L-Parameter

Setup: E nonarchimedean local field
G/E reductive group.

local Langlands correspondence: Canonical map

$$\{\text{irred. smooth } G(E)\text{-repr.}\} \longrightarrow \{L\text{-parameters}\}$$
$$\pi \longmapsto \varphi_\pi.$$

Usually, work with $\mathbb{C}$-coefficients.
($\Rightarrow$ canonical $\sqrt{q} \in \mathbb{C}$.)

L-group: $G/_E \rightsquigarrow$ dual group $\hat{G} / \mathbb{Z}$
$\circlearrowleft$
$\mathrm{Gal}(\bar{E}/E) \twoheadrightarrow Q$
action factors over finite quotient Q.

Definition ${}^L G := \hat{G} \rtimes Q$ alg. group $/\mathbb{Z}$.

L-group.

Definition. (Take 1) An L-parameter $/\mathbb{C}$ is a continuous map

$$\begin{tikzcd} \varphi : W_E \arrow[rr] \arrow[dr] & \circlearrowleft & {}^L G(\mathbb{C}) \arrow[dl] \\ & Q & \end{tikzcd},$$

equiv., a continuous 1-cocycle

$$W_E \longrightarrow \hat{G}(\mathbb{C}).$$

Remark. Continuity $\Longleftrightarrow$ factors over a discrete quotient W_E / I', $I' \subseteq I_E$ open finite index subgroup.

Deligne: It is better to also keep track of a monodromy operator N.

Definition (Take 2) An L-parameter$/\mathbb{C}$ is a pair (φ, N), where

$\varphi : W_E \to {}^L G(\mathbb{C})$ cont. group homom.,

$N \in \mathrm{Lie}\, \hat{g} \otimes \mathbb{C}$ s.th.

$\forall w \in W_E \quad \mathrm{Ad}(\varphi(w))(N) = q^{|w|} N$. ($\Rightarrow$ N nilpotent)

(or $q^{-|w|} N$?).

For $G = GL_n$, these are also called Weil-Deligne representations.

<u>Definition</u> (Take 3) An L-parameter$/\mathbb{C}$ is a pair (φ, r) where

$\varphi : W_E \to {}^L G(\mathbb{C})$ cont. group homom.

$r : SL_2 \to \hat{G}_{/\mathbb{C}}$ alg. repr.

s.th. r, φ commute. $\left(W_E \times SL_2 \to {}^L G. \right)$

Then $\varphi'(w) = \varphi(w)\, r \begin{pmatrix} q^{|w|/2} & \\ & q^{-|w|/2} \end{pmatrix}$

with $N = (\mathrm{Lie}\, r)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defines L-param. in sense of Take 2.

Each of Take 1, Take 2 and Take 3

naturally gives rise to a algebraic variety of L-parameters,

all distinct.

Parameters in sense of Take 2 & Take 3 are, up to $\hat{G}(\mathbb{C})$ - conjugation, in bijection, but scheme structures are different.

<u>Reason</u>: In Take 2, $N \neq 0$ can degenerate to $N=0$.

But in Take 3, SL_2 has "rigid" representation theory.

We <u>want</u> to have the degenerations, so Take 2 is the good one for us.

<u>Deligne's motivation</u> Fix $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. <u>$\ell \neq p$</u>.

<u>Definition</u> (Take 2') An L-parameter / $\overline{\mathbb{Q}}_\ell$ is a continuous group homomorphism

$$\begin{tikzcd} \varphi : W_E \arrow[r] \arrow[dr] & {}^L G(\overline{\mathbb{Q}}_\ell) \arrow[d] \\ & Q \end{tikzcd}$$

, i.e. a continuous 1-cocycle

$$W_E \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell).$$

<u>Thm</u> (Grothendieck, Deligne). Take 2 & Take 2' are equivalent in following sense:

[fix $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$ + Frobenius element $\Phi \in W_E$.

$\rightsquigarrow$ retract

$$W_E \to I_E \twoheadrightarrow \mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$$

with the composite t_ℓ.]

Any continuous group homom.

$$\begin{tikzcd} \varphi_\ell : W_E \arrow[r] \arrow[dr] & {}^L G(\overline{\mathbb{Q}}_\ell) \arrow[d] \\ & Q \end{tikzcd}$$

is of the form

$$\varphi_\ell(w) = \varphi(w)\exp\left(t_\ell(w)\cdot N\right)$$

for a unique L-param. (φ, N) in sense of Take 2.

<u>Key Point</u>. $W_E \to GL_n(\overline{\mathbb{Q}}_\ell)$ need not be trivial on an open subgroup $I' \subseteq I_E$; can only find such I' so that it factors over $I' \to \mathbb{Z}_\ell$.

$\mathrm{Hom}(\mathbb{Z}_\ell, GL_n(\overline{\mathbb{Q}}_\ell))$ are, on an open subgroup, given by $x \mapsto \exp(xN)$, N nilpotent matrix.

Thm depends on some choices.

We will adopt Take 2' as the definition.
$\Rightarrow$ forced to work over $\mathbb{Z}_\ell$.

<u>Goal:</u> Construct a moduli space of L-parameters, i.e. scheme locally of finite type

$$Z^1(W_E, \hat{G}) \quad / \mathbb{Z}_\ell$$

s.th. A-valued points (A any $\mathbb{Z}_\ell$-algebra) are the continuous group homom.

$$\begin{tikzcd} \varphi : W_E \arrow[rr] \arrow[dr] & & {}^L G(A) \arrow[dl] \\ & Q & \end{tikzcd}$$

, i.e. continuous 1-cocycles

$$W_E \to \hat{G}(A).$$

Dat - Helm - Kurinczuk - Moss , thm.

<u>Obvious question:</u> What topology on A?

<u>Construction.</u> Any $\mathbb{Z}_\ell$-module M can be endowed with the filt. colimit topology

$$M = \varinjlim_{\substack{M' \subseteq M \\ \text{fin.gen.}/\mathbb{Z}_\ell}} (M', \ell\text{-adic}).$$

$$\left[\begin{array}{l}\text{In language of condensed mathematics,} \\ \underline{M} = M_{disc} \otimes_{\mathbb{Z}_{\ell,disc}} \mathbb{Z}_\ell .\end{array}\right]$$

$\rightsquigarrow$ endowing A with this topology, moduli problem above is well-defined.

<u>Thm</u> There is a scheme $Z^1(W_E, \hat{G})/\mathbb{Z}_\ell$ param. L-parameters for G. It is a disjoint of affine schemes of finite type over $\mathbb{Z}_\ell$ that are flat, complete intersections, all of dimension $\dim G = \dim \hat{G}$.

<u>Note:</u> Can divide by conjugation action of $\hat{G}$ to get an Artin stack "$\mathrm{LocSys}_{\hat{G}}$".

Remark. The natural extension to animated $\mathbb{Z}_\ell$-algebras is the same moduli space.

Proof (Sketch). Any cont. 1-cocycle

$$\varphi : W_E \to \hat{G}(A)$$

is trivial on an open subgroup P of wild inertia.

$$Z^1(W_E, \hat{G}) = \bigcup_P Z^1(W_E/P, \hat{G})$$

transition maps are open + closed!

enough: All $Z^1(W_E/P, \hat{G})$ are affine, flat, complete intersection of dimension $= \dim \hat{G}$.

Trick: Pick $W \subset W_E/P$ dense discrete subgroup of following form: pick

$\sigma \in W_E$ Frobenius

$\tau \in I_E$ generator of tame inertia

Take subgroup generated by σ, τ, wild inertia.

$$1 \to I \to W \to \underset{\text{gen. by } \sigma.}{\mathbb{Z}} \to 1.$$

$$1 \to (\text{finite } p\text{-group}) \to I \to \underset{\text{gen. by } \tau.}{\mathbb{Z}[1/p]} \to 1.$$

<u>Claim</u>. $Z^1(W_E/P, \hat{G}) \longrightarrow Z^1(W, \hat{G})$

is an isomorphism.

<u>Proof of Claim</u>. For any A, need to see that

a cocycle $\varphi_0 : W \to \hat{G}(A)$ extends

uniquely to a cont. cocycle

$$\varphi : W_E/P \to \hat{G}(A).$$

uniqueness is clear, as $W \subset W_E/P$ dense.

<u>existence</u>: may enlarge E. to see: for any

$$\tau^{\mathbb{Z}[1/p]} \rtimes \sigma^{\mathbb{Z}} \longrightarrow GL_n(A),$$

$$\sigma^{-1}\tau\sigma = \tau^q.$$

the map

$$\mathbb{Z}[1/p] \longrightarrow GL_n(A), \quad n \mapsto \mathrm{im}(\tau)^n.$$

extends continuously to $\prod_{\ell \neq p} \mathbb{Z}_\ell$.

note: $\mathrm{im}(\tau)$ conj. to $\mathrm{in}(\tau)^q \Rightarrow$ all eigenv. are roots of unity of order prime to q

$\Rightarrow$ some power is unipotent.

But for unipotent matrices, all $\mathbb{Z}_\ell$-powers are well-defined.

$\square$ (Claim).

Claim $\Rightarrow$ $Z^1(W_E/P, \hat{G})$ is an affine scheme of finite type.

Can control deformation theory: "looks like complete intersection".

$\rightsquigarrow$ W_E/P has "cohom. dim. ($\leq$) 2.

to prove flat + of correct dim., enough to bound dimension of special fibre.

<u>Key Input:</u> <u>Thm</u> (Lusztig) There are only finitely many unipotent conjugacy classes.

($\rightsquigarrow$ "stratify according to unip. conj. class of τ.") $\square$.

A presentation of $\mathcal{O}(Z^1(W_E/P, \hat{G}))^{\hat{G}}$

Fix discretization $W \subset W_E/P$.

Then for any map $F_n \to W$ from a free group F_n,

get map

$$Z^1(W_E/P, \hat{G}) = Z^1(W, \hat{G}) \longrightarrow Z^1(F_n, \hat{G})$$
$$\| $$
$$\hat{G}^n.$$

<u>Proposition</u>. (Even in $\mathcal{D}(\mathbb{Z}_\ell)$)

$$\operatorname*{colim}_{(n, F_n \to W)} \mathcal{O}(\hat{G}^n) \xrightarrow{\sim} \mathcal{O}(Z^1(W_E/P, \hat{G})).$$

sifted colimit. (so agrees in module / algebras / ...)

<u>Corollary</u>. The map

$$\operatorname*{colim}_{(n, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \longrightarrow \mathcal{O}(Z^1(W_E/P, \hat{G}))^{\hat{G}}$$

global functions on <u>stack</u> of L-parameters

"spectral Bernstein center"

is a universal homeomorphism on spectra;

and an isom. after inverting ℓ.

(Use Haboush's theorem on geometric reductivity)

This will appear as "the algebra of excursion operators".

<u>Theorem</u>. Actually, (even in $\mathcal{D}(\mathbb{Z}_\ell)$) the map

$$\operatorname{colim}_{(n, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \longrightarrow \mathcal{O}(Z^1(W_E/P, \hat{G}))^{\hat{G}}$$

$\hat{G}$-action of simultaneous twisted conjugation.

is an isomorphism if

$\hat{G}_{der}$ s.c. $\Leftrightarrow$ $Z(G)$ connected and ℓ "is not too small": (ℓ "good").

- all ℓ type A
- all $\ell \neq 2$ type ${}^2A_n$, B_n, C_n, D_n, ${}^2D_n$
- all $\ell \neq 2, 3$ type ${}^3D_4$, ${}^6D_4$, E_6, E_7, F_4, G_2, ${}^2E_6$
- all $\ell \neq 2, 3, 5$ type E_8.

Can get rid of this assumption later.

Moreover, $\mathcal{O}(Z^1(W_E/P, \hat{G}))$ has a good $\hat{G}$-filtr.

$\Rightarrow H^i(\hat{G}, \mathcal{O}(Z^1(W_E/P, \hat{G}))) = 0$ for $i > 0$,

formation of $\hat{G}$-invariants commutes with any base change.

deformation theory:

T_2 = obstruction group $= H^2(W_E/P, \operatorname{Lie} \hat{g})$

T_1 = tangent group $= Z^1(W_E/P, \operatorname{Lie} \hat{g})$

$$\dim T_1 - \dim T_2 = \dim \hat{G} \quad .$$

(Euler char. $= 0$)

by local Tate duality.