Relative Fargues - Fontaine curve.

Aim: Classify vector bundles on "the" Fargues - Fontaine curve $X_{C,E}$.

Last Time: - Classification of line bundles

reduced to - Classification of VB semistable of slope 0.

Proof relies on putting "geometric structure" on

$H^0(X_{C,E}, \mathcal{E})$. for $\mathcal{E} \in VB(X_{C,E})$.

$H^1(\dots)$. "Banach - Colmez spaces".

Intermediate aim: Explain these "geometric structures"

$\shortparallel$

diamonds

\+ relative Fargues - Fontaine curve

$X_{S,E}$ for perfectoid space S.

"Étale Cohomology of Diamonds".

<u>Recall</u>. A perfectoid algebra $/\mathbb{F}_p$ is just a perfect Tate algebra $R/\mathbb{F}_p$.

Huber's sense: has top. nilp. unit. ϖ.

perfect Banach alg $/\mathbb{F}_p((\varpi))$.

A perfectoid space over $/\mathbb{F}_p$ is an adic space $X/\mathbb{F}_p$ covered by open $U = \mathrm{Spa}(R, R^+) \subseteq X$ s.th. R perfect Tate. i.e. R perfectoid

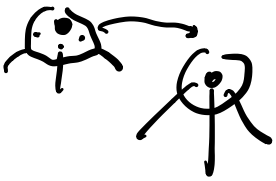
If $S = \mathrm{Spa}(R, R^+)/_{\mathbb{F}_q}$ is affinoid perfectoid, can mimic construction of FF curve, replacing $\mathcal{O}_C$ by R^+:

$$\boxed{\begin{array}{l} E \supseteq \mathcal{O}_E \ni \pi \\ \quad \mathbb{F}_q. \end{array}}$$

Pick $\varpi \in R$ pseudounif.

$$\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \supseteq \mathcal{Y}_{(R,R^+),E} \supseteq Y_{(R,R^+),E} \quad /E$$

// $\quad \varphi_E \quad$ ||



$\{[\varpi] \neq 0\}$ $\qquad\qquad$ $\{[\varpi] \neq 0, \pi \neq 0\}$

analytic adic spaces.

$\leftarrow$ locally Spa(Tate alg).

have continuous radius function

$$\text{rad}: \mathcal{Y}_{(R,R^+),E} \longrightarrow [0,\infty)$$

$$y \longmapsto \frac{\log|[\varpi](\tilde{y})|}{\log|\pi(\tilde{y})|}.$$

$$\phi_{R^+} \circlearrowright \mathcal{Y}_{(R,R^+),E} \longrightarrow (0,\infty) \ni q.$$

free, tot. disc.

$$\text{rad} \circ \phi_{R^+} = q \cdot \text{rad}.$$

<u>Def'n</u> $\quad X_{(R,R^+),E} = \mathcal{Y}_{(R,R^+),E} / \phi^{\mathbb{Z}}.$

adic space $/E$

"relative Fargues-Fontaine curve".

Does not map to S!

<u>Example</u>. $\quad E = \mathbb{F}_q((t))$.

$$\operatorname{Spa}(R,R^+) \subseteq \operatorname{Spa}(R^+,R^+)$$
$$\| $$
$$\{\varpi \neq 0\}.$$

Then $W_{\mathcal{O}_E}(R^+) = R^+[[t]]$.

$$\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R,R^+) \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q[[t]].$$

$$\cup\!| \qquad = \qquad \mathbb{D}_{\mathrm{Spa}(R,R^+)} \qquad \text{open unit disc.}$$

$$\hookuparrow$$

$$Y_{(R,R^+),E} = \mathbb{D}^*_{\mathrm{Spa}(R,R^+)} \qquad \text{punctured open unit disc.}$$

ϕ.

__Note:__

$$\begin{tikzcd}
Y_{(R,R^+),E} \arrow[r, phantom, "\subseteq"] \arrow[dr] & \mathcal{Y}_{(R,R^+),E} \arrow[d] \\
& \mathrm{Spa}(R,R^+)
\end{tikzcd}$$

in this case,

but not after quotienting by ϕ.

__Claim.__ For general perfectoid,

$$\begin{tikzcd}
\mathcal{Y}_{S,E} \arrow[r, phantom, "\supseteq"] & Y_{S,E} \arrow[d] \\
& X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}}
\end{tikzcd}$$

Can be glued from affinoid case.

<u>easy for $E = \mathbb{F}_q((t))$:</u> Then

$$
\begin{array}{ccc}
\mathcal{Y}_{S,E} & = & S \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q[[t]] \\
\cup\!\! | & & \cup\!\! | \\
Y_{S,E} & = & S \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathbb{F}_q((t)) \\
\downarrow & & \\
X_{S,E} & = & Y_{S,E} / \varphi^{\mathbb{Z}} .
\end{array}
$$

<u>Note:</u> $\mathrm{id} = \varphi$ on $|S| \rightsquigarrow |X_{S,E}| \to |S|$.

would like to argue similarly in p-adic case.

<u>"Diamond Equation"</u>.

$$
\begin{array}{ccc}
\mathcal{Y}^{\diamond}_{S,E} & = & S \times (\operatorname{Spa} \mathcal{O}_E)^{\diamond} \\
\cup\!\! | & & \cup\!\! | \\
Y^{\diamond}_{S,E} & = & S \times (\operatorname{Spa} E)^{\diamond} \\
|
\end{array}
$$

$$X_{S,E}^{\diamond} = Y_{S,E}^{\diamond}/\varphi^{\mathbb{Z}}.$$

Diamonds.

Idea. Diamonds = quotients of perfectoid spaces (of char. p) by pro-étale equivalence relation.

$$\{\text{Analytic adic spaces}/\operatorname{Spa}\mathbb{Z}_p\} \ni X \mapsto X^{\diamond} \in \text{Diamonds}$$

"generalizing tilting".

Analytic: locally Spa(Tate alg, -).

$\tilde{X} \to X$ pro-étale cover

$\tilde{X}$ perfectoid.

$$X \text{ "=" } \tilde{X}/R \qquad R \subseteq \tilde{X} \times \tilde{X}$$

R perfectoid, pro-étale equiv. rel.

$$\rightsquigarrow X^{\diamond} := \tilde{X}^{\flat}/R^{\flat}.$$

(Pro-)Étale maps of perfectoid spaces.

<u>Definition</u>. Let $f: Y \to X$ map of perfectoid spaces. (possibly of mixed char.)

1) f is <u>finite étale</u> if for any open affinoid perfectoid $U = \mathrm{Spa}(R, R^+) \subseteq X$ (equiv., for a cover by such U)

the preimage $V = f^{-1}(U) = \mathrm{Spa}(S, S^+) \subseteq Y$ is affinoid perfectoid, and

S finite étale R-algebra

+ $S^+ \subseteq S$ integral closure of R^+.

$\rightsquigarrow$ $\mathrm{Spa}(R, R^+)_{\text{fét}} \cong \mathrm{Spec}(R)_{\text{fét}}$.

(Implicit: If S f.ét. R-alg., then S again perfectoid.

$\sim$ Faltings's almost purity thm.)

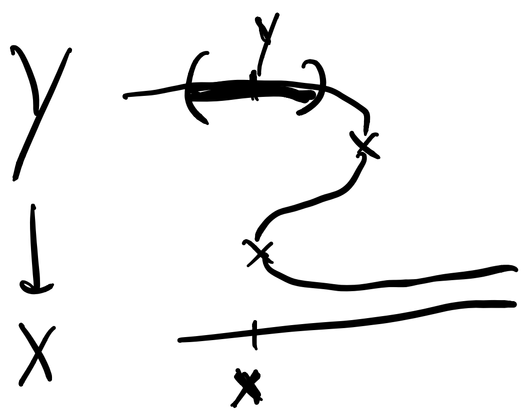
2) f is <u>étale</u> if it is locally on Y of form

$$Y \hookrightarrow Y' \xrightarrow{\text{fin. ét.}} U \hookrightarrow X.$$

open immersions.

Nontrivial, but true: Composites of étale maps are étale.

Analogous assertion fails for étale maps of schemes:



But works for analytic adic spaces. (Huber).

3) f is <u>pro-étale</u> if locally on X and Y

<u>affinoid pro-étale</u>:

$f = \varprojlim_i f_i$,

$$Y = \mathrm{Spa}(S,S^+) = \varprojlim_i \mathrm{Spa}(S_i,S_i^+) \longrightarrow X = \mathrm{Spa}(R,R^+)$$

cofilt. limit

all $f_i : \mathrm{Spa}(S_i,S_i^+) \longrightarrow \mathrm{Spa}(R,R^+)$

are étale.

$$S^+ = \left(\varinjlim S_i^+\right)^\wedge_\varpi, \quad S = S^+\left[\tfrac{1}{\varpi}\right].$$

<u>Example</u>.

$$\begin{tikzcd} Y = \mathrm{Spa}(C\langle T^{1/2p^\infty}\rangle) \arrow[d, "\text{extract } \sqrt{T}."] \\ X = \mathrm{Spa}(C\langle T^{1/p^\infty}\rangle) \end{tikzcd}$$

$p \neq 2$.

looks like it's ramified at origin.

<u>Claim</u>. $\exists$ pro-étale cover $\tilde{X} \to X$

affinoid.

s.th.

affinoid pro-étale.

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\text{pro-étale}"] \arrow[d] & Y \arrow[d] \\ \tilde{X} \arrow[r, "\text{pro-étale}"'] & X \end{tikzcd}$$

Let $\mathcal{U}_n \subseteq X$

small disc

$\mathcal{U}_n = \{|T| \le \frac{1}{p^n}\}$,

$\mathcal{U}_{n,n+1} = \{\frac{1}{p^{n+1}} \le |T| \le \frac{1}{p^n}\} \subseteq \mathcal{U}_n$.

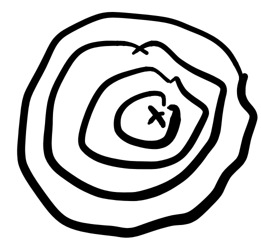
annulus.

all affinoid perfectoid.

For each n,

let $X_n = \mathcal{U}_{0,1} \sqcup \mathcal{U}_{1,2} \sqcup \dots \sqcup \mathcal{U}_{n-1,n} \sqcup \mathcal{U}_n$.

$$X_n \xrightarrow{\text{étale cover.}} X$$



$\tilde{X} = \varprojlim_n X_n \longrightarrow X$ affinoid pro-étale.

$$\tilde{X} \longrightarrow \pi_0 \tilde{X} = \mathbb{N} \cup \{\infty\}.$$

fibre of $\tilde{X}$ over $n \in \mathbb{N}$ is $\mathcal{U}_{n,n+1}$.

fibre over ∞ is $\operatorname{Spa} C$ = origin of X.

easy to see: $\tilde{Y} = Y \times_X \tilde{X} \to \tilde{X}$ affinoid pro-étale.

<u>Example.</u> Any Zariski closed immersion is $_\wedge$pro-étale.
affinoid

$$X = \mathrm{Spa}(R, R^+) \supseteq V(f) = Z = \mathrm{Spa}(S, S^+).$$

$$S = R/\overline{(f^{1/p^\infty})} \supseteq S^+ \text{ integral closure of } R^+.$$

Let $U_n = \{ |f| \leq \frac{1}{p^n} \} \subseteq X$ rational open subset.

$$V(f) = \bigcap_n U_n = \varprojlim_n U_n \subseteq X.$$

Here, $|f| = 0$ everywhere, so $f = 0$.
(perfectoid spaces are reduced, even uniform.)

<u>Example.</u> $f: \mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle \to \mathrm{Spa}\, C$

not pro-étale.

as fibre is positive-dim'l.

__Theorem.__ A map $f: Y \to X$ of affinoid perfectoid spaces is pro-étale locally on X affinoid pro-étale iff

for all geom. (rk 1) points

$$\mathrm{Spa}(C, \mathcal{O}_C) \longrightarrow X$$

the fibre product

$$Y \times_X \mathrm{Spa}(C, \mathcal{O}_C) \longrightarrow \mathrm{Spa}(C, \mathcal{O}_C)$$

is affinoid pro-étale; equiv., isom to

$$\underline{S} \times \mathrm{Spa}(C, \mathcal{O}_C) = \lim_i \left(S_i \times \mathrm{Spa}(C, \mathcal{O}_C)\right)$$

for a profinite set $S = \lim_i S_i$.

Call such f __quasi-pro-étale__.

__Def'n.__ A map $f: Y \to X$ is a __v-cover__ if for any qc $U \subseteq X$ $\exists$ qc $V \subseteq Y$

s.th. $|V| \to |U|$ is surjective.

pro-étale covers if v-covers + f quasi-pro-étale.

Thm

1) $X \mapsto \mathcal{O}_X(X)$ sheaves for v-topology
$\mathcal{O}_X^+(X)$ on perfectoid spaces.

2) For any perf'd space X,
$\mathrm{Hom}(-, X)$ is a sheaf for v-topology.

3) $X \mapsto VB(X)$ v-stack.

⋮

4) If X affinoid
$= \mathrm{Spa}(R, R^+)$

$$H_v^i(X, \mathcal{O}_X) = \begin{cases} R & i=0 \\ 0 & \text{for } i>0 \end{cases}$$

$$H_v^i(X, \mathcal{O}_X^+) \underset{\text{almost}}{=} \begin{cases} R^+ & i=0 \\ 0 & \text{for } i>0 \end{cases}$$

killed by ϖ^{1/p^n}, all n.

Sketch. First, prove that
$X \mapsto \mathcal{O}_X^+(X)$ sheaf for étale top — de Jong –

+ $H^i_{\acute{e}t}(X, \mathcal{O}_X^+) =_{\text{almost}} 0$ for $i > 0$.

X affinoid.

van der Put.
open
+ finite étale.
(Faltings' almost purity)

$\rightsquigarrow$ similar assertions for

$\underbrace{\mathcal{O}_X^+/\varpi}$.

this extends to aff.'d pro-étale things by filt. colimits.

$\rightsquigarrow$ get $\mathcal{O}_X^+/\varpi$ has good properties as pro-étale sheaf.

$\rightsquigarrow$ $\mathcal{O}_X^+ = \varprojlim_n \mathcal{O}_X^+/\varpi^n$. as well,

$\mathcal{O}_X = \mathcal{O}_X^+[\frac{1}{\varpi}]$.

pro-étale locally, v-covers are faithfully flat on $\mathcal{O}^+/\varpi$-level.

Then use f.flat descent. $\square$!

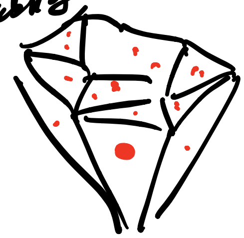
$$X \longrightarrow \pi_0 X$$

each conn. comp. $= \mathrm{Spa}(C, C^+)$.

profinite.

C complete alg. closed nonarch. field.

$\overset{\cup}{C^+}$ valuation subring.



$$X = \tilde{X}/R.$$