The Fargues-Fontaine curve, III

E nonarch. local field, $\mathcal{O}_E \ni \pi$ $\mathbb{F}_q$.

$C/\mathbb{F}_q$ complete nonarch alg closed field.

⇝ Fargues-Fontaine curve

$$\begin{array}{l} X_{C,E} = Y_{C,E}/\phi_C^{\mathbb{Z}} \quad \text{adic space } /E. \\ \| \\ X_C \\ \| \\ X \end{array}$$

$$Y_{C,E} \underset{\text{open}}{\subseteq} \mathrm{Spa}\, \underbrace{W_{\mathcal{O}_E}(\mathcal{O}_C)}_{\text{flat deformation of } \mathcal{O}_C \text{ to } \mathcal{O}_E.}$$

where $\pi \neq 0$ and $[\varpi] \neq 0$ $\varpi \in C$ pseudouniformizer.

$$\begin{array}{l} X^{cl}_{C,E} \subseteq |X_{C,E}|. \\ \| \\ \{\text{untilts } C^\#/E \text{ of } C\}/\phi_C^{\mathbb{Z}} \end{array}$$

- Any can. affinoid open subset $\mathrm{Spa}\, A \subseteq X_{C,E}$ has A principal ideal domain and $\mathrm{Spm}\, A = X^{cl}_{C,E} \cap |\mathrm{Spa}\, A| \subseteq |X_{C,E}|$.
- for any classical point $y \in Y^{cl}_{C,E}$, $\exists\, t \in \mathbf{C}$

$0 < |t| < 1$ s.th. $y = V(\pi - [t])$.

Classification Theorem for Vector Bundles.

Isocrystals. **Recall.** An isocrystal is a pair

(V, ϕ_V) where V fin.-dim'l $\breve{E}$-vector space.

($\breve{E}/E$ completion of max'l unramified extension)

$\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[\frac{1}{\pi}]$

$+ \ \phi_V : V \simeq V$

$\phi_{\breve{E}}$-linear automorphism

$\rightsquigarrow$ E-linear $\otimes$-category Isoc_E

Examples. 0) unit $(\breve{E}, \phi_{\breve{E}})$.

1) 1-dim'l objects : $(\breve{E}, b\phi_{\breve{E}})$ for some $b \in \breve{E}^\times$

any such is isom. to $(\breve{E}, \pi^n \phi_{\breve{E}})$ for a unique $n \in \mathbb{Z}$. n = slope of $(\breve{E}, \pi^n \phi_{\breve{E}})$.

Change of basis replaces b by

$$a^{-1} b \, \phi(a) = \underbrace{a^{-1} \phi(a)}_{\in \mathcal{O}_{\breve{E}}^\times} b. \qquad a \in \breve{E}^\times$$

"ϕ-conjugation"

$\in \mathcal{O}_{\breve{E}}^\times$, can be any possible element.

$$\mathrm{Hom}\big((\breve{E}, \pi^n \phi_{\breve{E}}), (\vec{E}, \pi^m \phi_{\breve{E}})\big) = \begin{cases} E & m = n \\ 0 & m \neq n. \end{cases}$$

$$\overset{\breve{E}^{\phi = \pi^{m-n}}}{\rotatebox{90}{=}}$$

2). If $\lambda = \frac{s}{r} \in \mathbb{Q}$ $(s,r) = 1$, $r > 0$,

let $$(V_\lambda, \phi_{V_\lambda}) = \left(\breve{E}^r, \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \\ \pi^s & & & 0 \end{pmatrix} \phi_{\breve{E}}\right)$$

<u>Thm</u> (Dieudonné - Manin)

$$\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^\lambda \longleftarrow \text{"isoclinic of slope } \lambda\text{"},$$

$\mathrm{End}(V_\lambda) = \underset{\text{central division alg.}}{D_\lambda}$ of inv. λ. $\quad \mathrm{Isoc}_E^\lambda = (\overset{\text{f.d.}}{E\text{-v.s.}}) \otimes V_\lambda.$

<u>Sketch of proof</u>. For any nonzero $V = (V, \phi_V) \in \mathrm{Isoc}_E$,

let $$\mu(V) = \frac{\deg(V)}{\underbrace{\mathrm{rk}(V)}_{\text{dim of underlying } \breve{E}\text{-v.s.}}}$$ — $\det V \in \mathrm{Isoc}_E$ of rk 1.

$(\breve{E}, \pi^n \phi_{\breve{E}}) \cong \det V$, $\quad \det V = \Lambda^{\mathrm{rk} V} V$

$\rightsquigarrow$ "Harder - Narasimhan formalism of slopes". $\quad n = \deg V.$

$\rightsquigarrow$ <u>(semi)stable objects</u>: (V, ϕ_V) <u>(semi)stable</u> if for all $\quad 0 \subsetneq (V', \phi_{V'}) \subsetneq (V, \phi_V)$

one has $\quad \mu(V') \underset{(<)}{\leq} \mu(V)$

stable.

$\rightsquigarrow$ any object $V = (V, \phi_V) \in \mathrm{Isoc}_E$ has a unique "Harder-Narasimhan filtration" decreasing separated exhaustive filtr. $\mathbb{Q}$-indexed

$$V^{\geq \lambda} \subseteq V \qquad (\text{in } \mathrm{Isoc}_E)$$

s.th. each

$$V^{\lambda} := V^{\geq \lambda} \Big/ \bigcup_{\lambda' > \lambda} V^{\geq \lambda'}$$

is semistable of slope λ.

In this simple case, also

$$\mu'(V) = -\mu(V) = \frac{-\deg(V)}{\mathrm{rk}(V)}$$

gives HN formalism

$\rightsquigarrow$ this filtration is canonically split, and

$$\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^{\lambda} \longleftarrow$$

"semistable of slope λ".

. $\lambda = 0$: want: $\mathrm{Isoc}_E^{0} \xleftarrow{\cong}$ f.d. E-v.s.

$$(W \otimes_E \breve{E}, \mathrm{id} \otimes \phi_{\breve{E}}) \longleftarrow\!\!\shortmid \quad W$$

equiv., for all $(V,\phi_V) \in \mathrm{Isoc}_E^{\circ}$, letting $W = V^{\phi_V = \mathrm{id}}$, one has

$$W \otimes \breve{E} \xrightarrow{\cong} V.$$

idea: Show ^{E}that V contains a ϕ_V-stable lattice $L \subseteq V$. $\phi_V(L) = L$.

$\uparrow$ finite free $\mathcal{O}_{\breve{E}}$-module generating V (over $\breve{E}$).

Then $L^{\phi_L = \mathrm{id}}$ is fin. free over $\mathcal{O}_E$ of correct rank, by Artin-Schreier theory.

For general λ, use that if $(V,\phi_V) \in \mathrm{Isoc}_{\breve{E}}^{\lambda}$, then $\underline{\mathrm{Hom}}((V_\lambda, \phi_{V_\lambda}), (V,\phi_V)) \in \mathrm{Isoc}_E^{\circ}$, use result for $\lambda = 0$. $\square$

Back to Fargues-Fontaine curve.

Note: $\overline{\mathbb{F}}_q \subseteq C$

$$\begin{tikzcd} \rightsquigarrow Y_{C,E} \arrow[r] \arrow[loop below, "\phi_C"] & \mathrm{Spa}\, \breve{E} \arrow[loop below, "\phi_{\breve{E}}"] \end{tikzcd}$$

$\rightsquigarrow$ pullback functor

$$\begin{tikzcd} \mathrm{Isoc}_{\breve{E}} \arrow[r] & \{ \phi_C\text{-equiv. } VB / Y_{C,E} \} \arrow[d, equal, "\text{descent}"] \\ V \arrow[r, mapsto] & \mathcal{E}(V) \quad VB(X_{C,E}). \end{tikzcd}$$

Let $\mathcal{O}_{X_{C,E}}(\lambda) := \mathcal{E}(V_{-\lambda}). \in VB(X_{C,E}).$

Thm (Fargues-Fontaine '10, all E; Hartl-Pink '04, $E = \mathbb{F}_q((t))$; Kedlaya '04, E p-adic)

Any vector bundle $\mathcal{E}$ on $X_{C,E}$ is isom. to a direct sum of $\mathcal{O}(\lambda)$'s. Equiv, the functor

$$\mathrm{Isoc}_E \longrightarrow VB(X_{C,E})$$

induces a bijection on isom. classes.

More precisely, 1) any $\mathcal{E}$ admits a Harder-Narasimhan filtration $\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$ s.th. each

$$\mathcal{E}^{\lambda} = \mathcal{E}^{\geq\lambda} / \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$$ is semistable of slope λ.

2) $\mathrm{Isoc}_E^{\lambda} \xrightarrow{\cong} VB(X_{C,E})^{\lambda}$ (semistable of slope λ).

3). The HN filtration splits (but not uniquely)

1) like for all smooth proj curves.

2) similar to $\mathbb{P}^1$, but not other curves.

3) similar to $g=0,1$, but not higher genus.

↯ $$H^0(X_{C,E}, \mathcal{O}(n)) = \begin{cases} \text{inf.-dim'l } E\text{-v.s.} & n>0 \\ E & n=0 \\ 0 & n<0 \end{cases}$$

$$H^1(X_{C,E}, \mathcal{O}(n)) = \begin{cases} 0 & n>0 \\ 0 & n=0 \\ \text{inf.-dim'l. } E\text{-v.s.} & n<0 \end{cases}$$

Let $C^\sharp/E$ some untilt of C

$$\operatorname{Spa} C^\sharp \overset{i}{\hookrightarrow} X_{C,E}$$

$$0 \to \underset{\mathcal{O}_{X_{C,E}}(-1)}{\underset{\wr}{\mathcal{I}}} \to \mathcal{O}_{X_{C,E}} \to i_* C^\sharp \to 0$$

$$\rightsquigarrow H^1(X_{C,E}, \mathcal{O}(-1)) = C^\sharp/E.$$

"Banach-Colmez Spaces" : built from f.d. $C^\sharp$-v.s.

+ f.d. E-v.s.

Similarly,

$$0 \to \mathcal{O}_{X_{C,E}} \to \mathcal{O}_{X_{C,E}}(1) \to i_* C^\sharp \to 0.$$

$$\rightsquigarrow 0 \to E \to H^0(X_{C,E}, \mathcal{O}(1)) \to C^\sharp \to 0.$$

Next Goal: Sketch new proof of classification theorem,

using heavily perfectoid spaces, diamonds, v-descent,

but no computations.

Some reductions: (Same in all known proofs)

1). Classification of line bundles.

for this, first prove that "$\mathcal{O}(1)$ is ample":

Thm (Kedlaya-Liu '15) For any vector bundle $\mathcal{E}$ on $X_{C,E}$, all $n \gg 0$, the bundle $\mathcal{E}(n) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is globally generated, and $H^1(X_{C,E}, \mathcal{E}(n)) = 0$.

(Note: $H^i(X_{C,E}, \mathcal{E}) = 0$ for $i \geq 2$ always.)

not hard, just need to do the right estimates.

Cor. Let $P = \bigoplus_{n \geq 0} H^0(X_{C,E}, \mathcal{O}(n))$,

(GAGA) $X^{alg}_{C,E} = \mathrm{Proj}(P)$. $|X^{alg}_{C,E}| = X^{cl}_{C,E} \cup \{\eta\}$.

$\rightsquigarrow$ natural map of locally ringed top. spaces

$$X_{C,E} \xrightarrow{f} X^{alg}_{C,E}$$

s.th. - $f^*: VB(X^{alg}_{C,E}) \cong VB(X_{C,E})$ equivalence

- preserves cohomology.

($X^{alg}_{C,E}$ regular scheme, noeth, Krull dim 1, locally spectrum of a P.I.D.).

Cor. Any line bundle $\mathcal{L} \in \mathrm{Pic}(X_{C,E})$ is isom. to $\mathcal{O}(D)$ for some divisor $D \in \bigoplus_{x \in X^{cl}_{C,E}} \mathbb{Z}$.

Prop. For any $x \in X^{cl}_{C,E}$, $\mathcal{O}(x) = \mathcal{I}_x^{-1} \cong \mathcal{O}_{X_{C,E}}(1)$.

(next time: uses Lubin-Tate theory.)

$$\mathcal{O}(1) \leftarrow\!\shortmid 1$$

⇝ Cor $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$, so $\mathrm{Pic}(X_{C,E}) \cong \mathbb{Z}$.

Can now define $\deg \mathcal{E}$ = image of $\det \mathcal{E} = \Lambda^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}$

$$\rotatebox{90}{$\in$}$$

$$\mathbb{Z} \cong \mathrm{Pic}(X_{C,E})$$

$$\underset{\mathcal{E} \neq 0}{\mu(\mathcal{E})} = \frac{\deg \mathcal{E}}{\mathrm{rk}\,\mathcal{E}}.$$

⇝ Harder-Narasimhan filtration. ⇝ 1).

essential remaining step: Classify bundles that are semistable of slope 0:

$$\text{f.d. } E\text{-v.s.} \xhookrightarrow{\cong} VB(X_{C,E})^0$$
$$W \longmapsto W \otimes_E \mathcal{O}_{X_{C,E}}.$$

Key. If $\overset{\neq 0}{\mathcal{E}} / X_{C,E}$ semistable of slope 0, then

$$H^0(X_{C,E}, \mathcal{E}) \neq 0.$$

Idea. We will ourselves to (a priori) enlarge C consider the functor

$$C'/C \longmapsto H^0(X_{C',E}, \mathcal{E}|_{X_{C',E}}).$$

Some functor on $\{C'/C\}$.

Want to think of 'this' as a geometric object whose C'-valued pts are $H^0(X_{C',E}, \mathcal{E}|\ldots)$.

Extend the functor to all perfectoid C-algebras

⇝ get sheaf on category of (affinoid) perfectoid spaces.

Example. Fix $C^\# / E$ untilt of C.

R perfectoid C-algebra. $\rightsquigarrow$ untilt $R^\sharp / C^\sharp$.

tilting equiv.

$$\begin{tikzcd}
\operatorname{Spa} R^\sharp \arrow[r, hook, "i_{R^\sharp}"] \arrow[d] & X_{R,E} \arrow[d] \\
\operatorname{Spa} C^\sharp \arrow[r, hook, "i_{C^\sharp}"] & X_{C,E}
\end{tikzcd}$$

$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{R^\sharp *}\mathcal{O}_{R^\sharp} \to 0$

$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{C^\sharp *}\mathcal{O}_{C^\sharp} \to 0$

$$H^1(X_{R,E}, \mathcal{O}(-1)) = R^\sharp / E$$

In equal char., $R^\sharp = R$, so get

$$H^1(\mathcal{O}(-1)) = \mathbb{A}^1_C / E \, . \qquad E \subset \mathbb{A}^1_C \text{ closed subset}$$

quotient of $\mathbb{A}^1_C$ by pro-étale equivalence relation.

This will be the general picture:

$H^0(-, \mathcal{E})$, $H^1(-, \mathcal{E})$ are "quotients of perfectoid spaces under pro-étale equivalence relations".

analogous to $\boxed{\text{diamonds}} =$

Artin's <u>algebraic spaces</u> = quotients of schemes under étale equiv. relations.

$n > 0$ in <u>equal char</u>.

$$H^0(X_{C,E}, \mathcal{O}(n)) \cong \mathfrak{m}_{\mathcal{O}_C}^n .$$

$H^0(\mathcal{O}(n))$ is repr. by $\mathbb{D}^n$

n-dim'l open unit disc.