Diamonds and the relative Fargues-Fontaine curve.

Recall. E nonarch local field. $\mathbb{F}_q$ res. field.

$S / \mathbb{F}_q$ perfectoid space.

Aim. Introduce rel. FF curve

$$X_{S,E} = Y_{S,E} / \phi^{\mathbb{Z}}$$

$$Y_{S,E} \text{ ``}= S \times \operatorname{Spa} E\text{''}$$

$$Y_{S,E}^{\diamond} = S \times (\operatorname{Spa} E)^{\diamond}.$$

not a functor $X \longmapsto X^{\diamond}$

$$\{\text{analytic adic spaces}/\mathbb{Z}_p\} \longrightarrow \{\text{diamonds}\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\{\text{perf'd spaces}\} \longrightarrow \{\text{perf'd spaces}/\mathbb{F}_p\}.$$

$$X \longmapsto X^{\flat}.$$

defined pro-étale morphisms of perfectoid spaces.

Pro-Étale local structure of perfectoid spaces:

Def'n. A perfectoid space X is (strictly) totally disconnected if it is qcqs (in fact, affinoid) and every

étale cover splits (strictly tot. disc.)

resp. open cover splits (tot. disc.)

Prop'n. X is (strictly) totally disconnected iff qcqs and all fibres of

$$X \longrightarrow \pi_0 X$$

profinite set

are of form

$$\mathrm{Spa}(K, K^+)$$

where K is perfectoid field. (= complete nondiscretely valued nonarch field s.th. Φ surj. on O_K/p)

and $\mathfrak{m}_{O_K} \subseteq K^+ \subseteq O_K$ valuation subring.

(resp. and K is alg. closed, in strictly tot. disc. case).

__Remark.__ $K^+/\mathfrak{m}_{\mathcal{O}_K} \subseteq \mathcal{O}_K/\mathfrak{m}_K = k$.

valuation subring. field.

$\rightsquigarrow |\mathrm{Spa}(K,K^+)| \cong |\mathrm{Spec}(\underbrace{K^+/\mathfrak{m}_{\mathcal{O}_K}}_{\text{valuation ring}})|$

totally ordered chain of specializations,

generic point

$\cong \mathrm{Spa}(K,\mathcal{O}_K) \hookrightarrow \mathrm{Spa}(K,K^+)$.

"unique rk 1 generalization of any point".

__Cor.__ Assume $X = \mathrm{Spa}(R,R^+)$ tot. disc.,

$f: Y = \mathrm{Spa}(S,S^+) \longrightarrow X$ any

aff'd adic space over X. Then

$$R^+/\varpi \longrightarrow S^+/\varpi \quad \text{flat}$$

for any pseudounif $\varpi \in R^+$.

(and faithfully flat if $|f|$ is surjective).

<u>Proof.</u> Can be checked on connected components. Then $(R,R^+) = (K,K^+)$, so K^+ valuation ring. Note: $S^+ \subseteq S = S^+[\frac{1}{\varpi}]$ is ϖ-torsion free, so S^+ is flat $/K^+$, hence S^+/ϖ flat over K^+/ϖ.

For faithful flatness, use
$$|\mathrm{Spa}(K,K^+)| \cong |\mathrm{Spec}(K^+/\varpi)|. \quad \square$$

This allows us to deduce v-descent results from pro-étale descent and faithfully flat descent.

(all maps $f: Y \to X$ s.th. X, Y qc, $|f|$ surjective)

<u>Definition.</u> A <u>diamond</u> is a pro-étale sheaf Y on $\mathrm{Perf} := \{\text{perfectoid spaces}/\mathbb{F}_p\}$ that can be written in form.

$$Y = X/R, \text{ where}$$

- X perfectoid space
- $R \subseteq X \times X$ equivalence relation repr. by a perfectoid sth. s.t. $R \to X$ pro-étale.

Here, use Yoneda embedding

$$\begin{aligned} \mathrm{Perf} &\hookrightarrow \{\text{pro-étale sheaves on Perf}\}. \\ X &\longmapsto \mathrm{Hom}(-,X). \end{aligned}$$

Some facts: · Category of diamonds has all fibre products products, cofilt. inv. limits (all nonempty limits), but no final object.

The final object would be $\mathrm{Spa}\, \mathbb{F}_p$, and cannot adjoin one through pro-étale covers.

not a perf'd space, as not analytic: no top. nilp. unit.

- If $f: Y \to X$ quasi-pro-étale map, then Y diamond $\Leftarrow$ X diamond.
  $\Rightarrow$ if f surj. as map of pro-étale sheaves.

<u>Def'n</u> A map $f: Y \to X$ of pro-étale sheaves on Perf is <u>quasi-pro-étale</u> if for all str. tot. disc. perf'd spaces X', $X' \to X$, the fibre product $f': Y' = Y \times_X X' \to X'$ is repr. in perf'd spaces, and pro-étale.

- Y diamond $\Leftrightarrow$ $\exists$ surj. quasi-pro-étale $X \to Y$, X perf'd space.

- Can introduce underlying top. space

$$|Y| = |X| / |R| .$$

(is indep't of presentation).

<u>Example</u>. Fix geom. base point

$$S = \mathrm{Spa}(C, \mathcal{O}_C).$$

$$\begin{array}{ccc} \mathrm{ProFin} & \hookrightarrow & \mathrm{Perf}_{/S}. \\ T = \lim_i T_i & \longmapsto & \lim_i (T_i \times \mathrm{Spa}(C, \mathcal{O}_C)) = \underline{T} \times \mathrm{Spa}(C, \mathcal{O}_C) \end{array}$$

$$= \mathrm{Spa}(\mathrm{Cont}(T, C), \mathrm{Cont}(T, \mathcal{O}_C)).$$

<u>Recall</u>. Any $T \in \mathrm{CHaus}$ compact Hausdorff

$\cong$

$\tilde{T}/R$, $\tilde{T}$ profinite set

$\qquad\qquad\qquad\qquad$ $\shortparallel$

$\qquad\qquad\qquad\qquad$ tot. disc. compact Hausdorff space

$R \subseteq \tilde{T} \times \tilde{T}$

closed equiv. rel.

$\rightsquigarrow$ CHaus $\hookrightarrow \{\text{diamonds}/S\}$.

$$T \longmapsto \left(X \in \mathrm{Perf}_{/S} \mapsto \mathrm{Cont}(|X|, T) \right).$$

$$\searrow \underline{T} \times \mathrm{Spa}(C, \mathcal{O}_C).$$

$$\cong$$

$$(\underline{T} \times \mathrm{Spa}(C,\mathcal{O}_C)) / (\underline{R} \times \mathrm{Spa}(C,\mathcal{O}_C)).$$

Def'n. 1) A diamond $Y = X/R$ is spatial if it is qcqs ($\Rightarrow$ can choose X, R qcqs.)

$|Y|$ is spectral.
- inv. limit of finite T_0 spaces.
- $\cong$ Spec A, for some ring A.
- has good basis of quasicompact open subsets.

and $|X| \longrightarrow |Y|$ is spectral.
(preimage of qc open is qc open.)

2) Y is locally spatial if it has an open cover by spatial $U \subset Y$.

$\rightsquigarrow$ $|Y|$ is locally spectral.

Y spatial $\Leftrightarrow$ Y loc. spatial & $|Y|$ qcqs.

In practice, all relevant diamonds are locally spatial.

<u>Remark</u>. {locally spatial diamonds} has all
- (fibre) products,
- all cofilt. limits with qcqs transition maps.

<u>Structure of a locally spatial diamond Y:</u>

underlying locally spectral $|Y|$

for each $y \in |Y|$, have localization

$\varprojlim_{U \ni y} U = Y_y \subseteq Y$ of Y at y,

$Y_y = \mathrm{Spa}(C, C^+)/\underline{G}$,

where C complete alg. closed nonarch. field.

$\mathfrak{m}_{\mathcal{O}_C} \subseteq C^+ \subseteq \mathcal{O}_C$ valuation subring,

G profinite group acting continuously & faithfully on C.

$$\{\text{Analytic adic spaces}/\mathbb{Z}_p\} \to \{\text{diamonds}\}.$$

$$X \longmapsto X^\diamond.$$

<u>Def'n</u>. For analytic adic space $X/\mathbb{Z}_p$,
<u>/Prop'n</u>.

the functor

$$X^\diamond : \underset{\text{Perf}}{\underset{\in}{S}} \longmapsto \left\{ S^\# \text{ untilt of } S + \atop /\mathbb{Z}_p \text{ map } S^\# \to X \right\}.$$

defines a locally spatial diamond.

Moreover, there are canonical equiv.

$$|X| \cong |X^\diamond|,$$
$$X_{ét} \cong X^\diamond_{ét}.$$

"$X \mapsto X^\diamond$ remembers top. information about X,
but forgets structure map to $\mathrm{Spa}\,\mathbb{Z}_p$".

If X perf'd, $X^\diamond \cong X^\flat$.

<u>Sketch</u>. If X perf'd,

$$\{S^\#, S^\# \to X\} \xrightarrow{\sim} \{S \to X^\flat\}.$$

by tilting equivalence.

$\rightsquigarrow$ $X^\diamond$ is represented by $X^\flat$.

$$\left.\begin{array}{l} |X^\diamond| \cong |X^\flat| \cong |X| \\ X^\diamond_{ét} \cong X^\flat_{ét} \cong X_{ét}. \end{array}\right\} \text{by tilting equiv. for top. spaces / étale sites.}$$

In general, use that any X admits pro-étale surjection from perf'd space $\tilde{X} \longrightarrow X$.

Colmez, Faltings.

(locally $X = \mathrm{Spa}(A, A^+)$ If $A/\mathbb{Q}_p$ for simplicity, adjoining x^{1/p^∞} is pro-étale whenever $x \in A^\times$, for example $x \in 1 + pA^+$.

This defines cofil'd pro-étale perfectoid cover.) $\square$.

<u>Remark</u>. <u>Prop'n</u> (Kedlaya-Liu).

$$\left\{\begin{array}{c}\text{seminormal rigid-analytic} \\ \text{spaces} / \mathbb{Q}_p\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{diamonds} \\ / (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond\end{array}\right\}$$

$$X \longmapsto X^\diamond.$$

is fully faithful.

<u>Note:</u> $(\mathrm{Spa}\, \mathbb{Q}_p)^\diamond(S) = \{ S^\# \text{ untilt}/\mathbb{Q}_p \text{ of } S \}$.

parametrizes untilts of S.

Back to relative Fargues–Fontaine curve:

If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{/\mathbb{F}_q}$.

$$\rightsquigarrow \mathcal{Y}_{S,E} = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi]=0\}.$$

$$\cup\!\mathsf{I}$$

$$Y_{S,E} = \{\pi \neq 0\}.$$

<u>Thm.</u> ("Diamond Equation").

$$Y_{S,E}^\diamond \cong S \times (\mathrm{Spa}\, E)^\diamond.$$

Equiv: Given perf'd space $T/\mathbb{F}_q$, untilts $T^\# / Y_{S,E}$ is the same as an untilt $T^\#/E$ + map $T \to S$.

Sketch. Given $T^\#/E$, need to see that maps $T^\# \longrightarrow Y_{S,E} \; /E$ are "the same" as maps $T \longrightarrow S / \mathbb{F}_q$.

Let $T^\# = \mathrm{Spa}(A, A^+)$.

maps $T^\# \longrightarrow Y_{S,E} \subseteq \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$

are given by maps

$$W_{\mathcal{O}_E}(R^+) \longrightarrow A^+$$

s.th. $[\varpi], \pi \longmapsto$ units of A.

'automatic' as $T^{\#}/E$.

Adjunction between W_{O_E} (perf. rings) & tilting: $W_{O_E}(R^+) \longmapsto A^+$

$[\varpi] \longmapsto$ unit of A

maps $T = \mathrm{Spa}(A^\flat, A^{\flat+}) \to S = \mathrm{Spa}(R, R^+)$ $\cong$ $\begin{cases} R^+ \longrightarrow (A^+)^\flat = \varprojlim_{x \mapsto x^p} A^+/\pi. \\ \varpi \longmapsto \text{unit of } A^\flat. \end{cases}$

$\square$

$\rightsquigarrow$ Canonical map.

$$|Y_{S,E}| \cong |Y_{S,E}^\diamond| \cong |S \times (\mathrm{Spa}\, E)^\diamond|$$
$$\downarrow$$
$$|S|$$

<u>Prop'n</u>. For $S' \subseteq S$ open aff'd subset,

$Y_{S',E} \hookrightarrow Y_{S,E}$ open immersion with $$|Y_{S',E}| = |Y_{S,E}| \times_{|S|} |S'|. \quad \square$$

$\rightsquigarrow$ can glue $Y_{S,E}^{\circlearrowleft \varphi_S}$ for general perf'd spaces $S/\mathbb{F}_q$,

s.th. $$Y_{S,E}^{\diamond} \cong S \times (\mathrm{Spa}\, E)^{\diamond}.$$

" $Y_{S,E}$ is "the" analytic adic space $/E$ with $Y_{S,E}^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond} / (\mathrm{Spa}\, E)^{\diamond}$ ".

<u>**Def'n**</u>. $$X_{S,E} = Y_{S,E} / \varphi^{\mathbb{Z}}$$

"relative Fargues–Fontaine curve".

$$X_{S,E}^{\diamond} \cong S/\varphi_S^{\mathbb{Z}} \times (\mathrm{Spa}\, E)^{\diamond}.$$

$$(\mathrm{Spa}\, E)^\diamond \qquad \left((\mathrm{Spa}\, E)^\diamond \times (\mathrm{Spa}\, E)^\diamond \right) / \Sigma_2 .$$

$$\begin{array}{cc} \| & \| \\ \mathrm{Div}^1_Y & \mathrm{Div}^2_Y \end{array}$$

<u>Prop</u>n. All Diamonds are v-sheaves.

For any adic space $X / \mathbb{Z}_p$,

Can define $X^\diamond$ as a v-sheaf:

$$S \longmapsto \{ S^\# \text{ untilt} + S^\# \to X \}.$$

$$|X^\diamond| \longrightarrow |X|$$

usually far from an isom.

Ian Gleason.

$$H^1(\mathcal{O}(-1)) = (\mathbb{A}^1_E)^\diamond / \underline{E}.$$

$$S \mapsto H^1(X_{S,E}, \mathcal{O}(-1))$$