Geometric Satake

E nonarch local field, res. field $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_q$, $\breve{E}$.

G/E reductive group. $\mathrm{Div}^1_X = (\mathrm{Spd}\,\breve{E})^\diamond/\varphi^{\mathbb{Z}}$
v-sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

$\rightsquigarrow$ Beilinson - Drinfeld Grassmannian

$$\mathrm{Gr}_G^I \longrightarrow (\mathrm{Div}^1_X)^I \quad , \text{ any finite set } I.$$

$(L^+G)^I \curvearrowright \mathrm{Gr}_G^I$

$$\mathcal{H}ck_G^I = (L^+G)^I \backslash \mathrm{Gr}_G^I \longrightarrow (\mathrm{Div}^1_X)^I.$$

notion of perverse + ULA sheaves.

<u>Definition</u> (Satake category) Λ ring, $n\Lambda = 0$, $(n,p)=1$.

$$\mathrm{Sat}_G^I(\Lambda) = \mathrm{Perv}^{\mathrm{ULA}}_{\mathrm{flat}}(\mathcal{H}ck_G^I, \Lambda)$$

$$\Big\downarrow F^I = \bigoplus_i R^i\pi_{G*} \qquad \pi_G : \mathrm{Gr}_G^I \to (\mathrm{Div}^1_X)^I.$$

$$\mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \cong \mathrm{Rep}_{W_E^I}(\Lambda) \longleftarrow \text{on finite proj. } \Lambda\text{-modules.}$$

"Better" description of F^I: Use constant term functor

$$CT_B : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Sat}^I_T(\Lambda)$$

$$\cong Rp_! q^* [\deg].$$

$B \subset G$ Borel

via pull-push along

$$\begin{tikzcd} & \mathrm{Gr}^I_B \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}^I_G & & \mathrm{Gr}^I_T . \end{tikzcd}$$

Varying B, get functor to

$$\mathrm{LocSys}\big((\mathrm{Div}^1_X)^I \times \mathcal{F}\ell^\diamond, \Lambda\big) \cong \mathrm{LocSys}\big((\mathrm{Div}^1_X)^I, \Lambda\big)$$

simply connected! classical flag variety (param. Borels)

$\Rightarrow$ independent of B!

$$\pi_{T*} \circ CT_B \cong \bigoplus R^i \pi_{G*} = F^I$$

by hyperbolic localization.

Fusion product on Sat^I_G symmetric monoidal,

$F^I : \mathrm{Sat}_G^I \to \mathrm{Rep}_{W_E^I}(\Lambda)$ a symm. monoidal fibre functor.

$\overset{+\varepsilon}{\rightsquigarrow}$ $\exists$ Hopf algebra

$$\mathcal{H}^I \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda)$$

s.th. $$\mathrm{Sat}_G^I \cong \mathrm{CoMod}_{\mathcal{H}^I}(\mathrm{Rep}_{W_E^I}(\Lambda)).$$

"Künneth formula" : $\mathcal{H}^I = \bigotimes_{i\in I} \mathcal{H}^{\{i\}}$.

$\mathcal{H}^{\{i\}} \;\hat{=}\;$ affine flat group scheme $\check{G}_\Lambda/\Lambda$

+ cont. W_E-action.

formation comp. with base change in Λ, so may assume $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

<u>Thm</u> $\wedge$ $\check{G}_{\mathbb{Z}/\ell^n\mathbb{Z}} \cong \hat{G}_{\mathbb{Z}/\ell^n\mathbb{Z}}$, W_E-equiv.,

$\exists$ canonical isom.

if the W_E-action on $\hat{G}$ has a cyclotomic twist (see below).

$\check{G}$: geometrically constructed group, coming from $\mathrm{Perv}(\mathrm{Gr}_G)$.

$\hat{G}$: abstractly constructed dual group.

<u>Cor.</u>

$$\begin{tikzcd} \mathrm{Sat}_G^I(\Lambda) \arrow[rr, "\cong"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Rep}(\hat{G}_\Lambda^I) \arrow[dl] \\ & \mathrm{Rep}_{W_E^I}(\Lambda) & \end{tikzcd}$$

internally in $\mathrm{Rep}_{W_E^I}(\Lambda)$.

$\rightsquigarrow$ appropriately defined,

$$\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}\left(\left(\hat{G}_\Lambda \rtimes W_E\right)^I\right).$$

Dual group.

G/E (or any field).

$\rightsquigarrow$ "universal Cartan" T of G:

have flag variety $\mathcal{F}l / E$ parametrizing Borels $B \subset G$; each has its torus quotient $T_\cdot / \mathcal{F}l$

$T \triangleq \mathbb{Z}$-local system $X^*(T) / \mathcal{F}l$.

simply conn.

arises uniquely from $\mathbb{Z}$-local system $X^*(T) / (\operatorname{Spec} E)_{et}$.

arises uniquely via base change from a torus T / E.

$X^*(T_{\bar{E}}) \circlearrowleft \operatorname{Gal}(\bar{E}/E)$ so get fin. free $\mathbb{Z}$-module

$$\begin{array}{ccc} X^{* \circlearrowleft W} & \supseteq & X^*_+ \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Gal}(\bar{E}/E) & & \end{array}$$ dominant cocharacters.

+ Weyl group W acts on X^*.

also get set of simple reflection

$$\begin{array}{ccc} S & \subset & W \\ \circlearrowleft & & \circlearrowleft \\ & & \mathrm{Gal}(\bar{E}/E) \end{array}$$

and for any simple reflection $s \in S$,

have simple root $\alpha_s \in X^*$,

root space $U_{\alpha_s} \subseteq G_{\bar{E}}$,

(if $T \subseteq B \subseteq G_{\bar{E}}$ fixed.)

+ simple coroot $\alpha_s^\vee \in X_* := (X^*)^\vee$

$$\begin{tikzcd} \mathbb{G}_m \subseteq SL_2 \arrow[r] \arrow[dr, bend right] & G_{\bar{E}} . \\ & T_{\bar{E}} \arrow[u, phantom, "\cup|"] \end{tikzcd}$$

$\rightsquigarrow$ root datum $\left(X^*, X_*, \underset{\text{roots}}{\Phi}, \underset{\text{coroots}}{\Phi^\vee}, X^*_+, X_*^+\right)$.

$$\circlearrowright$$

$$\mathrm{Gal}(\bar{E}/E).$$

<u>Observation</u> Exchanging X^* & X_* also gives a root datum, and this functor

$$(\text{reductive groups}) \longrightarrow (\text{root data})$$

has a canonical splitting given by Chevalley group schemes.

$\rightsquigarrow$ Chevalley group scheme

$$\hat{G} \;/\; \mathbb{Z} \;\circlearrowleft\; \mathrm{Gal}(\bar{E}/E).$$

corr. to $\left(X_*, X^*, \Phi^\vee, \Phi, X_*^+, X^*_+\right)$

This is the "dual group" of G.

$\hat{G}$ comes with

$$\hat{T} \subseteq \hat{B} \subseteq \hat{G}$$

($Gal(\bar{E}/E)$ stable)

s.th. $X^*(\hat{T}) = X_*$

+ for any simple reflection $s \in S$,

$$\mathrm{Lie}\ \hat{U}_{\alpha_s^\vee} \underset{\psi_s}{\cong} \mathbb{Z}$$

and the ψ_s are $Gal(\bar{E}/E)$-invariant.

<u>cyclotomic twist:</u> will work over $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^n\mathbb{Z}$

instead of $\mathbb{Z}$, then replace ψ_s with isom.

$$\mathrm{Lie}\ \hat{U}_{\alpha_s^\vee} \underset{\psi_s'}{\cong} \mathbb{Z}_\ell(1).$$

Tate twist: repr. of $Gal(\bar{E}/E)$

$\rightsquigarrow$ changed the $Gal(\bar{E}/E) \supseteq W_E$ -action on

$\check{G}$.

to prove Thm, need to find

$$\check{T} \subseteq \check{B} \subseteq \check{G}_{\mathbb{Z}/\ell^n\mathbb{Z}} \quad W_E\text{-stable}$$

(geom. constructed)

s.th.

- $X^*(\check{T}) = X_*$
- $\check{G}$ reductive of correct type
- canonical isom. $\mathrm{Lie}\ \check{U}_{\check{\alpha}_s} \cong \mathbb{Z}/\ell^n\mathbb{Z}(1)$.

(This is slightly finer information than in Mirkovic-Vilonen.)

<u>Proof of Thm</u>. Isomorphism canonical, so may extend E to assume G split.

Will now fix splitting of G; in particular

$$T \subseteq B \subseteq G.$$

(Independence of this choice is proved essentially as above: $\rightsquigarrow \mathcal{F}\ell$ is simply connected.)

$$CT_B : \mathrm{Sat}_G \longrightarrow \mathrm{Sat}_T$$

symm. monoidal & comm. with fibre functors.

$$\mathrm{Sat}_T = \mathrm{Perv}^{ULA}_{flat}(\mathcal{H}ck_T) = \bigoplus_{X_*(T)} \mathrm{Rep}_{W_E}(\Lambda).$$

$$\mathcal{H}ck_T = \bigsqcup_{X_*(T)} [\mathrm{Div}^1_X / L^+T].$$

$$\cong \mathrm{Rep}(\check{T}) \qquad X^*(\check{T}) = X_*(T).$$

$$\rightsquigarrow \text{map} \quad \check{T} \hookrightarrow \check{G}.$$

$$\begin{tikzcd} \check{T} \arrow[r, hook] & \check{G} \\ \mathbb{G}_m \arrow[u] \arrow[ur] & \end{tikzcd}$$

comes from $\bigoplus_i R^i \pi_{G*}$ F_η

$\mathbb{G}_m$ defines attracting "parabolic"

$$\check{T} \subseteq \check{B} \subseteq \check{G}.$$

$$X^*(\check{T}) = X_*(T).$$

Case of rk 1 groups:

$$G \longrightarrow G_{ad} \cong PGl_2.$$

$$Gr_G = Gr_{G_{ad}} \underset{\mathbb{Z}/2\mathbb{Z}}{\times} \pi_1(G)$$

$$\Rightarrow \check{G} \cong \check{G}_{ad} \overset{\mu_2}{\times} \check{Z}$$

$$X^*(\check{Z}) = \pi_1(G).$$

reduce to adjoint case.

$G = PGl_2.$

$$\mathrm{Gr}_{PGl_2} \overset{i_\mu}{\supseteq} \mathrm{Gr}_{PGl_2, \mu} \cong (\mathbb{P}^1)^\diamond$$

$$A = i_{\mu *} \Lambda[1] \in \mathrm{Sat}_G(\Lambda) = \mathrm{Rep}(\check{G})$$

$$F(A) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)$$
$$= \Lambda \oplus \Lambda(-1).$$

$$\check{G} \longrightarrow \mathrm{Aut}(F(A)) = Gl_2 \quad /\Lambda.$$

Know: Over $\mathbb{Q}_\ell$ + $\mathbb{F}_\ell$,

$$\mathrm{Irr\,Rep}(\check{G}) \longleftrightarrow X_*^+ = \mathbb{Z}_{\geq 0}.$$

$$IC_\lambda \longleftarrow\!\shortmid \lambda$$

$$\mathrm{Sat}_G(\mathbb{Z}_\ell) = \varprojlim_n \mathrm{Sat}_G(\mathbb{Z}/\ell^n\mathbb{Z})$$

$$\mathrm{Sat}_G(\mathbb{Q}_\ell) = \mathrm{Sat}_G(\mathbb{Z}_\ell)[\tfrac{1}{\ell}]$$

$\mathrm{Sat}_G(\mathbb{Q}_\ell)$ is semisimple: all obj's direct sums of IC_λ's.

via comparison to Witt vector affine Grassm.

$\Rightarrow$ $\check{G}_{\mathbb{Q}_\ell}$ reductive + connected, rk = 1.

$\rightsquigarrow$ $\check{G} \longrightarrow SL_2 \subseteq GL_2$. $/\mathbb{Z}_\ell$,

$$\begin{tikzcd} \check{G} \arrow[r] & SL_2 \\ \mathbb{G}_m \arrow[u] \arrow[ur] & \end{tikzcd}$$

must be an isom. $/\mathbb{Q}_\ell$.

Over $\mathbb{F}_\ell$,

$\check{G}_{\mathbb{F}_\ell} \longrightarrow SL_{2,\mathbb{F}_\ell}$ must be surj.,

as otherwise image is torus or Borel or normalizer of torus.

This would lead to too many irred repr. of $\check{G}_{\mathbb{F}_\ell}$.

Thus, $\mathcal{O}(SL_2) \longrightarrow \mathcal{O}(\check{G})$ $/\mathbb{Z}_\ell$

isom. in char. 0, both are flat $/\mathbb{Z}_\ell$,

injective mod ℓ.

$\Rightarrow$ is isomorphism!

$$\check{G} = SL_2 = SL(F(A)).$$
$$= SL(\Lambda \oplus \Lambda(-1))$$

$\rightsquigarrow$ cyclotomic twist in root group.

<u>Back to general G:</u> CT_P exists for

any parabolic P, defines symm. mon. functor
$\downarrow$
M

$$\begin{tikzcd} Sat_G \arrow[r, "CT_P"] \arrow[dr, "F_G"'] & Sat_M \arrow[r, "CT_{B_M}"] \arrow[d, "F_M"] & Sat_T \arrow[dl, "F_T"] \\ & Rep_{W_E} & \end{tikzcd}$$

with CT_B the composite $Sat_M \to Sat_T$ drawn above.

Use this for minimal parabolics.

$$\rightsquigarrow \quad \underset{\hat{T}}{\overset{\shortparallel}{\check{T}}} \hookrightarrow \underset{\hat{M}}{\overset{\shortparallel}{\check{M}}} \hookrightarrow \check{G}.$$

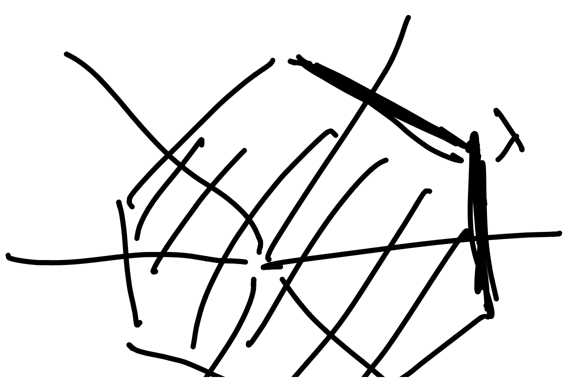
In part., for any simple reflection s, get

$$\mathbb{G}_m \subseteq SL_2 \to \hat{M} \longrightarrow \check{G}$$

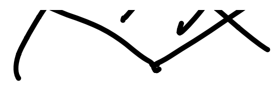
$\hat{=} \; \alpha_s \in X^* = X_*(\check{T})$.

<u>Know:</u> $\check{G}_{\mathbb{Q}_\ell}$ connected + reductive, and has at least as many roots/coroots as $\hat{G}$. Does not have more, by looking at weights appearing in

$F(IC_\lambda)$ = weights of irred repr. of $\check{G}$ of weight λ.



$\Rightarrow \check{G}_{\mathbb{Q}_\ell} \cong \hat{G}_{\mathbb{Q}_\ell}$, canonical, as simple root subgroups are pinned!



$$\hat{M} \hookrightarrow \check{G}$$

defined integrally!

In particular,

$$\check{G}(\breve{\mathbb{Z}}_\ell) \subseteq \check{G}(\breve{\mathbb{Q}}_\ell) \simeq \hat{G}(\breve{\mathbb{Q}}_\ell)$$

contains $\hat{M}(\breve{\mathbb{Z}}_\ell)$

for any minimal Levi.

$$\Rightarrow \quad \hat{G}(\breve{\mathbb{Z}}_\ell) \subseteq \check{G}(\breve{\mathbb{Z}}_\ell) \subseteq \hat{G}(\breve{\mathbb{Q}}_\ell).$$

hyperspecial maximal compact, w/ $\hat{G}$ reductive.

still a bounded subgroup.

$$\Rightarrow \quad \check{G}(\breve{\mathbb{Z}}_\ell) = \hat{G}(\breve{\mathbb{Z}}_\ell).$$

$\overset{+\varepsilon}{\Longrightarrow}$ $\check{G} = \hat{G}$ as integral models of

$$\check{G}_{\mathbb{Q}_\ell} \cong \hat{G}_{\mathbb{Q}_\ell}.$$

This finishes the proof of geometric Satake. $\square$

<u>Lemma</u> (Prasad-Yu). H reductive $/\mathbb{Z}_\ell$ ($= \hat{G}$),

H' affine flat group scheme of finite type, $\rho: H \to H'$

that is a closed immersion in generic fibre.

Assume that $\ell \neq 2$, or that no almost simple factor of $H_{\overline{\mathbb{Q}}_\ell}$ is isomorphic to SO_{2n+1} (e.g. derived group of H simply connected).

Then ρ is a closed immersion.

How to apply? Can assume G adjoint $\Rightarrow$ $\hat{G}$ simply connected.

Pick a repr. $\check{G} \longrightarrow GL_N$

that is a closed imm. on generic fibre.

$$\hat{G}_{\mathbb{Q}_\ell} \cong \check{G}_{\mathbb{Q}_\ell} \hookrightarrow GL_{N,\mathbb{Q}_\ell}$$

and $\hat{G}(\breve{\mathbb{Z}}_\ell) \cong \check{G}(\breve{\mathbb{Z}}_\ell) \subseteq GL_N(\breve{\mathbb{Z}}_\ell)$.

as $\hat{G}$, GL_N smooth, this gives map

$$\hat{G} \longrightarrow GL_N \quad / \mathbb{Z}_\ell.$$

Lemma $\Rightarrow$ closed immersion.

$$\begin{tikzcd} \check{G} \arrow[r] \arrow[dr, dashed, "\exists"'] & GL_N \\ & \hat{G} \arrow[u, hook] \end{tikzcd} \quad / \mathbb{Z}_\ell.$$

iso on generic fibre, surj. over $\mathbb{F}_\ell$.

$\Rightarrow$ $\qquad \check{G} \xrightarrow{\simeq} \hat{G} \;/\mathbb{Z}_\ell$.

same obsevr. as for SL_2

$$\hat{\rho} : \mathbb{G}_m \longrightarrow \hat{G}_{ad}$$

$$\begin{array}{ccccccc} W_E & \xrightarrow{cyc} & \mathbb{Z}_\ell^\times & \xrightarrow{\hat{\rho}} & \hat{G}_{ad}(\mathbb{Z}_\ell) & \to & \mathrm{Aut}(\hat{G}_{\mathbb{Z}_\ell}) . \\ w & \longmapsto & q^{|w|} & \searrow & \uparrow & & \\ & & & & \hat{G}(\mathbb{Z}_\ell) & & \end{array}$$

cyclotomic twist = twist of usual action by this.

Can be lifted to $W_E \xrightarrow{2\hat{\rho}(\sqrt{q}^{|w|})} \hat{G}(\mathbb{Z}_\ell[\sqrt{q}])$

So over $\mathbb{Z}_\ell[\sqrt{q}]$, can undo the twist.

$$\mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \cong \mathrm{Rep}_{W_E^I}(\Lambda).$$

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$: follows from

$$\pi_1^{\text{ét}}((\mathrm{Div}^1_X)^I) = \mathrm{Gal}(\bar{E}/E)^I.$$

was proved in Berkeley lectures.

Key input: $X_{C,E} \times_E \bar{E}$ simply connected, all C alg. closed.