The Fargues–Fontaine curve, II.

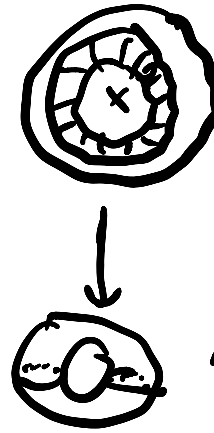
E nonarch. local field, π, $\mathbb{F}_q$.

$C/\mathbb{F}_q$ complete alg closed nonarch. field.

If $E = \mathbb{F}_q((t))$, consider

$$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}^*_C \quad \text{punctured open unit disc.}$$

$$\circlearrowright \phi_C$$



$\mathbb{D}^*_C$

$X_{C,E}$

Def. Fargues–Fontaine curve

$$X_{C,E} = \mathbb{D}^*_C / \phi_C^{\mathbb{Z}}.$$

Classical Points.

Tate '70.

$$\left\{ \begin{matrix} \text{rigid-analytic} \\ \text{varieties}/C \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{adic spaces "locally} \\ \text{of finite type" over } \operatorname{Spa} C \end{matrix} \right\}.$$

$$X(C) \longleftarrow\!\shortmid \; X$$

$X(C) \subseteq |X|$ "classical points".

Locally, $X = \mathrm{Spa}\ A$, $A = C\langle T_1, \dots, T_n \rangle / I$

$$\begin{array}{l} X = \mathrm{Spa}\ A \\ \downarrow \\ \mathrm{Sp}\ A, \quad |\mathrm{Sp}\ A| = X(C) = \mathrm{Spm}\ A \end{array}$$

$$\| \\ \{(x_1, \dots, x_n) \in C^n \mid |x_i| \le 1,\ \forall f \in I\ f(x_1, \dots, x_n) = 0\}.$$

Thm. (Huber) $(\mathrm{Sp}\ A)_{\mathrm{rig}} \cong \mathrm{Spa}\ A$

Grothendieck top. on Sp A.

: qc adm. opens $=$ qc opens of Spa A

adm covers $=$ covers.

For $\phi_C \circlearrowright |\mathbb{D}^*_C|$, classical points are

$$x \mapsto x^{1/q} \circlearrowright \{x \in C \mid 0 < |x| < 1\}.$$

For any connected affinoid $\mathrm{Spa}\ A \subseteq \mathbb{D}^*_C$, A is a principal ideal domain.

max'l ideal corr. to x is gen. by $T - x$.

By descent, can also define classical points of

$$X_{C,E} \supseteq \{0<|\pi|<1\}/\phi.$$
$$\Downarrow$$
$$X^{cl}_{C,E}.$$

Again, any conn. affinoid subset of $X_{C,E}$ is the Spa (principal ideal domain).

Now consider case $E/\mathbb{Q}_p$. Still take $C/\mathbb{F}_q$.

<u>Question</u>. What is

$$\text{" } \mathrm{Spa}\, E \underset{\mathrm{Spa}\, \mathbb{F}_q}{\times} \mathrm{Spa}\, C \text{" } ?$$

<u>Idea</u>. In char. p, deformed any $\mathbb{F}_q$-alg. R to $\mathbb{F}_q[[t]]$ by taking $R[[t]]$.

<u>Note</u>: If R <u>perfect</u> $\mathbb{F}_q$-alg, then there is $\mathbb{L}_{R/\mathbb{F}_q} = 0$.

a unique (up to unique isom) lift $\tilde{R}/\mathcal{O}_E$ that is flat, π-ad. complete, with $\tilde{R}/\pi = R$.

One choice is $\tilde{R} = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

using p-typical Witt vectors

$$W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E = W_{\mathcal{O}_E}(R)$$

"ramified Witt vectors".

Teichmüller map

$$[\cdot] : R \longrightarrow \tilde{R} = W_{\mathcal{O}_E}(R)$$
$$x \longmapsto \lim_{n \to \infty} \tilde{x}_n^{p^n}$$

mult., not additive

Let $\tilde{x}_n \in \tilde{R}$ any lift of x^{1/p^n}.

Any element of $\tilde{R}$ admits a unique expression as $\sum_{n \geq 0} \pi^n [r_n]$ $r_n \in R$.

$\rightsquigarrow$ analogue of

$$\operatorname{Spa} \mathbb{F}_q[[t]] \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_C = \operatorname{Spa} \mathcal{O}_C[[t]]$$

in mixed char. is

$$\text{“}\operatorname{Spa} \mathcal{O}_E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_C\text{”} := \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C).$$

analogue of

$$\operatorname{Spa} \mathbb{F}_q((t)) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}^*_C$$

is

$$\text{“}\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C\text{”} := Y_{C,E} \circlearrowleft \phi_C$$

$$\operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C) \supseteq \{\pi \neq 0, [\varpi] \neq 0\}$$

$$\circlearrowleft \phi_C$$

<u>Def'n</u>. The <u>Fargues-Fontaine curve</u> is

$$X_{C,E} = Y_{C,E} / \phi_C^{\mathbb{Z}} \quad \text{over } \operatorname{Spa} E.$$

<u>Thm</u>. 1) There is a notion of classical (FF, Kedlaya) points $Y^{cl}_{C,E} \subseteq Y_{C,E}$ s.th. for any connected affinoid $\operatorname{Spa} A \subseteq Y_{C,E}$, A is a principal ideal domain, and

$$\mathrm{Spm}\, A \xrightarrow{\cong} \mathrm{Spa}\, A \cap Y^{cl}_{C,E} \subseteq Y_{C,E}.$$

2) For any classical point $y \in Y^{cl}_{C,E}$, there is some $x \in C$, $0<|x|<1$, s.th.

$$y = V(\pi - [x]).$$

This element x is not unique.

3). For any classical point $y \in Y^{cl}_{C,E}$, the complete residue field at y is a complete alg closed nonarch field $\underset{\substack{\cup \\ E}}{C(y)}$ with a distinguished isom.

$$C(y)^{\underset{\uparrow\,\text{tilt}}{\flat}} \cong C.$$

This gives bijection

$$Y^{cl}_{C,E} \cong \{\text{untilts } C^{\sharp}/E \text{ of } C\}.$$

<u>Tilting</u>. For complete alg. closed nonarch field K s.th. $|p|_K < 1$, one can define

a complete alg closed nonarch field

$$K^\flat = \varprojlim_{x\mapsto x^p} K \quad \text{(as top. multiplicative monoid).}$$

of char. p.

$$\begin{array}{ccccc} K^\flat & = & \varprojlim_{x\mapsto x^p} K & & \\ \cup\!| & & \cup\!| & & \\ \mathcal{O}_{K^\flat} & = & \varprojlim_{x\mapsto x^p} \mathcal{O}_K & \xrightarrow{\sim} & \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p. \end{array}$$

cf. def. of Teichmüller map.

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want to sketch proof of Thm.

<u>Step 1.</u> Construct injective map

$$\{C^\#/E \text{ untilt of } C\} \longrightarrow |Y_{C,E}|.$$

Say $C^\#$ untilt of C, so

$$\begin{array}{ccccc} \mathcal{O}_C & \cong & \varprojlim_{x\mapsto x^q} \mathcal{O}_{C^\#} & \longrightarrow & \mathcal{O}_{C^\#} \\ \in & & & & \in \\ x & & \longmapsto & & x^\# \end{array}$$

$$\theta : W_{\mathcal{O}_E}(\mathcal{O}_C) \twoheadrightarrow \mathcal{O}_{C^\#} \qquad \text{"Fontaine's map".}$$

$$\sum_{n\geq 0} [x_n]\pi^n \longmapsto \sum_{n\geq 0} x_n^\# \pi^n .$$

$$\begin{array}{ccc} \rightsquigarrow \quad \mathrm{Spa}\, \mathcal{O}_{C^\#} & \hookrightarrow & \mathrm{Spa}\, W_{\mathcal{O}_E}(\mathcal{O}_C) \\ \cup\!\! \mid & & \cup\!\! \mid \\ \mathrm{Spa}\, C^\# & \hookrightarrow & Y_{C,E} \end{array}$$

image $y \in Y_{C,E}$, compl. res. field at y is $C^\#$.

$\rightsquigarrow$ injection

$$\{C^\#/E \text{ untilt of } C\} \hookrightarrow Y_{C,E}.$$

Define $Y^{cl}_{C,E}$ = image.

<u>Aside</u>. If (A, A^+) Huber pair, $x \in \mathrm{Spa}(A, A^+)$

$\rightsquigarrow |\cdot|_x : A \to \Gamma_x \cup \{0\}$

$\mathfrak{p}_x = \{f \in A \mid |f|_x = 0\} \subseteq A$ prime ideal.

$\rightsquigarrow \widehat{\mathrm{Frac}(A/\mathfrak{p}_x)} =: K(x)$.

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2). Tilting for $Y_{C,E}$.

Let $E_\infty = E(\pi^{1/p^\infty})^\wedge$

$= \left(\bigcup_n E(\pi^{1/p^n})\right)^\wedge$

"perfectoid field":

$\mathcal{O}_{E_\infty}/p \circlearrowleft x \mapsto x^p$ is surjective.

$\rightsquigarrow$ tilt $E_\infty^\flat \cong \mathbb{F}_q((t^{1/p^\infty})) \ni$

$$\begin{array}{c} \| \\ \varprojlim_{x \mapsto x^p} E_\infty \ni (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots) = t \end{array}$$

$$\mathcal{O}_C \otimes_{\mathbb{F}_q} \mathbb{F}_q[t^{1/p^\infty}]_{/\pi} = \mathcal{O}_C \otimes_{\mathbb{F}_q} \mathbb{F}_q[t^{1/p^\infty}]/t.$$

__Claim.__ $\left(Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty\right)^\flat$ $\leftarrow$ perfectoid space.

$$\cong \left(W_{\mathcal{O}_E}(\mathcal{O}_C) \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}\right)^\flat /\!/ R.$$

$$/\!/R$$

$$\mathrm{Spa}\, \mathcal{O}_C[\![t^{1/p^\infty}]\!] \supseteq \mathbb{D}_C^* \times_{\mathrm{Spa}\, \mathbb{F}_q((t))} \mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty})).$$

Moreover, classical points biject under this correspondence.

$$\rightsquigarrow |\mathbb{D}_C^*| = |\mathbb{D}_C^* \times_{\mathrm{Spa}\, \mathbb{F}_q((t))} \mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty}))| \cong$$

Tilting

invariance under perfection

$$\cong |Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty| \to |Y_{C,E}|$$

$$\begin{array}{ccc} \mathbb{D}_C^{*,cl} & \twoheadrightarrow & Y_{C,E}^{cl} \\ \| & & \\ \{0 < |x| < 1,\ x \in C\} & \longmapsto & V(\pi - [x]). \end{array}$$

Aside: Perfectoid Spaces.

Def. 1) A perfectoid Tate ring is a complete Tate ring A ($\exists \varpi \in A$ top. nilpotent unit, $\exists A_0 \subseteq A$ open, ϖ adic) $A = A_0[\frac{1}{\varpi}]$.

if $\exists \varpi$ s.th.

$\varpi^p \mid p$ in A°, A° ϖ-adic, ($\Leftarrow$ $A^\circ = A_0$ ring of def'n)

$\Rightarrow$ p top. nilpotent.

$x \mapsto x^p \curvearrowright A^\circ/p$ is surjective.

2) A perfectoid space is an adic space X covered by $\mathrm{Spa}(A, A^+)$ with A a perfectoid Tate ring.

Example. $A = E_\infty$, C, $\mathbb{F}_q((t^{1/p^\infty}))$,

$$C\langle T^{1/p^\infty}\rangle.$$

If $A/\mathbb{F}_p$ Tate ring, then

A perfectoid $\Leftrightarrow$ A perfect (i.e. $\phi: A \to A$, $x \mapsto x^p$ isom.)

Tilting extends to perfectoid rings :
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$
$\qquad\qquad\qquad\qquad$ perfectoid Tate rings

$$A \longmapsto A^\flat = \varprojlim_{x \mapsto x^p} A \quad \text{(with suitable addition)}$$

$$A^\flat \ni f \longmapsto f^\# \in A$$

$\left(\underline{\text{ex}}.\ E_\infty \langle T^{1/p^\infty} \rangle^\flat = E_\infty^\flat \langle T^{1/p^\infty} \rangle\right)$,

and to perfectoid spaces

$$\begin{array}{ccc} X & \longmapsto & X^\flat \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ \mathrm{Spa}(A, A^+) & \longmapsto & \mathrm{Spa}(A^\flat, A^{\flat +}) \end{array}$$

__Thm.__ 1) $|X| \xrightarrow{\cong} |X^\flat|$.

$$x \longmapsto x^\flat : \quad |f(x^\flat)| = |f^\#(x)|$$

If $X = \mathrm{Spa}(A, A^+)$, $f \in A^\flat$
$X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$.

"Tilting preserves top. spaces"

$$\begin{array}{ccc} X & & \mathrm{Spec}\,\mathbb{Z}_p \\ \| & & \uparrow \\ X^\flat & & \mathrm{Spec}\,\mathbb{F}_p \end{array}$$

2) Given perf'd space X,

$$\{\text{perf'd spaces } Y/X\} \longrightarrow \{\text{perf'd spaces } Y'/X^\flat\}$$

$$Y \longmapsto Y^\flat$$

is an equivalence of categories.

3) (Bhatt-S.) If $X = \mathrm{Spa}(A, A^+)$

$X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$, then

the Zariski closed subsets of X and $X^\flat$ correspond.

$$\begin{array}{ccccc} & Z & \subseteq & |X| & \text{Zariski closed} \\ \text{easy.} \Uparrow & \| & \Updownarrow & \| & \text{(vanishing locus of} \\ & Z^\flat & \subseteq & |X^\flat| & \text{Zariski closed. same ideal)} \end{array}$$

<u>Challenge:</u> $X = \mathrm{Spa}\, \mathbb{C}^\# \langle T^{1/p^\infty} \rangle \supseteq Z = V(T-1)$.

Show that $Z^\flat$ Zariski closed.

$$\begin{array}{c} Z^\flat \\ \cap\!\!| \\ \mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle \end{array}$$