Geometric Satake

Correction to last time: (regarding hyperbolic localization)

(minor): for schemes, stratifications need not exist as described.

ok if X normal.

HL ok if stratif. exist.

(major): claimed

$$R\Gamma(X, A) \cong R\Gamma(X^0, L(A)).$$

$$X \circlearrowleft \mathbb{G}_m \qquad X^0 = X^{\mathbb{G}_m} \underset{\text{closed}}{\subseteq} X.$$

proper/k.

$$L : D_{et}(X/\mathbb{G}_m) \to D_{et}(X^0).$$

This is false!

Example. $X = \mathbb{P}^1 \overset{j}{\hookleftarrow} (\mathbb{A}^1)^- \circlearrowleft \mathbb{G}_m.$

$$A = j_! \Lambda .$$

<u>What I thought:</u> Have stratification

$$X = \bigcup_{i=1}^{m} X_i^+$$

$\rightsquigarrow$ filtration of $R\Gamma(X, A)$ with graded pieces

$$R\Gamma_c(X_i^+, A|_{X_i^+})$$
$$\wr\!\wr$$
$$R\Gamma_c(X_i^0, \underbrace{R(p_i^+)_! \, A|_{X_i^+}}_{L(A)|_{X_i^0}}).$$

$$\text{graded} = \bigoplus_{i=1}^{m} R\Gamma_c(X_i^0, L(A))$$
$$= R\Gamma(X^0, L(A)).$$

$$\begin{array}{ccc} X_i^+ & \xrightarrow{q_i^+} & X \\ \big\downarrow {\scriptstyle p_i^+} & & \\ X_i^0 & & \end{array}$$

Also have stratification

$$X = \bigcup_{i=1}^{m} X_i^-$$

$\rightsquigarrow$ filtration of $R\Gamma(X, A)$ with graded pieces

$$R\Gamma_{X_i^-}(X, A) = R\Gamma(X_i^-, R(q_i^-)^! A)$$

$$= R\Gamma\Big(X_i^0, \underbrace{R(p_i^-)_* R(q_i^-)^! A}_{L(A)|_{X_i^0}.}\Big)$$

$$\begin{tikzcd} X_i^- \arrow[r, hook, "q_i^-"] \arrow[d, "p_i^-"] & X \\ X_i^0 & \end{tikzcd}$$

$$\rightsquigarrow \text{ graded } = \bigoplus_{i=1}^{m} R\Gamma(X_i^0, L(A)) = R\Gamma(X^0, L(A))$$

I thought these filtrations are opposite, and thus induce a splitting.

Unfortunately, the filtrations are the same.

Beilinson-Drinfeld Grassmannians

Assume $G / \mathcal{O}_E$ is split reductive.

(In general, use étale localization to reduce to this case.)

<u>Recall</u>. moduli space of degree d Cartier divisors on $\mathcal{Y}_S = S \dot{\times} \operatorname{Spa} \mathcal{O}_E$

(for $S = \operatorname{Spa}(R, R^+)$, $\varpi \in R$ pseudounif.,
$\mathcal{Y}_S = \operatorname{Spa} W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\}$.)

$$\mathrm{Div}^d_{\mathcal{Y}} = (\mathrm{Div}^1_{\mathcal{Y}})^d / \Sigma_d$$

small v-stack;

$$= (\operatorname{Spa} \mathcal{O}_E)^{\diamond, d} / \Sigma_d .$$

$\mathrm{Div}^1_{\mathcal{Y}} \to *$ repr. in loc. spat. diam.

"param. d points on $\operatorname{Spa} \mathcal{O}_E$".

Given S, section of $\mathrm{Div}^d_Y(S)$,

get relative Cartier divisor $D_S \subset Y_S$.
(/S)

If $S = \mathrm{Spa}(R, R^+)$ affinoid, let

$$\mathbb{B}^+ = \text{completion of } \mathcal{O}(Y_S) \text{ along } \mathcal{I}(D_S)$$

Notation incompatible with [Fargues-Fontaine]

$$= W_{\mathcal{O}_E}(R^+)\left[\frac{1}{[\varpi]}\right]^{\wedge}_{\xi}, \text{ where } D_S = V(\xi).$$

$$\mathbb{B} = \mathbb{B}^+\left[\frac{1}{\xi}\right].$$

<u>Definition</u>. L^+G, LG / Div^d_Y:

$$S = \mathrm{Spa}(R, R^+) / \mathrm{Div}^d_Y \longmapsto G(\mathbb{B}^+) \text{ resp. } G(\mathbb{B}).$$

Beilinson-Drinfeld Grassm.

$$\mathrm{Gr}_{G, \mathrm{Div}^d_Y} = LG/L^+G,$$

local Hecke stack

$$\mathcal{H}ck_{G,\mathrm{Div}^d_{\mathcal{Y}}} = L^+G \backslash Gr_{G,\mathrm{Div}^d_{\mathcal{Y}}}$$

$$= L^+G \backslash LG / L^+G,$$

small v-stack over $\mathrm{Div}^d_{\mathcal{Y}}$.

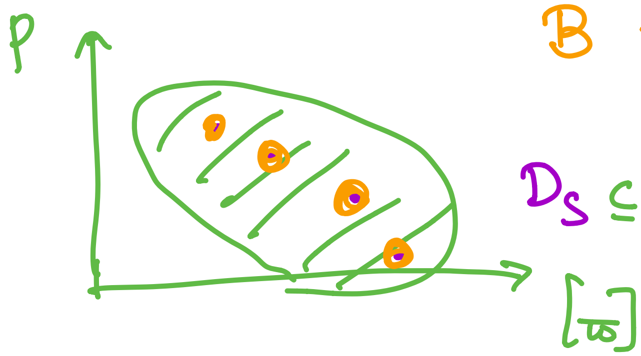
<u>Prop'n.</u> $Gr_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ param. G-torsors $\mathcal{E}$ over B^+ + trivialization over B;

equiv., param. G-torsors $\mathcal{E}$ over $\mathcal{Y}_S$ + merom. trivialization over $\mathcal{Y}_S \setminus D_S$.

Beauville-Laszlo

$\mathcal{H}ck_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ param. G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over B^+ + isom. over B; equiv., param. G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over $\mathcal{Y}_S$ + merom. isom.

over $\mathcal{Y}_S \setminus D_S$.



$B = \hat{\mathcal{O}}(\mathcal{Y}_S{}^{\wedge}_{D_S} \setminus D_S)$

$D_S \subseteq \mathcal{Y}_S$

For $S \longrightarrow \mathrm{Div}^d_{\mathcal{Y}}$ any small v-stack,

let $Gr_{G, S/\mathrm{Div}^d_{\mathcal{Y}}} := Gr_{G, \mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$.

<u>Schubert varieties</u>.

Assume $S = \mathrm{Spa}\, C$ geometric point.

$S \longrightarrow \mathrm{Div}^d_{\mathcal{Y}}$ corr. to a collection of

d untilts $C_1^\#, \dots, C_d^\#$ of C

If some agree, can remove them and

$Gr_{G, S/\mathrm{Div}^d_Y}$ does not change; so

assume distinct.

$\xi_1, \dots, \xi_d \in W_{O_E}(O_C)$

so that $O_{C_i^\#} = W_{O_E}(O_C)/(\xi_i)$.

$\xi = \xi_1 \cdots \xi_d$.

$T \subset B \subset G$

(G split).

<u>Prop'n.</u> $|Hck_{G, S/\mathrm{Div}^d_Y}| \xleftarrow{\sim} X_*^+(T)^d$

$\longleftarrow\!\shortmid \ (\mu_1, \dots, \mu_d)$

orbit of

$\mu_1(\xi_1) \cdots \mu_d(\xi_d) \in LG(S) = G(B)$

$$Hck_{G, S/\mathrm{Div}^d_Y} = \prod_{i=1}^{m} {}^{/S} \ Hck_{G, S/\mathrm{Div}^1_Y}$$

implicit map given by $S_i^\#$.

$\mathcal{H}ck_{G, S/Div^d_{\mathcal{Y}}, (\mu_1 \ldots \mu_d)}$, define L^+G- orbits

$$Gr_{G, S/Div^d_{\mathcal{Y}}, (\mu_1 \ldots \mu_d)} \subseteq Gr_{G, S/Div^d_{\mathcal{Y}}},$$

closure relations:

$$Gr_{G, S/Div^d_{\mathcal{Y}}, \leq (\mu_1 \ldots \mu_d)} := \overline{Gr_{G, S/Div^d_{\mathcal{Y}}, (\mu_1 \ldots \mu_d)}}$$

$$= \bigcup_{(\mu'_1, \ldots, \mu'_d) \leq (\mu_1, \ldots, \mu_d)} Gr_{G, S/Div^d_{\mathcal{Y}}, (\mu'_1 \ldots \mu'_d)}$$

in dominance order

quotients $\mathcal{H}ck_{G, S/Div^d_{\mathcal{Y}}, (\mu_1 \ldots \mu_d)}$,

$\leq$

Can also define this in families:

Over $S = (\mathrm{Div}^1_Y)^d \to \mathrm{Div}^d_Y$,

$\mu_1, \dots, \mu_d \in X_*^+$, can define

$$\mathrm{Gr}_{G, S/\mathrm{Div}^d_Y, (\leq)(\mu_1, \dots, \mu_d)} \subseteq \mathrm{Gr}_{G, S/\mathrm{Div}^d_Y}$$

by applying previous def'n fibrewise.

When points collide, need to add corresponding μ_i's.

<u>Thm</u>. $\mathrm{Gr}_{G, S/\mathrm{Div}^d_Y, \leq(\mu_1 \dots \mu_d)} \subseteq \mathrm{Gr}_{G, S/\mathrm{Div}^d_Y}$ closed, proper + repr. in spatial diamonds over S. (finite dim'l).

This is the main theorem of Berkeley course '2014.

<u>Remark</u>. No explicit pro-étale charts are known! OK if base change to

$$(\mathrm{Spa}\, E)^{\diamond, d} = \mathrm{Div}^d_Y.$$

$\rightsquigarrow$ Master Thesis of Bence Hevesi.

<u>Prop</u>. On open schubert cells, away from diagonals the L^+G-action is transitive.

<u>Cor</u>. The strata of $\mathrm{Hck}_{G,S/\mathrm{Div}^d_Y}$ are, away from diagonals, of form

$$[S / \underbrace{\text{some large group}}_{\text{ext. of fin. dim'l cohom. smooth}}].$$

group (like $G^\diamond$)

+ inf.-dim'l "unipotent" group.

(like $\ker(L^+G \to G^\diamond)$)

$\rightsquigarrow$ on level of D_{et}, all strata behave

like Artin v-stacks.

Similarly, the L^+G-action on each

$Gr_{G,S/Div^d_Y, \leq(\mu_1 \dots \mu_d)}$ factors over

a quotient $(L^+G)_{\leq(\mu_1 \dots \mu_d)} \leftarrow L^+G$.

fin.-dim'l when smooth

+ kernel is "unipotent".

$$\Rightarrow D_{ét}(Hck_{G,S/Div^d_Y, (\leq \mu_\bullet)}, \Lambda) \cong D_{ét}(\underbrace{(L^+G)_{\leq \mu_\bullet} \backslash Gr_{\dots}}_{\text{Artin v-stack.}}, \Lambda).$$

Def'n. $D_{et}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_Y}, \Lambda)^{bd}$ ← "bounded"

$$\begin{gathered} D_{et}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_Y}, \Lambda)^{bd} \\ \| \\ \bigcup_{\mu_\bullet} D_{et}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_Y, \le \mu_\bullet}, \Lambda) \subseteq D_{et}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_Y}, \Lambda). \end{gathered}$$

This is a monoidal category:

Def'n. (Convolution): For

$A, B \in D_{et}(\mathcal{H}ck, \Lambda)^{bd}$,

$$\begin{tikzcd} \mathcal{H}ck \times \mathcal{H}ck & L^+G \backslash LG \overset{L^+G}{\times} LG/L^+G \arrow[l, "\pi"'] \arrow[d, "m"] \\ & L^+G \backslash LG / L^+G = \mathcal{H}ck. \end{tikzcd}$$

$$A \star B := Rm_* \pi^*(A \boxtimes B).$$

Note: m is ind-proper, as fibres are affine Grassmannians, so proper base change ensures that this is associative.

Perverse Sheaves on $\mathcal{H}ck$

Def'n. Fix $S \to \mathrm{Div}^d_{\mathcal{Y}}$.

Let

$$ {}^{p}D^{\leq 0}_{\text{ét}}\left(\mathcal{H}ck_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda\right)^{bd} \subseteq D(\mathcal{H}ck)^{bd} $$

be the full subcategory of all

$$ A \in D(\mathcal{H}ck)^{bd} $$

s.th. for all geom. pts $\mathrm{Spa}(C,C^+) \to S$,

all $\mu_1, \dots, \mu_m \in X_*^+$ (m = # distinct untilts of $\mathrm{Spa}(C,C^+)$ corr. to $\mathrm{Spa}(C,C^+) \to S \to \mathrm{Div}^d_{\mathcal{Y}}$),

$$A\big|_{\mathcal{H}ck_{G,\,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}},\,(\mu_1\dots\mu_m)}} \in D^{\le -d(\mu_\bullet)}(\Lambda)$$

$$d(\mu_\bullet) = \sum_{i=1}^{m} \langle 2\rho, \mu_i\rangle.$$

This is the direct analogue of notion of "relative perversity".

${}^pD^{\ge 0}$ = right orth. to ${}^pD^{\le 0}[1]$.

<u>Thm</u>. This defines a t-structure. Moreover, $A \in {}^pD^{\ge 0}$ iff. for all geom pts. $\mathrm{Spa}(C, C^+) \to S$ as above,

$$A\big|_{\mathcal{H}ck_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}}}} \in {}^pD^{\ge 0}.$$

$\Updownarrow$

!-restriction to all Schubert cells lives

in $D^{\geq -d(\mu_\bullet)}$.

In particular, pullback under $S' \hookrightarrow S$ is

t-exact.

<u>Remark</u>. For schemes, in [Hansen-S.], analogous

result uses perversity of nearby cycles.

This $\wedge$ false in p-adic geometry!

+ Artin vanishing

<u>Example</u>. $\mathbb{A}^1_k \overset{i}{\subseteq} \widehat{\mathbb{A}}^1_{\mathcal{O}_C} \overset{j}{\supseteq} \mathbb{B}_C$

Artin vanishing would suggest

$$R\Gamma(\mathbb{B}_C, A) \in D^{\leq 1}.$$

But this fails for $A = j'_! \Lambda$, $j' : \mathbb{B}'_C \hookrightarrow \mathbb{B}_C$

$\{T, |T| \le 1/2\}$.

Then
$$R\Gamma(B_c, A)$$
$$= R\Gamma_c(B'_c, \Lambda) = \Lambda[-2].$$

Similarly, in this ex,
$$R\psi A = i^* Rj_* A = \text{skyscraper sheaf } \Lambda[-2] \text{ at origin.}$$

not perverse!

<u>Proof Sketch</u>. Use Hyperbolic Localization.

Thm is easy when $G = T$ torus.

Then $\mathrm{Gr}_{T, S/\mathrm{Div}^d_Y, \le(\mu_1, \dots, \mu_d)} \xrightarrow[\text{finite}]{} S$,

t-structure = usual t-structure.

Key Lemma.

$$\begin{tikzcd}
 & Gr_B \arrow[dl, "q"'] \arrow[dr, "p"] & & \mathbb{Z} \\
Gr_G & & Gr_T \arrow[ur, "\deg"] &
\end{tikzcd}$$

$$Gr_G \circlearrowleft \mathbb{G}_m \qquad \begin{tikzcd}
 & X^+ \arrow[dl] \arrow[r] & X^0 . \\
X & &
\end{tikzcd}$$

let $CT_B := Rp_! q^* [\deg]$:

$$D_{et}(Hck_G, \Lambda)^{bd} \longrightarrow D_{et}(Hck_T, \Lambda)^{bd}.$$

Then CT_B is <u>t-exact</u>.

+ <u>conservative</u>.

Can then deduce desired results on G from results on T.

<u>Sketch proof of Key Lemma</u>. Can reduce to case of geometric points. Then $D_{et}(Hck, \Lambda)^{bd}$ has stratif. in terms of $D_{et}(Hck_{(\mu_1 \dots \mu_d)}, \Lambda)^{bd}$.

generated by $D(\Lambda)$

So suffices to check

$$CT_B({}^pD^{\leq 0}) \subseteq {}^pD^{\leq 0}$$
$${}^pD^{\geq 0} \subseteq {}^pD^{\geq 0}$$

on standard objects

$$j_{\mu!}\Lambda[d_\mu], \qquad Rj_{\mu *}\Lambda[d_\mu].$$

These sheaves are ULA on $Gr_{G,S}/S$

+ hyperbolic localization preserves ULA'ness.

$\Rightarrow$ Cohom. locally constant, can reduce

to case of geom. point in char. p.

But for $S = (\mathrm{Spa}\,\mathbb{F}_q)^\diamond \to \mathrm{Div}^d_Y$,

$$Gr_{G, S/\mathrm{Div}^d_Y} = \left(Gr^{\mathrm{Witt}}_G\right)^\diamond.$$

Wit vector affine Grassm.
of Zhu.

+ six operations are
compatible for
Schemes vs. assoc. v-sheaves.

$\Rightarrow$ reduce to same statement for
$\mathrm{Gr}_{G}^{\mathrm{Witt}}$ [Zhu].

$\square$.