Geometric Satake

Currently, $G / \mathcal{O}_E$ split reductive group.

$$\begin{matrix} G \\ \cup \\ B \\ \cup \\ T \end{matrix}$$

$\mathrm{Div}^d_{\mathcal{Y}}$ moduli space of degree d Cartier divisors on

$$D_S \subset \mathcal{Y}_S = S \dot{\times} \mathrm{Spa}\, \mathcal{O}_E .$$

Beilinson-Drinfeld Grassmannian

$$\begin{tikzcd} L^+G \circlearrowright \mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}} \arrow[r] & \mathrm{Div}^d_{\mathcal{Y}} . \\ \mathrm{Gr}_{G, S/\mathrm{Div}^d_{\mathcal{Y}}} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & S \arrow[u] \end{tikzcd}$$

Param. G-torsors on $\mathcal{Y}_S$ with merom. trivialization on $\mathcal{Y}_S \setminus D_S$.

Beauville-Laszlo: —"— on $\mathcal{Y}_S|^{\wedge}_{D_S}$. —"— $\mathcal{Y}_S|^{\wedge}_{D_S} \setminus D_S$.

local Hecke stack

$$\mathcal{H}ck_{G, S/\mathrm{Div}^d_{\mathcal{Y}}} = L^+G \backslash \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$$

$$= \left(L^+G \backslash LG / L^+G\right) \times_{\mathrm{Div}^d_{\mathcal{Y}}} S.$$

$D_{ét}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{bd}$ ← bounded support

has relative$/S$ perverse t-structure.

$\rightsquigarrow$ $\mathrm{Perv}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ abelian category, functorial in S.

+ monoidal structure $\star$ = convolution product on $D_{ét}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{bd}$.

<u>Prop'n</u>. Pullback to Gr induces a fully faithful functor

$$\mathrm{Perv}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{bd}.$$

__Sketch.__ L^+G is connected. $\square$.

__Proposition.__ If $A,B \in {}^pD^{\leq 0}(\mathrm{Hck})^{bd}$,

then also $A * B \in {}^pD^{\leq 0}(\mathrm{Hck})^{bd}$.

If A,B perverse and A __flat__ perverse

(i.e. $A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} M$ perverse, any Λ-module M)

then also $A * B$ perverse.

__Sketch.__ Statement can be proved on geometric fibres; then Hck decomposes into product over untilts; reduce to $d=1$, and

$$A = j_{\mu_1 !}\Lambda, \quad B = j_{\mu_2 !}\Lambda.$$

reduce to $S = \mathrm{Div}^1_{\mathcal{Y}}$. Then everything ULA, so reduce to special fibre, where it follows from work of Zhu on Witt vector affine Grassmannian

Alternatively, use fusion product to conclude (do not need Witt vector affine Grassmannian or "semismallness of convolution" here.) $\square$.

$\rightsquigarrow$ convolution product on flat perverse sheaves.

ULA sheaves on Hck.

<u>Definition</u>. An object $A \in D_{et}(Hck_{G, S/Div^d_y}, \Lambda)^{bd}$ is ULA $/S$ if its pullback to $Gr_{G, S/Div^d_y}$ is ULA $/S$.

$\Big\}$ $Hck = L^+G \backslash LG / L^+G$ has switching symmetry.

Being ULA is invariant under this symmetry.

<u>Proposition</u>. For $d=1$, $A \in D_{et}(Hck_{G,S/Div^d_y}, \Lambda)^{bd}$

is ULA iff for all μ,

$$A|_{S \xrightarrow{[\mu]} \mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}} \in D_{\acute{e}t}(S,\Lambda)$$

locally constant with perfect fibres.

"local system".

<u>Cor</u> This class of sheaves is preserved under all six operations one can build from Hck & its strata. (Rj_*, $Ri^!$, ..., $R\mathcal{H}om(-,-)$.)

<u>Cor</u>.

$$\begin{tikzcd}
(\mathrm{Spa}\, C)^\diamond \arrow[r, hook] \arrow[dr] & (\mathrm{Spa}\, \mathcal{O}_C)^\diamond \arrow[d] & (\mathrm{Spa}\, k)^\diamond \arrow[l, hook'] \arrow[dl] \\
& \mathrm{Div}^1_{\mathcal{Y}} = (\mathrm{Spa}\, \mathcal{O}_E)^\diamond &
\end{tikzcd}$$

induces equiv.

$$D^{ULA}_{\acute{e}t}(\mathrm{Hck}_{G,\mathrm{Spa}\, C}, \Lambda)^{bd} \xleftarrow{\sim} D^{ULA}_{\acute{e}t}(\mathrm{Hck}_{G,\mathrm{Spa}\, \mathcal{O}_C}, \Lambda)^{bd}$$

$$\Big|\wr$$

$$\downarrow$$
$$D_{ét}^{ULA}(\mathcal{H}ck_{G,\mathrm{Spa}\,\mathbb{F}_2}, \Lambda)^{bd}.$$

<u>Sketch of proof of Prop</u>. Key point is that

$j_{\mu!}\Lambda$ is ULA. $S = \mathrm{Div}_Y^1$.

$$j_\mu : \mathrm{Gr}_{G,\mu} \hookrightarrow \mathrm{Gr}_{G,\leq\mu} \quad /S.$$

check this after pullback to affine flag variety.

$$\mathrm{Iw} \circlearrowright \mathcal{F}\ell_G = LG/\mathrm{Iw} \longrightarrow \mathrm{Gr}_G = LG/L^+G$$

$(G/B)^\diamond$ - bundle.

$$\begin{tikzcd} \mathrm{Iw} \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & L^+G \arrow[d, "ev"] \\ B \arrow[r, hook] & G \end{tikzcd}$$

$$\mathcal{F}\ell_G = \bigcup_{w \in \tilde{W}} \mathcal{F}\ell_{G,w}$$

$\tilde{W}$ extended affine Weyl group.

$$\begin{matrix} \mathcal{F}\ell_G \\ \uparrow \\ W \\ \uparrow \end{matrix}$$

Iw-orbits.

$$\begin{tikzcd}
1 \arrow[d] \\
\tilde{W} = N(T)(B_{dR}) / T(B_{dR}^+). \\
X_*(T) = T(B_{dR}) / T(B_{dR}^+) \arrow[u] \\
1 \arrow[u]
\end{tikzcd}$$

$$1 \to W_{aff} \to \tilde{W} \to \Omega \to 1.$$

Can reduce instead to

$$j_{w!} \Lambda \quad , \; j_w : \mathcal{F}\ell_{G,w} \hookrightarrow \mathcal{F}\ell_{G,\leq w}.$$

<u>Key Point</u>. $\exists$ Demazure - Bott - Samelson resolution.

$$\begin{tikzcd}
\mathcal{F}\ell_{G,w} \arrow[r, hook, "j_w"] \arrow[d, hook, "\tilde{j}_w"'] & \mathcal{F}\ell_{G,\leq w} \\
\widetilde{\mathcal{F}\ell}_{G,\dot{w}} \arrow[ur] &
\end{tikzcd}$$

for $w = s_{i_1} \cdots s_{i_\ell} \cdot \omega \qquad \omega \in \Omega \subset \tilde{W}$

$\Omega \cong$ closed orbits.

$\ell = \ell(w) \qquad s_{i_r}$ simple affine reflections.

$\mathcal{F}\ell_{G,\omega} =$ points.

$$\widetilde{\mathcal{F}\ell}_{G,\dot w} = \mathcal{P}_{s_{i_1}} \overset{\mathcal{I}w}{\times} \mathcal{P}_{s_{i_2}} \overset{\mathcal{I}w}{\times} \cdots \overset{\mathcal{I}w}{\times} \mathcal{P}_{s_{i_\ell}} / \mathcal{I}w.$$

$$\mathcal{P}_{s_{i_j}} / \mathcal{I}w \cong (\mathbb{P}^1)^{\diamond}.$$

iterated $(\mathbb{P}^1)^{\diamond}$-bundle.

$\mathcal{P}_s \subseteq L^+ G$ parahoric subgroups.

corr. to simple affine reflection s.

(If s usual reflection,

$$\begin{tikzcd}
\mathcal{P}_s \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\square"] & L^+ G \arrow[d] \\
P_s \arrow[r, hook] & G
\end{tikzcd}$$

minimal parabolic .)

enough: $\tilde{j}_{w!}\Lambda$ ULA on $\widetilde{\mathcal{F}\ell}_{G,\dot w}$

cohom. smooth, boundary behaves like normal crossing divisor.

remove factor from w.

All strata smooth $\Rightarrow$ $\tilde{j}_{w!}\Lambda$ is ULA. $\square$

<u>Proposition</u>. $S \to \mathrm{Div}_Y^d$ arbitrary. Consider

$$CT_B = Rp_! q^* [\deg] : D_{et}(\mathrm{Hck}_{G,S/\mathrm{Div}_Y^d}, \Lambda)^{bd} \ni A$$
$$\downarrow$$
$$D_{et}(\mathrm{Hck}_{T,S/\mathrm{Div}_Y^d}, \Lambda)^{bd}$$

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}_G & & \mathrm{Gr}_T \arrow[d, "\deg"] \\ & & \mathbb{Z} \end{tikzcd}$$

Then A is ULA iff

$CT_B(A)$ is ULA.

iff $R\pi_{T*} CT_B(A) \in D_{et}(S,\Lambda)$

is locally constant with perfect values.

$$\pi_T : \mathrm{Gr}_{T,S/\mathrm{Div}_Y^d} \longrightarrow S.$$

$\square$

Back to preservation of perversity:

Wanted: $j_{\mu_1 !}\Lambda * j_{\mu_2 !}\Lambda$ still in ${}^{p}D^{\leq 0}$.

let $\widetilde{Gr}_{G,(Div_y^1)^2} \xrightarrow{\pi_2} Gr_{G,(Div_y^1)^2}$

$\Vert$

$\{(\mathcal{E}_1, \mathcal{E}_2, \text{two untilts } S_1^\#, S_2^\#) \,/\, \mathcal{E}_1$ triv. away from $S_1^\#$,

$\mathcal{E}_2 \cong \mathcal{E}_1$ away from $S_2^\#\}$.

isom. away from diagonal. Over diagonal, get

$$\widetilde{Gr}_{G, Div_y^1} \xrightarrow{\pi} Gr_{G, Div_y^1}$$

conv. aff. Grassm.

want: $R\pi_* (j_{\mu_1 !}\Lambda \,\widetilde{\boxtimes}\, j_{\mu_2 !}\Lambda) \in {}^{p}D^{\leq 0}$.

globalizes to

$$R\pi_{2*}\big(\underbrace{j_{\mu_1 !}\Lambda \ \tilde{\boxtimes}\ j_{\mu_2 !}\Lambda}_{\text{ULA on } \widetilde{\mathrm{Gr}}_{G,(\mathrm{Div}^1_Y)^2}}\big) \quad \text{ULA on } \mathrm{Gr}_{G,(\mathrm{Div}^1_Y)^2}.$$

<u>Want:</u> $\in {}^p D^{\leq 0}$.

Can be checked after applying

$R\pi_{T*}\, CT_B[\deg]$; result is in

$D_{et}((\mathrm{Div}^1_Y)^2, \Lambda)$, locally constant.

Away from diagonal, it is just the tensor product by Künneth. $\underbrace{R\pi_{T*}\, CT_B(j_{\mu_1 !}\Lambda) \overset{L}{\otimes} R\pi_{T*}\, CT_B(j_{\mu_2 !}\Lambda)}_{\in D^{\leq 0}}$

Complement of diagonal dense $\Rightarrow$ same over diagonal.

$\square$

From now on, work again over $(\mathrm{Div}^1_X)^d$.

G/E, any reductive group. X in place of Y.

will use previous results via implicit étale localization to reduce to G split.

<u>Definition</u>. (Satake category). Let I finite set, Λ any ring killed by some n prime to p.

$$\mathrm{Sat}^I_G(\Lambda) = \mathrm{Perv}^{ULA}_{flat}(\mathcal{H}ck_{G,(\mathrm{Div}^1_X)^I}, \Lambda)$$

$\mathcal{H}ck^I_G = \mathcal{H}ck_{G,(\mathrm{Div}^1_X)^I}$

flat perverse sheaves A on $\mathcal{H}ck^I_G$ that are ULA.,

with fibre functor (exact + conservative).

locally free, fin. proj. $/\Lambda$.

$$F : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda)$$

version of Drinfeld's lemma. $\longrightarrow$ <u>Thm</u> $\cong$ $\mathrm{Rep}_{W_E^I}(\Lambda)$

$$A \longmapsto \bigoplus_{i\in\mathbb{Z}} R^i\pi_{G*} A$$

$\cong$

$$\pi_G : \mathrm{Gr}_G^I \longrightarrow (\mathrm{Div}^1_X)^I \ . \ 2$$

$$\mathrm{Gr}_G^I = \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} .$$

if G split

$R\pi_{T*} CT_B(A)$.

using hyperbolic localization.

Fusion product:

functor

(put some signs in symmetry constraints)

exterior tensor product.

$$\begin{tikzcd}
* : \mathrm{Sat}_G^{I_1}(\Lambda) \times \dots \times \mathrm{Sat}_G^{I_m}(\Lambda) \arrow[d] \arrow[dr] & \\
\mathrm{Sat}_G^{I_1 \sqcup \dots \sqcup I_m}(\Lambda) \arrow[r, hook, two heads] & \mathrm{Sat}_G^{I_1, \dots, I_m}(\Lambda)
\end{tikzcd}$$

use convolution affine Grassmannians to show factorization.

$\Rightarrow$ fusion = convolution.

defined as $\mathrm{Sat}_G^{I_1 \sqcup \dots \sqcup I_m}$, but over

$(\mathrm{Div}^1_X)^I \setminus$ locus where $x_i = x_j$, then here, $\mathrm{Gr}_G^I|_{\dots}$ ↗ i, j lie in different I_k's.

$$\mathrm{Gr}_G^I|_{\dots} \cong \mathrm{Gr}_G^{I_1} \times \dots \times \mathrm{Gr}_G^{I_m}|_{\dots}.$$

$$\ast : \mathrm{Sat}_G^I(\Lambda) \times \dots \times \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I \sqcup \dots \sqcup I}(\Lambda) \xrightarrow{\text{diagonal restr.}} \mathrm{Sat}_G^I(\Lambda)$$

turns each

$\mathrm{Sat}_G^I(\Lambda)$ into a symm. monoidal cat., functorial in Λ, compatibly with its original monoidal structure.

$\Longrightarrow$ fusion = convolution. for category-theoretic reasons.

<u>Prop'n</u> $F = \bigoplus R^i \pi_{G*}$ is a symmetric monoidal functor.

$$F: \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda).$$

<u>Thm</u>. $F: \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda)$ satisfies all required properties for a Tannakian reconstruction, so that $\exists$ Hopf algebra

$$\mathcal{H}^I \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda),$$

$$\mathrm{Sat}^I_G(\Lambda) \cong \mathrm{CoMod}_{\mathcal{H}^I}(\mathrm{Rep}_{W_E^I}(\Lambda)).$$

Clear in char. 0; integrally, have to work a little bit.

<u>key</u>: $\mathrm{Sat}^I_G(\Lambda)$ is a highest weight category. $\square$

<u>Prop'n</u>. $\mathcal{H}^I = \bigotimes_{i \in I} \mathcal{H}^{\{i\}}$.

Then $\mathcal{H}^{\{i\}} \mathrel{\hat{=}}$ affine group scheme $\check{G}/\Lambda$ with continuous W_E-action.

Thm $\check{G} \cong \hat{G}$ dual group,

W_E-equivariant if the pinning of $\hat{G}$ includes cyclotomic twist:

$\exists$ canonical isom. $\mathrm{Lie}\, \hat{U}_a \cong \Lambda(1)$.

(can be trivialized if $\sqrt{q} \in \Lambda$.)

Cor $\mathrm{Sat}_G(\Lambda) \cong \mathrm{Rep}(\hat{G})$

internally in $\mathrm{Rep}_{W_E^{(I)}}(\Lambda)$.

$\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}(\hat{G}^I)$

$$\begin{tikzcd}
\mathrm{Sat}_G(\Lambda) \arrow[rr, phantom, "\cong"] \arrow[dr] & & \mathrm{Rep}_\Lambda(\underset{\substack{\parallel \\ {}^L G.}}{\hat{G} \rtimes W_E}). \arrow[dl] \\
& \mathrm{Rep}_{W_E}(\Lambda). &
\end{tikzcd}$$