qquad \text{from last time.}$$

In general, say for $E = \mathbb{Q}_p$,

$$\mathcal{BC}(\mathcal{O}(\lambda)) \cong \tilde{G}_S \ , \ \text{where}$$

$\tilde{G}$ = univ. cover of p-div. group $G/\overline{\mathbb{F}_q}$ with Dieudonné module $= D_{-\lambda}$.

Vanishing of H^1: direct computation using $X_S = Y_{S,[1/A]}/\ldots$.

4) Use (pro-étale locally on S)

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\#} \to 0.$$

$$\leadsto 0 \to H^0(\mathcal{O}) \to \underset{\substack{\| \\ \tilde{G}(S^\#)}}{H^0(\mathcal{O}(1))} \xrightarrow[\log_{\tilde{G}}]{} R^\# \to H^1(\mathcal{O}) \to 0.$$

Now use that $\log_{\tilde{G}}$ pro-étale locally surj.,

$$\ker = \underline{E}.$$

1) + 2): Bootstrap from 3) using various exact

sequences.

<u>e.g.</u> : $\mathcal{BC}(\mathcal{O}(-1)[1]) \cong (\mathbb{A}^1_E)^\diamond / \underline{E}$:

for $S/(\mathrm{Spa}\, E_\infty^{LT})^\diamond$.

Use

$$0 \to \mathcal{O}_{X_S}(-1) \to \mathcal{O}_{X_S} \to \mathcal{O}_{S^\#} \to 0.$$

$$\begin{array}{ccccccc} 0 \to & H^0(\mathcal{O}) & \to & H^0(\mathcal{O}_{S^\#}) & \to & H^1(\mathcal{O}(-1)) \to & H^1(\mathcal{O}) \\ & \| & & \| & & & \downarrow \\ & \underline{E}(S) & & (\mathbb{A}^1_E)^\diamond(S) & & \text{proét locally} & 0. \\ & & & & & \| & \\ & & & & & 0. & \end{array}$$

□.

Classification of Vector Bundles

Back to $S = \mathrm{Spa}\, C$. geometric point.

Thm $\mathrm{Isoc}_E / \cong \;\longrightarrow\; VB(X_C)/\cong$

bijection. :

Any $\mathcal{E} \in VB(X_C)$ is isom. to

$\bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_C}(\lambda)^{n_\lambda}$. for unique $n_\lambda \in \mathbb{Z}$

Step 1. $\mathcal{O}_{X_C}(1)$ is ample: For any $\mathcal{E}$, all $n \gg 0$,

Kedlaya–Liu $\mathcal{E}_C(n)$ globally generated + $H^1(X_C, \mathcal{E}(n)) = 0$

(works for any affinoid S).

(uses presentation $X_S = Y_{S,[1,q]} / (Y_{S,[1,1]} \overset{\simeq}{\underset{\varphi}{}} Y_{S,[q,q]})$

+ explicit estimates .)

Step 2. $\mathrm{Pic}(X_C) \cong \mathbb{Z}$.

$\mathcal{O}_{X_C}(n) \leftarrow n$

Step 1 $\Rightarrow$ any $\mathcal{L} \in \mathrm{Pic}(X_C)$ is generically trivial.

on the schematic curve.

$\rightsquigarrow$ $\mathcal{L} \cong \mathcal{O}(D)$ some divisor D on schematic curve.

All closed pts on schematic curve $\cong$ untilts,

and $\mathcal{O}(\text{untilt}) \cong \mathcal{O}(1)$ by last lecture

$\rightsquigarrow$ $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$.

$\Rightarrow$ $\mathbb{Z} \twoheadrightarrow \mathrm{Pic}(X_C)$. As $H^0(\mathcal{O}(-n)) \underset{n>0}{=} 0$.

isom.

Step 3. Build Harder - Narasimhan formalism.

rk, deg: $VB(X_C) \to \mathbb{Z}$ $\rightsquigarrow \mu = \frac{\deg}{rk}$ slope.

$\rightsquigarrow$ Harder - Narasimhan filtrations.

Using $H^1(X_C, \mathcal{O}_{X_C}(\lambda)) = 0$ for $\lambda \geq 0$,

reduce to case of semistable $\mathcal{E}_C$.

+ ε : semistable $\mathcal{E}_C$ of slope 0.

Step 4: Goal: Any semistable $\mathcal{E}_C$ of slope 0 is isom.

to $\mathcal{O}_{X_C}^n$.

enough to show this after possibly enlarging C.

(v - descent: If true over C'/C ($\Rightarrow \underset{v\text{-covers}}{\mathrm{Spa}\, C' \to \mathrm{Spa}\, C}$),

torsor of isom. $\mathcal{E}_C \cong \mathcal{O}^n$ $\underline{GL_n(E)}_v$-torsor over $\mathrm{Spa}\, C$.

Any such torsor is split.)

Also, can assume by induction that Thm is

true in smaller rank.

Consider minimal $\begin{matrix} d \geq 0 \\ d \in \mathbb{Z} \end{matrix}$ s.th. there exists

an injection

$$0 \to \mathcal{O}_{X_C}(-d) \hookrightarrow \mathcal{E} \to \overline{\mathcal{E}} \to 0.$$

$d = 0$: Then $\overline{\mathcal{E}}$ semistable of slope 0,

so $\overline{\mathcal{E}} \cong \mathcal{O}_{X_C}^{n-1}$ by induction,

$H^1(X_C, \mathcal{O}_{X_C}) = 0 \to$ extension splits.

$\mathcal{E} \cong \mathcal{O}^n$.

$d \geq 2$: rather simple contradiction.

Key case $d = 1$: Get

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0.$$

rk $n-1$, deg 1, slopes ≥ 0.

induction $\to$ $\overline{\mathcal{E}} \cong \mathcal{O}_{X_C}^i \oplus \mathcal{O}_{X_C}\left(\frac{1}{n-1-i}\right)$.

Key case: $\overline{\mathcal{E}} \cong \mathcal{O}\left(\frac{1}{n-1}\right)$.

reduced to following Lemma:

Lemma. Let $\mathcal{E}_C$ be an extension

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E}_C \to \mathcal{O}_{X_C}\left(\tfrac{1}{n}\right) \to 0.$$

Then, after possibly enlarging C,

$$H^0(X_C, \mathcal{E}_C) \neq 0.$$

Remark. Reduction to this lemma goes back to Hartl-Pink '04.

Proof of Lemma. Assume contrary.

Then for all $S \in \mathrm{Perf}_{/C}$,

$$H^0\left(X_S, \mathcal{O}_{X_S}(1/n)\right) \hookrightarrow H^1\left(X_S, \mathcal{O}_{X_S}(-1)\right),$$

i.e.

$$\mathcal{BC}\left(\mathcal{O}(1/n)\right) \xhookrightarrow{\cong} \mathcal{BC}\left(\mathcal{O}(-1)[1]\right).$$

$$\begin{array}{ccc} \mathcal{BC}(\mathcal{O}(1/n)) & & \mathcal{BC}(\mathcal{O}(-1)[1]) \\ \wr\wr & & \wr\wr \\ \widetilde{\mathbb{D}}_C & & (\mathbb{A}^1_{C^\flat})^\diamond / \underline{E}. \end{array}$$

perfectoid open unit disc

injection. <u>not</u> perfectoid, at least if $E/\mathbb{Q}_p$.

also necessarily surjective: image cannot be contained in classical points. (those are tot. disc.)

$\Rightarrow$ contains some non-classical point.

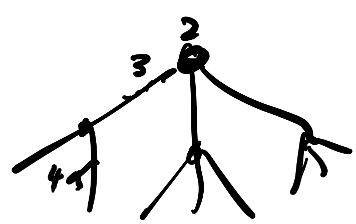
$\Rightarrow$ after possibly enlarging C, image contains nonempty open subset.

$\Rightarrow$ contains nonempty open nbhd of 0

contracting $\Rightarrow$ action of $\times\pi$ image contains everything. $\square$.

$x \in |(\mathbb{A}^1_C)^{ad}|$

$C(x)$.



$2 \,\hat{=}$ generic point of disc

$\mathbb{B}(x, r)$. $\qquad$ $C(x)$

$$\tilde{x} \in \left(\mathbb{A}^1_{C(x)}\right)^{ad} = \left(\mathbb{A}^1_C\right)^{ad} \times_{\operatorname{Spa} C} \operatorname{Spa} C(x).$$

$\mathbb{B}(\tilde{x}, <r) \subset$ contained in preimage of

$\{x\} \subseteq \left|\left(\mathbb{A}^1_C\right)^{ad}\right|$.

E'/E $\quad$ deg d. $\qquad$ $\pi_{E'/E} : X_{S,E'} \to X_{S,E}$. $\quad X_{S,E'} = X_{S,E} \otimes_E E'$.

$$\mathcal{O}_{X_{S,E}}\left(\tfrac{1}{d}\right) \cong \pi_{E'/E *} \mathcal{O}_{X_{S,E'}}(1).$$

$\mathbb{A}^n \setminus \{0\}$. $\qquad$ $\mathcal{BC}(\mathcal{L}) \setminus \{0\}$.

$$\begin{tikzcd} \pi_{HT} : \mathcal{M}^*_{ell,\infty,\mathbb{C}_p} \arrow[r] & \mathbb{P}^1_{\mathbb{C}_p} \\ \mathcal{M}^*_{ell,\infty,\text{antican}}(\varepsilon) \arrow[u, phantom, sloped, "\subset"] \arrow[r] & \mathbb{B}(0,\varepsilon). \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

as adic spaces.

aff'd perf'oid.

$\rightsquigarrow \quad f \in \mathcal{O}\left(\mathcal{M}^*_{ell,\infty,\text{antican}}(\varepsilon)\right).$

generates a closed ideal,

but after passing to ordinary locus,

not any more.