Untilts, $\mathcal{O}(1)$, and Lubin–Tate theory.

E nonarch local field, $\mathcal{O}_E \ni \pi$, $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_q$

$\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[\frac{1}{\pi}]$ compl. of max. unr. ext.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ perf'oid space.

$$\rightsquigarrow \quad \begin{array}{c} Y_{S,E} \\ \downarrow \\ X_{S,E} \end{array} \quad \begin{array}{l} Y_{S,E}^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond}. \\ \\ = Y_{S,E} / \phi_S^{\mathbb{Z}} \end{array}$$

" Untilts = degree 1 Cartier divisors on $Y_{S,E}$"

Given an untilt $S^\# / E$ of S,

locally $S = \mathrm{Spa}(R, R^+)$
$S^\# = \mathrm{Spa}(R^\#, R^{\#+})$

and there is a canonical surjection

$$\theta : W_{\mathcal{O}_E}(R^+) \twoheadrightarrow R^{\sharp +}$$

$$\Big(\hat{=} \qquad R^+ \longrightarrow R^{\sharp+}/\pi$$

$$\hat{=} \qquad R^+ = \varprojlim_{\phi} R^{\sharp+}/\pi \longrightarrow R^{\sharp+}/\pi \Big).$$

$\ker(\theta) = (\xi)$, ξ nonzero-divisor in $W_{\mathcal{O}_E}(R^+)$.

↑ general structural result on integral perfectoid rings = perfect prisms

$$\rightsquigarrow \quad \underset{\substack{\| \\ V(\xi)}}{S^\sharp} \hookrightarrow \underset{\substack{\| \\ Y_{S,E}}}{\operatorname{Spa} W_{\mathcal{O}_E}(R^+)} \setminus \{\pi = 0 \text{ or } [\varpi] = 0\}.$$

presents $S^\sharp$ as a Cartier divisor on $Y_{S,E}$.

Def'n. let X uniform analytic adic space.

↑ R is uniform, spectral norm is a norm, i.e.

↑ locally $\operatorname{Spa}(R, R^+)$, R Tate.

$R^\circ \subseteq R$ bounded $\Leftrightarrow$ $R^+ \subseteq R$ bounded.

A <u>closed Cartier divisor</u> on X is an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, locally free of rk 1, s.th. $\forall$ affinoid $U \subseteq X$,

$$\mathcal{I}(U) \to \mathcal{O}_X(U) \text{ has closed image.}$$

$\rightsquigarrow$ ($\underset{\substack{Z = \\ \subseteq X.}}{V(\mathcal{I})}$, $\mathcal{O}_X/\mathcal{I}$, valuations) defines an adic space itself.

<u>Prop'n.</u> 1) $V(\xi) = S^\#$ $\hookrightarrow$ $Y_{S,E}$ is a closed Cartier divisor.

2) Also $S^\#$ $\hookrightarrow$ $Y_{S,E} \to X_{S,E}$ defines closed Cartier divisor.

Prop 11.3.1. in Berkeley Lectures.

<u>Definition.</u> Let Div^1_Y, Div^1_X : $\mathrm{Perf}_{\overline{\mathbb{F}}_q} \to$ Sets

be the functors taking S to the set of closed Cartier divisors on $Y_{S,E}$ (resp. $X_{S,E}$) that locally on S arise as

$$S^\# \hookrightarrow Y_{S,E}$$

$$(\text{resp. } S^\# \hookrightarrow X_{S,E})$$

for untilt $S^\# / E$.

"moduli space of degree 1 Cartier divisors".

<u>Prop'n.</u> 1) $\mathrm{Div}^1_Y = (\mathrm{Spa}\,\breve{E})^\diamond$

2) $\mathrm{Div}^1_X = \mathrm{Div}^1_Y / \phi_E^{\mathbb{Z}} = (\mathrm{Spa}\,\breve{E})^\diamond / \phi_E^{\mathbb{Z}}$.

In 2), note

$$(\mathrm{Spa}\,\breve{E})^\diamond = \underbrace{\overset{\phi_E \circlearrowleft}{(\mathrm{Spa}\,E)^\diamond}}_{\substack{\text{functor on} \\ \phi_S \circlearrowright\, S \in \mathrm{Perf}_{\mathbb{F}_q}}} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$$

Proof. 1) By def'n, $(\mathrm{Spa}\,\breve{E})^\diamond \twoheadrightarrow \mathrm{Div}^1_Y$

as any Cartier div. param. by Div^1_Y locally comes from an untilt $S^\#/\breve{E}$.

But conversely, a closed Cartier divisor on $Y_{S,E}$ determines $Z \hookrightarrow Y_{S,E}$, locally on S this gives an untilt of S.

$\rightsquigarrow$ Z is untilt $S^\#$ of S.
$/\breve{E}$

$\rightsquigarrow$ $(\mathrm{Spa}\,\breve{E})^\diamond \xrightarrow{\sim} \mathrm{Div}^1_Y$.

2) Take quotient by Frobenius. $\square$

$$\lightning \quad X^\diamond_{S,E} = S/\varphi_S^{\mathbb{Z}} \times (\mathrm{Spa}\,E)^\diamond$$

$$\underline{\mathrm{Div}^1_X} = (\mathrm{Spa}\,\breve{E})^\diamond/\varphi_E^{\mathbb{Z}}.$$

"mirror curve". only a diamond.
not quasiseparated, not locally spatial.

$\mathcal{O}(1)$ + Lubin–Tate theory.

Recall: $\mathcal{O}_{X_{S,E}}(1)$ is the line bundle on $X_{S,E}$ corr. to isocrystal $(\breve{E}, \pi^{-1}\sigma)$

In particular,

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})^{\phi_S = \pi}.$$

<u>Goal:</u> If $S^\#$ any untilt / E of S, then $\mathcal{I}_{S^\#} \subseteq \mathcal{O}_{X_{S,E}}$ is (at least after pro-étale localization on S) isomorphic to $\mathcal{O}_{X_{S,E}}(-1)$.

(In this sense, $S^\# \hookrightarrow X_{S,E}$ is of degree 1.)

to do this, need to construct maps

$$\mathcal{O}(-1) \to \mathcal{I}_{S^\#} \hookrightarrow \mathcal{O}$$

$\hat{=}$ maps $\quad \mathcal{O} \longrightarrow \mathcal{O}(1)$

i.e. $\quad H^0(X_{S,E}, \mathcal{O}(1))$.

We will give a formula for $H^0(X_{S,E}, \mathcal{O}(1))$ in terms of a Lubin-Tate formal group.

<u>Recall</u>. A Lubin-Tate group is a 1-dim'l formal group $G / \mathcal{O}_{\breve{E}}$ with an action of $\mathcal{O}_E$ s.th. the two induced actions on $\mathrm{Lie}\ G$ agree, and "of <u>height 1</u>".

<u>Thm</u>. Such a G is unique up to isomorphism.

<u>Example</u>. $\quad E = \mathbb{Q}_p$.

$$\rightsquigarrow \quad G = \text{formal multiplicative group}.$$

$$= \mathrm{Spf}\ \breve{\mathbb{Z}}_p[[X]].$$

$$X +_G Y = (1+X)(1+Y) - 1.$$

$$[\pi]_G(X) = \pi X + a_2 X^2 + \dots$$

mod π: first nonzero coeff. is necessarily $a_{q^h} X^{q^h}$ $h = 1, 2, \ldots, \infty$.

h = height of G.

In general, $G \times_{\mathcal{O}_{\breve{E}}} \breve{E} \underset{\log_G}{\cong} \mathbb{G}_{a,\breve{E}}$ additive group

One can choose $G \cong \mathrm{Spf}\, \mathcal{O}_{\breve{E}}[\![X]\!]$ so that

$$\log_G(X) = X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \ldots$$

$\leadsto$ $X +_G Y = \underset{\log_G^{-1}}{\underset{\wr}{\exp_G}}\Big(\underbrace{\log_G(X) + \log_G(Y)}\Big)$

$\in\,!$ $\mathcal{O}_{\breve{E}}[\![X, Y]\!]$.

<u>Connection to local class field theory:</u>

For any $n \geq 1$,

$G[\pi^n] \subseteq G$ kernel of mult. by π^n on G.

$\parallel$

$\mathrm{Spf}\, \mathcal{O}_{\breve{E}}[\![X]\!] / \big([\pi^n]_G(X)\big)$.

($E = \mathbb{Q}_p$: get μ_{p^n} = group of p^n-th roots of unity.)

$$G[\pi^n] \times_{\mathcal{O}_{\breve{E}}} \breve{E} = \coprod_{i=0}^{n} \operatorname{Spec} \breve{E}_i$$

$(\mathcal{O}_E/\pi^n)^\times$

$\breve{E}_0 = \breve{E}$, $\breve{E}_0 \subset \breve{E}_1 \subset \dots \subset \breve{E}_n \subset \dots$

adjoin a primitive π^n-torsion point.

<u>Thm</u>. The maximal abelian extension of E is $\bigcup \breve{E}_n$. (up to completion issues)

$$\mathrm{Gal}(\breve{E}_n / \breve{E}) = (\mathcal{O}_E/\pi^n)^\times .$$

Also, $\breve{E}_\infty$ = completion of $\bigcup \breve{E}_n$ is perfectoid. $\square$

<u>"Universal cover" $\tilde{G}$ of G:</u>

<u>Definition</u>. $\tilde{G} := \varprojlim_{[\pi]_G} G$

$$\begin{tikzcd} & \tilde{G} \arrow[dl, "f_2"'] \arrow[d, "f_1"] \arrow[dr, "f_0"] & \\ G \arrow[r] & G \arrow[r] & G. \end{tikzcd}$$

Prop'n. $\quad \tilde{G} \cong \mathrm{Spf}\, \mathcal{O}_{\breve{E}}[\![\tilde{X}^{1/p^\infty}]\!]$.

Proof. enough: $\tilde{G} \times_{\mathcal{O}_{\breve{E}}} \overline{\mathbb{F}}_q \cong \mathrm{Spf}\, \overline{\mathbb{F}}_q[\![\tilde{X}^{1/p^\infty}]\!]$.

But $[\pi]_G(X) \equiv X^q \mod \pi$. $\square$

have maps $\tilde{G} \xrightarrow{f_n} G$ proj. maps.

$$\mathcal{O}_{\breve{E}}[\![\tilde{X}^{1/p^\infty}]\!] \longleftarrow \mathcal{O}_{\breve{E}}[\![X_n]\!]$$

then $\tilde{X} = \lim\limits_{n\to\infty} X_n^{q^n}$.

$\rightsquigarrow$ $$\underbrace{\tilde{G}^{ad} \times_{\mathcal{O}_{\breve{E}}} \breve{E}}_{} \longrightarrow \underbrace{G^{ad} \times_{\mathcal{O}_{\breve{E}}} \breve{E}}_{} \xrightarrow{\log_G} \mathbb{G}_a$$

infinite cover of open unit disc. $\qquad$ open unit disc.

given by $\sum\limits_{i\in\mathbb{Z}} \pi^i \tilde{X}^{q^{-i}} \in \mathcal{O}(\tilde{G}^{ad} \times_{\mathcal{O}_{\breve{E}}} \breve{E})$.

<u>Proposition</u>. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

$\rightsquigarrow X_{S,E}$, $\mathcal{O}_{X_{S,E}}(1)$. Let $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ untilt of S. Then.

top. nilp. elements of R.

$$\begin{array}{ccc} \tilde{G}^{ad}(S^\#) = \tilde{G}(R^{\#+}) \cong R^{\circ\circ} & & \tilde{x} \\ \searrow & & \downarrow \\ & H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}}) \ni \sum_{i \in \mathbb{Z}} \pi^i [\tilde{x}^{q^{-i}}] & \end{array}$$

induces an isomorphism.

$$\tilde{G}^{ad}(S^\#) \xrightarrow{\sim} H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1))$$
$$\| $$
$$H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})^{\phi_S = \pi}.$$

Under this isomorphism, the evaluation

$$H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}}) \longrightarrow R^\#$$

at $S^\# \subseteq Y_{S,E}$

corresponds to logarithm map

$$\begin{tikzcd} \tilde{G}^{ad}(S^\#) \arrow[r, "\log_{\tilde{G}}"] & R^\#. \\ \tilde{G}(R^{\#+}) \arrow[u, phantom, "\cong"] \arrow[ur] & \end{tikzcd}$$

In particular,

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) \cong \mathrm{Hom}(S, \mathrm{Spa}\, \mathbb{F}_q[[\tilde{x}^{1/p^\infty}]]).$$

<u>Remark</u>. If $n \leq [E:\mathbb{Q}_p]$ (resp. all n if E equal char.)

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(n)) \cong \mathrm{Hom}(S, \mathrm{Spa}\, \mathbb{F}_q[[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]])$$

"the functor

$$S \longmapsto H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(n)) \text{ is}$$

'repr.' by $\operatorname{Spa} \overline{\mathbb{F}}_q[[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]]$, "

'an n-dim'l perfectoid open unit disc'

$n > [E:\mathbb{Q}_p]$: this functor not representable.

$\rightsquigarrow$ Banach-Colmez spaces,

give interesting examples of diamonds.

<u>Proof</u>. - Commutation with $\log_{\overline{G}}$ follows from formulas.

- clear that one gets map to

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y, \mathcal{O})^{\phi_S = \pi}.$$

- if $E \cong \mathbb{F}_q((t))$,

$H^0(Y_{S,E}, \mathcal{O})$ = certain power series

$t = \pi$ $\qquad \sum_{n\in\mathbb{Z}} \pi^n \cdot r_n \qquad r_n \in R,$

subject to convergence on punctured open unit disc.

condition $\phi_S = \pi$: $\quad r_{n+1}^q = r_n.$

$\rightsquigarrow$ everything determined by $r_0 = r \in R$.

so that $\sum_{n\in\mathbb{Z}} \pi^n r^{1/q^n}$ converges.

This happens $\Leftrightarrow$ $r \in R^{\circ\circ}$ top. nilpotent.

- if $E/\mathbb{Q}_p$, can be deduced from a result in Dieudonné theory in [SW13]. $\square$

$\lightning$ If E is p-adic, one cannot describe

$H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})$ as certain sums

$$\sum_{n\in\mathbb{Z}} \pi^n [r_n] \qquad r_n \in R.$$

$$0 \to \bigcup_n G^{ad}_{\breve{E}}[\pi^n] \longrightarrow G_{\times \breve{E}}^{ad} \xrightarrow{\log_G} \mathbb{G}_{a,\breve{E}} \to 0.$$

as étale sheaves.

$$0 \to V_\pi G^{ad}_{\breve{E}} \to \tilde{G}^{ad}_{\breve{E}} \xrightarrow{\log_{\tilde{G}}} \mathbb{G}_{a,\breve{E}} \to 0$$

rational π-adic Tate module.

E ↺ $2/\mathbb{Q}$

on geometric points.

$$V_\pi G^{ad}_{\breve{E}} \setminus \{0\} = \bigsqcup_{n\in\mathbb{Z}} \mathrm{Spa}\, \breve{E}_\infty .$$

In particular, given an untilt $S^\# / \breve{E}_\infty$, get a distinguished section

$$\begin{tikzcd} s \arrow[d, mapsto] & \in & \tilde{G}^{ad}_{\check{E}}(S^\#) \arrow[d, "\log_{G}"] \\ 0 & \in & G_{0,\check{E}}(S^\#) \end{tikzcd}$$

corr. to chosen compatible π^n-torsion points.

Thus, under the isom. of Prop'n, get map

$$\begin{tikzcd} \mathcal{O}_{X_{S,E}} \arrow[r] \arrow[dr, "0"'] & \mathcal{O}_{X_{S,E}}(1) \arrow[d] \\ & \mathcal{O}_{S^\#} \end{tikzcd}$$

corr. to s,

evaluate at chosen untilt.

$\rightsquigarrow$ map $\mathcal{O}_{X,S} \longrightarrow \mathcal{I}_{S^\#}(1)$.

<u>Prop'n</u>. This map $\mathcal{O}_{X,S} \to \mathcal{I}_{S^\#}(1)$ is an isomorphism.

Proof: after confusing identifications, follows from $\ker(\log_{\tilde{G}}) \setminus \{0\}$

$$\| \\ \bigsqcup_{n \in \mathbb{Z}} \mathrm{Spa}\, \breve{E}_\infty .$$

reduce to univ. case $S^\# = \mathrm{Spa}\, \breve{E}_\infty$.

Then $Y_{S,E} = \tilde{G}^{ad}_{\breve{E}} \setminus \{0\}$.

Both are $\mathrm{Spa}\, \mathcal{O}_{\breve{E}}[\![\tilde{X}^{1/p^\infty}]\!] \setminus \left\{ \begin{matrix} \pi = 0 \text{ or} \\ \tilde{X} = 0 \end{matrix} \right\}$.

$\square$.

$\rightsquigarrow$ For any untilt $S^\# / \breve{E}$, get identification $\mathcal{I}_{S^\#} \cong \mathcal{O}(-1)$ after pro-étale cover $S^\# \times_{\breve{E}} \breve{E}_\infty \longrightarrow S^\#$.

$G_{/\overline{\mathbb{F}}_q}$ any π-div. O_E-module

$$\Big\Vert ?.$$

Dieudonné module $(V_{/\breve{E}}, \phi_V)$.

$\tilde{G} \cong \mathrm{Spf}\, \overline{\mathbb{F}}_q[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]$.

$$\tilde{G}(S)$$

$$\Big\Vert 2$$

$$H^0(X_{S,E}, \mathcal{E}(V)).$$

$$\mathrm{Div}^1_Y = (\mathrm{Spa}\, E)^\diamond$$

$$\mathrm{Div}^1_X = (\mathrm{Spa}\, E)^\diamond / \phi_E^{\mathbb{Z}}.$$

$$Y_{S,E} \hookrightarrow \mathcal{Y}_{S,E} \underset{\{\pi \neq 0\}}{\subseteq}$$

<u>Prop'n</u> (Kedlaya-Liu) ϕ-equiv. VB on $\mathcal{Y}_{S,E}$ are equiv. to $\mathbb{Z}_p$-local systems on S.

$$\mathbb{L} \mapsto (\mathbb{L} \otimes \mathcal{O}_{\mathcal{Y}_{S,E}}, \mathrm{id} \otimes \phi)$$

$$\mathcal{E}^{\phi=1} \leftarrow (\mathcal{E}, \phi)$$