# Construction of L-parameter

$E$ nonarch. local field, $G/E$ reductive group.

$\ell \neq p$ $\qquad \hat{G} / \mathbb{Z}_\ell$ dual group

$W_E \circlearrowright \hat{G}$

Fix $\sqrt{q}$ $\quad (\mathbb{Z}_\ell \rightsquigarrow \mathbb{Z}_\ell[\sqrt{q}])$.

## Representation theory side

$D(G(E), \mathbb{Z}_\ell)$ (derived) category of smooth $G(E)$-repr.

$\downarrow$

$D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$

variant of $D_{ét}$ that works for all $\mathbb{Z}_\ell$-alg. $\Lambda$.
(uses 'solid 6-functor formalism')

local Langlands.

## Galois side

(spectral side)

Artin stack

$Z^1(W_E, \hat{G}) / \hat{G}$

of L-parameters.

irred. object $\longmapsto$ point.

e.g. $\pi$ irred smooth repr. of $G(E)$ $\longmapsto$ L-parameter $\varphi_\pi$.

"$\pi \mapsto \varphi_\pi$ should vary algebraically":

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$Z(G)$.

<u>Definition</u>. The Bernstein center of $G$ is the (comm.) algebra of endomorphisms of the identity functor on the category of smooth $G(E)$-representations.

for each $\pi$, give $f(\pi): \pi \to \pi$,
comm. with all $\pi \to \pi'$.

In particular, if $f \in Z(G)$, $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$,
($\rightsquigarrow \mathrm{End}(\pi) = \overline{\mathbb{Q}}_\ell$)

get scalar $f(\pi) \in \overline{\mathbb{Q}}_\ell$.

$$Z(G)_{\overline{\mathbb{Q}}_\ell} \hookrightarrow \{\text{functions on } \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)\}.$$

should be thought of as "the algebraic functions on the set $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$".

<u>Want</u>: for any $f \in \mathcal{O}(Z^1(W_E, \hat{G}))^{\hat{G}}$,

the map $\pi \mapsto f(\varphi_\pi)$ should be "algebraic", i.e. lie in Bernstein center.

<u>Definition</u>. The <u>spectral Bernstein center</u> is

$$Z^{spec}(G) := \mathcal{O}(Z^1(W_E, \hat{G}))^{\hat{G}}.$$

Also consider

$$Z^{geom}(G) = \text{"Bernstein center of } D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)\text{"}$$
$$= \mathrm{End}\left(\mathrm{id}_{D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)}\right) \to Z(G).$$

<u>Main Thm</u>. $\exists$ canonical map

(Fargues-S.)

(small assumption on $\ell$)

(all ok for $G = GL_n$.)

$$\psi : Z^{spec}(G) \to Z^{geom}(G)/\mathbb{Z}_\ell.$$

In particular, for each

$A \in D_{lis}(\mathrm{Bun}_G, L)$ $\qquad L/\mathbb{Z}_\ell$ alg closed field

$\mathrm{End}(A) = L$, (for example, $A = j_! [\pi]$, $\pi$ irr. smooth repr.)

$\exists !$ (up to $\hat{G}(L)$-conjugation)

$$\varphi_A : W_E \longrightarrow \hat{G}(L), \text{ "semisimple"}$$

s.th. $\forall f \in Z^{spec}(G)$,

$$f(\varphi_A) = \psi(f)(A) \quad \in L.$$

Properties of this correspondence:

Proposition. The map $\pi \mapsto \varphi_\pi$ has following properties:

(i) for tori, it agrees with usual LLC.

(ii) compatible with twisting, central characters

(iii) compatible with duals

(iv) if $G' \to G$ map inducing isom. of adjoint groups, $\pi$ irr. repr. of $G(E)$, $\pi'$ irr. constituent of $\pi|_{G'(E)}$, then $\varphi_{\pi'}$ = image of $\varphi_\pi$ under

$\hat{G} \to \hat{G}'$.

(v) compatible with products.

(vi) compatible with Weil restrictions of scalars.

(vii) compatible with parabolic induction.

(viii) agrees with usual correspondence for $GL_n$.

[Only place where (implicitly) global methods are used.]

(ix) compatible with Hecke functors on $\mathrm{Bun}_G$.

(x) compatible with cohomology of moduli spaces of local shtukas, e.g. local Shimura varieties, e.g. Rapoport-Zink spaces.

<u>Cor</u> (Thm of Helm–Moss) For $G = GL_n$, the map

$$Z^{spec}(G)_{\mathbb{Q}_\ell} \longrightarrow Z(G)_{\mathbb{Q}_\ell} \quad \text{defined by usual LLC}$$

is defined integrally, i.e. induces map

$$Z^{spec}(G) \longrightarrow Z(G).$$

"compatibility of LLC with $\ell$-adic congruences".

## Construction of $\psi: Z^{spec}(G) \to Z^{geom}(G)$

Have following: ($\infty$-) category $\mathcal{C} := D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$

and for any finite set $I$, an exact monoidal functor $\quad (W_E \twoheadrightarrow \underset{\text{finite}}{Q} \circlearrowright \hat{G}.)$

$$\mathrm{Rep}_{\mathbb{Z}_\ell}(\hat{G} \rtimes Q)^I \longrightarrow \mathrm{End}(\mathcal{C})^{W_E^I}$$
$$V \longmapsto T_V$$

$W_E^I$ - equiv. endofunctors.

linear over $\mathrm{Rep}_{\mathbb{Z}_\ell}(Q^I)$, functorially in $I$.

This comes from the Hecke action.

We will only this kind of abstract data.

<u>Proposition</u>. For any $A \in \mathcal{C}^\omega$, $\exists P \subset W_E$ open in wild inertia s.th. $\forall I$, $V \in \mathrm{Rep}(\hat{G} \rtimes Q)^I$, the $W_E^I$-action on $T_V(A)$ factors over $(W_E/P)^I$.

$\rightsquigarrow$ can replace $W_E$ by $W_E/P$ above,

then (as last time) by discretization $W \subset W_E/P$.

<u>Last time</u>:

<u>Thm</u> $\operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \xrightarrow{\sim} \mathcal{O}(Z^1(W, \hat{G}))^{\hat{G}}$

$$\begin{tikzcd}
\operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \arrow[rr, "\sim"] \arrow[dr, dashed] & & \mathcal{O}(Z^1(W, \hat{G}))^{\hat{G}} \arrow[dl, dashed] \\
& Z^{\mathrm{geom}}(G) = \operatorname{End}(\mathrm{id}_G) &
\end{tikzcd}$$

enough to produce these maps.

will be done by "excursion operators" (V. Lafforgue).

<u>Definition</u>. 1) An <u>excursion datum</u> is a tuple

$(I, V, \alpha: \mathbb{1} \to V|_{\Delta(\hat{G})}, \beta: V|_{\Delta(\hat{G})} \to \mathbb{1}, (\gamma_i)_{i \in I})$.

$I$: finite set

$V \in \operatorname{Rep}(\hat{G} \rtimes Q)^I$

$\gamma_i \in W$.

2) Given excursion data, the excursion operator

is the following element of $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$: for any $A \in \mathcal{C}$,

$$A = T_{\mathbb{1}}(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_{\mathbb{1}}(A) = A.$$

<u>Proposition</u>. These excursion operators define a map

$$\begin{tikzcd}
\operatorname*{colim}_{(n, F_i \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \arrow[rr] \arrow[dr, "\wr"] & & \mathrm{End}(\mathrm{id}_{\mathcal{C}}). \\
& \mathcal{O}(Z^1(W, \hat{G}))^{\hat{G}} \arrow[ur] &
\end{tikzcd}$$

$\mathrm{End}(A) = L$

<u>Corollary</u>. The L-parameter $\varphi_A$ are characterized as follows: for all excursion data, the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i))_{i\in I}} V \xrightarrow{\beta} L$$

agrees with scalar

$$A \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} A.$$

# The spectral action

**Theorem.**

(Nadler–Yun, Gaitsgory–Kazhdan–Rozenblyum–Varshavsky, FS.)

The $\infty$-categorical data from above are equivalent to an action of

$$\mathrm{Perf}\left(Z^1(W_E, \hat{G})/\hat{G}\right) \quad \text{on} \quad \mathcal{D}_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell).$$

$$\begin{tikzcd}
\mathrm{Rep}\left((\hat{G} \rtimes Q)^I\right) \arrow[r, "\text{pullback + tensor}"'] & \mathrm{Perf}\left(Z^1(W_E, \hat{G})/\hat{G}\right)^{W_E^I} \arrow[d] \\
& \mathrm{End}(\mathcal{C})^{W_E^I}.
\end{tikzcd}$$

What does this mean for "elliptic" L-parameters?

Assume for simplicity $G$ semisimple, coeff. $= \overline{\mathbb{Q}}_\ell$.

Say $\varphi$ elliptic if it defines an isolated comp. of $Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}$.

$$[*/S_\varphi] \underset{\text{open+closed}}{\subset} [Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}].$$

$S_\varphi \subset \hat{G}$ centralizer

$\overset{\wr}{}$ finite.

$\rightsquigarrow$ get corresponding

$$D^{\varphi}_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \underset{\bigoplus}{\subseteq} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell).$$

$\circlearrowleft$ $\mathrm{Rep}(S_\varphi)$. $\qquad$ $\circlearrowleft$ spectral action.

Compatibility of spectral action + Hecke action:

Given $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G} \rtimes Q)$,

$$V|_{S_\varphi \times W_E} = \bigoplus_{i=1}^{m} W_i \boxtimes r_i.$$

$W_i \in \mathrm{Irr\,Rep}\, S_\varphi \quad r_i \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$.

$$T_V(A) = \bigoplus_{i=1}^{m} \mathrm{Act}_{W_i}(A) \otimes r_i \in (D^{\varphi}_{lis})^{W_E}.$$

$A \in D_{lis}^{\varphi}$

This "is" the Kottwitz conjecture.

<u>Prop'n.</u> All $A \in D_{lis}^{\varphi}$ are concentrated on semistable locus, (ev compatibility with parabolic induction) and corr. to supercusp. repr.

$$\Rightarrow D_{lis}^{\varphi} = \bigoplus_{b \in B(G)_{basic}} \bigoplus_{\substack{\pi \text{ irr. supercusp.} \\ \text{repr. of } G_b(E) \\ \varphi_\pi = \varphi}} D(\overline{\mathbb{Q}}_\ell) \cdot [\pi].$$

$\circlearrowright$

$\mathrm{Rep}(S_\varphi)$.

If $S_\varphi$ abelian, characters of $S_\varphi$ will permute these $\pi$'s.

"Jacquet-Langlands".

Also, $T_V([\pi]) = \pi$ - isotypic comp. of cohom. of some moduli space

$\|$

$$\parallel \qquad \text{of local shtukas.}$$

$$\bigoplus_{i=1}^{m} \mathrm{Act}_{W_i}([\pi]) \otimes r_i.$$

This matches the Kottwitz conjecture.

Expect parametrization of all $\pi$ with $\varphi_\pi = \varphi$:

<u>Conjecture</u>. Assume $G$ quasisplit. Fix Whittaker data. Then there is a unique <u>generic</u> $\pi_0$ with $\varphi_{\pi_0} = \varphi$, and

$$\begin{aligned} \mathrm{Perf}([*/S_\varphi]) &\xrightarrow{\sim} \left(\mathcal{D}_{\mathrm{lis}}^{\varphi}\right)^{\omega} = \bigoplus_{b} \bigoplus_{\pi} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)\cdot[\pi]. \\ W &\longmapsto \mathrm{Act}_W([\pi_0]) \end{aligned}$$

is an equivalence of categories.

$$\Rightarrow \mathrm{Irr\,Rep}\, S_\varphi \xrightarrow{\sim} \{\pi\}$$

(This is Kaletha's /Kottwitz formulation of LLC using $B(G)_{\mathrm{basic}}$.)

Back to all of $D_{lis}$:

Main Conjecture (Categorical geometric Langlands)

Fix Whittaker data: $U \subset B \subset G$, $\psi : U(E) \to \breve{\mathbb{Z}}_\ell^\times$ nondeg. character,

$\rightsquigarrow$ $c\text{-}\mathrm{Ind}_{U(E)}^{G(E)} \psi$. smooth representation of $G(E)$. "universal generic rep."

There is an equivalence

$$D^b_{coh, qc\,supp.}\left(Z^1(W_E, \hat{G})_{\breve{\mathbb{Q}}_\ell}/\hat{G}\right) \cong D_{lis}(\mathrm{Bun}_G, \breve{\mathbb{Q}}_\ell)^\omega$$

linear over $\mathrm{Perf}\left(Z^1(W_E, \hat{G})_{\breve{\mathbb{Q}}_\ell}/\hat{G}\right)$, taking

$$\mathcal{O}_{[Z^1(W_E, \hat{G})_{\breve{\mathbb{Q}}_\ell}/\hat{G}]} \text{ to } [c\text{-}\mathrm{Ind}_{U(E)}^{G(E)} \psi]$$

Integrally: Same for

$$\mathcal{D}^b_{\mathrm{coh},\, \mathrm{Nilp},\, \mathrm{qc\,supp}}\left(Z^1(W_E, \hat{G})_{\breve{\mathbb{Z}}_\ell}/\hat{G}\right) \simeq D_{\mathrm{lis}}(\mathrm{Bun}_G, \breve{\mathbb{Z}}_\ell)^\omega .$$

"nilpotent singular support" (cf. Arinkin - Gaitsgory).

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$A$

$$\begin{tikzcd} Z^{\mathrm{spec}}(G) \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{End}(A) \arrow[dl] \\ & \mathrm{End}(T_V(A)) & \end{tikzcd}$$

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<u>Dream.</u> Can one compare t-structures?

On $D_{\mathrm{lis}}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)$ use perverse t-structure.

Which t-structure on $D^b_{\mathrm{coh}}(Z^1(W_E, \hat{G})_{\bar{\mathbb{Q}}_\ell}/\hat{G})$ should this correspond to?

<u>Question.</u> Is there a natural bij.

$$\bigsqcup_{b \in B(G)} \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G_b(E) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\varphi \text{ Frob-ss L-param}/\text{conj.}} \mathrm{Irr\,Rep}\, S_\varphi \quad ?$$

$\|\wr$ irr. perv. sheaves

$\|\wr$ ?? irr. "perv. coh." sheaves

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compatibility with parabolic induction:

<u>two steps</u>: 1). $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$.

there are two ways to assoc. L-param.: from $\mathrm{Bun}_G$, or from $\mathrm{Bun}_{G_b}$.

These are compatible.

<u>Sketch</u>: may assume $b$ very instable wrt. $P_b$.

But then Hecke operator for $G$ "reduce to" Hecke operators for $M$, if Hecke operator "not too large"

$\Rightarrow$ excursion operators agree.

2) Use comp. with Hecke operators, write induced

repr. as Hecke image of some repr. on $\mathrm{Bun}_G^b$.

( use "Hodge-Newton reducible case".)