Geometric Satake

Thm (roughly) $\left(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbb{Z}_\ell), *\right)$
$\cong (\mathrm{Rep}\, \hat{G}, \otimes)$.

Two preparations: 1) Reminder on perverse sheaves.

2) Hyperbolic Localization.

Perverse Sheaves.

Usual setting: X separated scheme of fin. type over an alg. closed field k.

Λ ring killed by some n, $n \in k^\times$. (noeth.)

(or $\overline{\mathbb{Q}}_\ell$, –).

$D_{et}(X, \Lambda) = D(X_{et}, \Lambda)$, compactly generated, Compact objects $=: D^b_{c, ftd}(X_{ét}, \Lambda)$: bounded complexes, constr. cohomology sheaves, finite Tor dimension $/\Lambda$.

$$\cap\!|$$

$D^b_c(X_{ét}, \Lambda)$: bounded complexes with constr. cohom. sheaves.

<u>Definition</u>. 1) ${}^pD^{\leq 0}_{ét}(X,\Lambda) \subset D_{ét}(X,\Lambda)$ full subcategory of all

$$A \in D_{ét}(X,\Lambda)$$

s.th. for all geom. pts. $\bar{x} \longrightarrow X$,

$$A_{\bar{x}} \in D^{\leq -d(\bar{x})}(\Lambda)$$

where $d(\bar{x}) = \dim \overline{\{x\}} = \operatorname{trdeg} k(\bar{x})/k$.

2) ${}^pD^{\leq n} := {}^pD^{\leq 0}[-n]$.

3) ${}^pD^{\geq 0}$ right orth. of ${}^pD^{\leq -1}$, i.e.

$B \in {}^pD^{\geq 0} \iff$ for all $A \in {}^pD^{\leq -1}$,

$$\mathrm{Hom}(A, B) = 0.$$

4) ${}^p D^{\geq n} := {}^p D^{\geq 0}[-n].$

<u>Theorem</u>. 1) $({}^p D^{\geq 0}, {}^p D^{\leq 0})$ defines a <u>t-structure</u> on $D_{ét}(X, \Lambda)$

truncation

$\rightsquigarrow \exists$ functors ${}^p\tau^{\geq n}, {}^p\tau^{\leq n} : D_{ét}(X, \Lambda) \to {}^p D^{\geq n}, {}^p D^{\leq n}$.

left resp. right adjoint to inclusion, ?

<u>and</u> $\quad {}^p\tau^{\leq 0} A \longrightarrow A \to {}^p\tau^{\geq 1} A$

dist. triangle.

2) $A \in D_{ét}(X, \Lambda)$ lies in ${}^p D^{\geq 0}_{ét}(X, \Lambda)$

iff. $\forall\, \bar{x} \xrightarrow{i_{\bar{x}}} X$ geom. pts,

$$R i_{\bar{x}}^! A \in D^{\geq -d(\bar{x})}(\Lambda).$$

$$\bar{x} \xrightarrow{j_{\bar{x}}} \overline{\{x\}} \xhookrightarrow{i} X .$$
$$\overline{\{x\}} = Z$$

$$Ri_{\bar{x}}^! A := j_{\bar{x}}^* Ri^! A .$$

3) It induces a t-structure on

$D^b_c(X, \Lambda)$ (equiv., ${}^p\tau^{\geq 0}$, ${}^p\tau^{\leq 0}$ preserve this subcategory).

They do not preserve $D^b_{c,\mathrm{ftd}}(X,\Lambda)$: Already for $X = \operatorname{Spec} k$, truncation of perfect Λ-complexes need not be perfect.

OK if Λ regular.

<u>Definition</u>. $\mathrm{Perv}(X,\Lambda) := {}^pD^{\geq 0} \cap {}^pD^{\leq 0}$,

is an abelian category.

heart of t-structure.

Examples. 1) $i: \operatorname{Spec} k \hookrightarrow X$,

then $i_* \Lambda$ perverse.

2) X smooth, of equidim d, then

$\Lambda[d]$ perverse.

Thm $\operatorname{Perv}(X,\Lambda) \subset D^b_c(X,\Lambda)$ is an artinian category, if $\Lambda = \overline{\mathbb{F}}_\ell$, every object has finite length, irred. objects are in bijection with closed irr. subsets $Z \subseteq X$ + irred. repr. of the absolute Galois group of $k(Z)$ on $\overline{\mathbb{F}}_\ell$-v.s.

Sketch. Given $i: Z \hookrightarrow X$, such an irred. repr.,

get dense open $j: U \hookrightarrow Z$ +

irr. $\overline{\mathbb{F}}_\ell$-local system $\mathbb{L}$ on U. , U smooth.

$$j_! \mathbb{L}[d_Z] \in {}^p D^{\leq 0}(Z, \overline{\mathbb{F}}_\ell)$$

$$Rj_* \mathbb{L}[d_Z] \in {}^p D^{\geq 0}(Z, \overline{\mathbb{F}}_\ell)$$

$${}^p j_! \mathbb{L}[d_Z] = {}^p \tau^{\geq 0}(j_! \mathbb{L}[d_Z])$$ truncation to heart

$$\downarrow$$

$${}^p Rj_* \mathbb{L}[d_Z] = {}^p \tau^{\leq 0}(Rj_* \mathbb{L}[d_Z]).$$

image in $\mathrm{Perv}(Z, \overline{\mathbb{F}}_\ell)$ is by definition

$IC(Z, \mathbb{L})$ "intersection complex".

Then $i_* IC(Z, \mathbb{L}) \in \mathrm{Perv}(X, \overline{\mathbb{F}}_\ell)$.

These are the irred. objects.

Relative perversity

<u>Setting</u>. $f: X \longrightarrow S$ separated, of finite type,

S arbitrary.

<u>Goal</u>: Define notion of "perversity $/S$".

<u>Definition</u>. 1). ${}^{p/S}D^{\leq 0}(X, \Lambda) \subseteq D_{et}(X,\Lambda)$ full subcategory of all $A \in D_{et}(X,\Lambda)$ s.th. for all geom pts. $\bar{s} \to S$,

$$A|_{X_{\bar{s}}} \in {}^{p}D^{\leq 0}(X_{\bar{s}}, \Lambda).$$

equiv: for all geom. pts

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[d] & X \arrow[d] \\ \bar{s} \arrow[r] & S \end{tikzcd},$$

$$A_{\bar{x}} \in D^{\leq -d(\bar{x}/\bar{s})}(\Lambda).$$

2) ${}^{p/S}D^{\geq 0}(X,\Lambda)$ = right orthogonal of

$${}^{p/S}D^{\leq -1}.$$

<u>Theorem</u> (Hansen–S., upcoming). 1) This defines a t-structure on $D_{ét}(X,\Lambda)$.

2) $A \in D_{ét}(X,\Lambda)$ lies in ${}^{p/S}D^{\geq 0}(X,\Lambda)$ iff. for all $\bar{s} \to S$ geom. pt.,

$$A|_{X_{\bar{s}}} \in {}^{p}D^{\geq 0}(X_{\bar{s}}, \Lambda).$$

3) It induces a t-structure on

$$D^b_c(X,\Lambda).$$

<u>Cor.</u> (of 2)) Pullback under
$$\begin{array}{ccc} S' & \to & S \\ \uparrow & \square & \uparrow \\ X' & \to & X \end{array}$$
induces t-exact functors.

$${}^{p/S}D^{\leq 0}(X,\Lambda) \to {}^{p/S'}D^{\leq 0}(X',\Lambda).$$

≥ 0 ≥ 0

Cor. There is a notion of "family of perverse sheaves on X/S"

$$\mathrm{Perv}(X/S, \Lambda) := {}^{p/S}D^{\geq 0} \cap {}^{p/S}D^{\leq 0}.$$

Perverse Sheaves in p-adic geometry

Warning: I do not know how to define the "correct" dimension of a point of $\mathbb{B}^2_{\mathbb{C}_p}$.

Examples 1). $|\mathbb{B}^{ad}_{\mathbb{C}_p}|$:

Classical pts: dim 0.

All other rk 1 pts should be of dim 1.

What about rk 2 pts? Either 0 or 1, depending on perspective.

two choices are exchanged under Verdier duality.

Example 2. $\left|\mathbb{B}^2_{\mathbb{C}_p}\right|$ There is no classification of rk 1 pts, and "top. transcendence degree" has weird behaviour:

$$\exists \text{ tower } \quad \overbrace{\underbrace{\mathbb{C}_p \subseteq K_1}_{\text{top.tr.deg.1.}} \underbrace{\subseteq K_2}_{\text{top.tr.deg 1.}}}^{} .$$

$$\text{top. tr. deg. 1.}$$

Cf. Temkin, "Topological Transcendence Degree".

⇝ no hope for completely general theory

of perverse sheaves here.

But we only need relative one for

$$\mathcal{H}ck_G \longrightarrow \mathrm{Div}^1$$

$\rightsquigarrow$ only need to define dimensions of points of

$$\mathcal{H}ck_G \times_{\mathrm{Div}^1} \mathrm{Spd}\, C$$

But then, have Cartan stratification,

$$\mathcal{H}ck_G = L^+G \backslash \mathrm{Gr}_G,$$

$$\mathrm{Gr}_G = \bigcup \mathrm{Gr}_{G,\mu}$$

decomp. into L^+G-orbits,

$$\dim \mathrm{Gr}_{G,\mu} = \langle 2\rho, \mu \rangle.$$

(for any possible notion of dim.)

Hyperbolic Localization.

Usual setup: k alg closed field.

$$X/k \text{ proper scheme,} \quad \mathbb{G}_m \circlearrowright X.$$

$\rightsquigarrow$ fixed pts $X^0 = X^{\mathbb{G}_m} \underset{\text{closed}}{\subseteq} X$.

+ two stratifications

$$X = \bigcup_{i=1}^{m} X_i^+ \qquad X^0 = \bigsqcup_{i=1}^{m} X_i^0.$$

$$X = \bigcup_{i=1}^{m} X_i^-$$

loc. closed

open + closed

all $\mathbb{G}_m$-stable

$X^+ := \bigsqcup X_i^+$, $X^- := \bigsqcup X_i^-$.

s.th. $\mathbb{G}_m$-action extends to maps

$$\begin{array}{ccc} (\mathbb{A}^1)^+ \times X_i^+ & \longrightarrow & X_i^+ , \quad \text{contracting} \\ \cup\!\!\mid & & \cup\!\!\mid \\ 0 \times X_i^+ & \longrightarrow & X_i^0 \end{array}$$

$$\begin{array}{ccc} (\mathbb{A}^1)^- \times X_i^- & \longrightarrow & X_i^- \\ \cup\! \mid & & \cup\! \mid \\ 0 \times X_i^- & \longrightarrow & X_i^0 \end{array}$$

X_i^+ = locus where $\lim_{t \to 0} t \cdot x$ exists, and lies in X_i^0,

X_i^- = — " — $\lim_{t \to \infty} t \cdot x$ — " — " —

<u>Example.</u> $\mathbb{G}_m \circlearrowright X = \mathbb{P}^1$

$$X^0 = \{0, \infty\} = \underset{i=1}{\{0\}} \sqcup \underset{i=2}{\{\infty\}}.$$

$$\begin{array}{ccccc} X^+ = & \mathbb{A}^1 & \sqcup & \{\infty\}. \\ & \parallel & & \parallel \\ & X_1^+ & \sqcup & X_2^+ \\ X^- = & 0 & \sqcup & (\mathbb{A}^1)^- \\ & \parallel & & \parallel \\ & X_1^- & \sqcup & X_2^-. \end{array}$$

<u>Example</u>. $\mathbb{G}_m \circlearrowright \mathbb{P}^1 \times \mathbb{P}^1$

$t \cdot (a_1, a_2) = (t^{-1}a_1, ta_2)$.

"hyperbolic action"

<u>Goal of hyperbolic localization:</u>

Describe cohomology of $\mathbb{G}_m$-equiv. sheaves on X in terms of local information at $X^0 \subseteq X$.

<u>Thm</u>. $\exists$ functor

(Braden).

$$L : D_{\acute{e}t}(X/\mathbb{G}_m, \Lambda) \to D_{\acute{e}t}(X^0, \Lambda)$$

s.th. $R\Gamma(X, A) \cong R\Gamma(X^0, L(A))$.

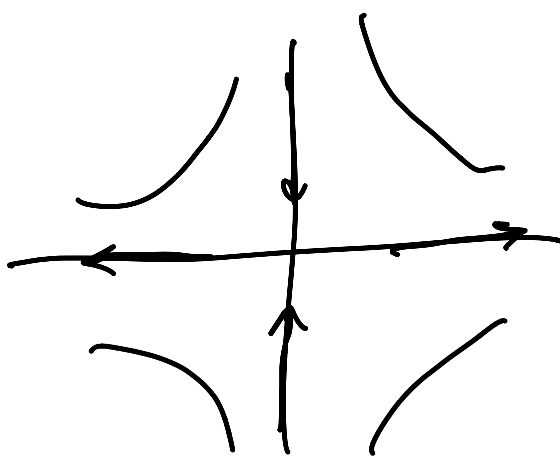
In fact, L admits four descriptions:

$$\begin{tikzcd}
X_i^0 \arrow[r, "i^-"'] \arrow[d, "i^+"] & X_i^- \arrow[l, bend right, "p^-"'] \arrow[d, "q^-"] \\
X_i^+ \arrow[u, bend left, "p^+"] \arrow[r, hook, "q^+"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
R(p^-)_! (q^-)^* & R(i^-)^! (q^-)^* \arrow[l, "\sim"'] & (i^+)^* R(q^+)^! \arrow[l, "\sim"'] \\
& & R(p^+)_* R(q^+)^! . \arrow[u, "2"']
\end{tikzcd}$$

<u>Example</u>. For $\mathbb{G}_m$-equiv. A on $\mathbb{A}^2$

$\mathbb{G}_m \circlearrowleft$ $t\cdot(a_1,a_2) = (t^{-1}a_1, t a_2)$



$$\begin{tikzcd}
0 \arrow[r, "i^-"] \arrow[d, "i^+"] & \mathbb{A}^1\times\{0\} \arrow[d, "q^-"] \\
0\times\mathbb{A}^1 \arrow[r, "q^+"'] & \mathbb{A}^2
\end{tikzcd}$$

$$\Rightarrow (i^+)^* R(q^+)^! A \longrightarrow R(i^-)^! (q^-)^* A.$$

<u>Example</u>. $X = \mathbb{P}^2 \circlearrowright \mathbb{G}_m$, $A = \Lambda$.

$$R\Gamma(\mathbb{P}^1, \Lambda) = \Lambda[0] \oplus \Lambda[-2].$$

$$L(A)_{\{0\}} = R\Gamma_c(\mathbb{A}^1, \Lambda) = \Lambda[-2]$$
$$= R\Gamma_{\{0\}}(\mathbb{A}^1, \Lambda) = \Lambda[-2]$$

$$L(A)_{\{\infty\}} = R\Gamma_c(\{0\}, \Lambda) = \Lambda[0]$$

$$R\Gamma(X^0, L(A)) = \Lambda[0] \oplus \Lambda[-2]. \quad \checkmark.$$

<u>Ex</u>. $X = \underset{\substack{\| \\ G/P}}{\text{flag variety}} \circlearrowright G \supseteq \underset{\text{dominant}}{\mathbb{G}_m}.$

$$X^{\mathbb{G}_m} = X^T = W/W_P$$

$$\rightsquigarrow \quad R\Gamma(X,\Lambda) = \bigoplus_{\substack{w \in W/W_P \\ \cong \\ W^P}} \Lambda[-2\ell(w)]$$

We will use this for $X = \mathrm{Gr}_{G, \leq \mu} \subseteq \mathrm{Gr}_G$
$\circlearrowleft$
$\mathbb{G}_m \underset{\text{dominant}}{\subseteq} L^+G$

to understand cohomology of L^+G-equiv. perv. sheaves on Gr_G.

Hyperbolic Localization for diamonds.

Setup. $f : X \to S$ proper map of small v-stacks repr. in spatial diamonds, dim. trg. $f < \infty$.

+ action of $\mathbb{G}_m$ on X/S (trivial on S).

$[\mathbb{G}_m(R, R^+) = R^\times]$

<u>Hypothesis</u>. Have $\mathbb{G}_m$-equivariant stratifications as above

$X = \bigcup X_i^+$, $\quad X = \bigcup X_i^-$, etc.

<u>Theorem</u>. In this situation, for all $A \in D_{ét}(X/\mathbb{G}_m, \Lambda)$, the maps

$$\begin{array}{ccccc} R(p^-)_! (q^-)^* A & \xleftarrow{\sim} & R(i^-)^! (q^-)^* A & \xleftarrow{\sim} & (i^+)^* R(q^+) A \\ & & & & \uparrow 2 \\ & & & & R(p^+)_* R(q^+)^! A \end{array}$$

are isomorphisms, defining "hyperbolic localization functor"

$L_{X/S} : D_{ét}(X/\mathbb{G}_m, \Lambda) \longrightarrow D_{ét}(X^0, \Lambda)$.

$L_{X/S}$ comm. w/ all (co)limits (in ∞-category land)

comm. w/ all base changes $S' \to S$.

...

$+$ $f : X \to S$
$\cup$ $\nearrow f^\circ$
X°

$$\boxed{Rf_* \cong Rf^\circ_* L_{X/S}}$$

<u>Sketch of proof</u> <u>Claim</u>: Everything follows from following geometric principle:

If $Y \circlearrowleft \mathbb{G}_m$, $[Y/\mathbb{G}_m]$ qcqs. $/S$. (say S = Spd $\mathbb{C}$)

Y loc. spatial diamond, partially proper $/S$ - dim.trg $< \infty$.

$\leadsto$ Y has two ends, and

for all $A \in D_{et}([Y/\mathbb{G}_m], \Lambda)$,

$$R\Gamma_{\partial\text{-}c}(Y, A) = 0.$$

comp. support at one end,

no support at other end.

$$\mathbb{G}_m \circlearrowright \mathbb{P}^1,$$

diff. between $R\Gamma_c(\mathbb{A}^1, A)$

& $R\Gamma_{\{0\}}(\mathbb{A}^1, A)$

is $R\Gamma_{\partial\text{-}c}(\mathbb{G}_m, A)$.

$$X = \mathbb{P}^1 \underset{j}{\supseteq} U = \mathbb{A}^1 \quad A = j_! \Lambda.$$

$$R\Gamma(\mathbb{P}^1, A) = R\Gamma_c(\mathbb{A}^1, \Lambda) = \Lambda[-2].$$

$$L(A)_{\{0\}} = \Lambda[-2]$$

$$L(A)_{\{\infty\}} = 0.$$

$$\begin{tikzcd}
X_\eta \arrow[r, hook, "j"] \arrow[d] & X \arrow[d] & X_s \arrow[l, "i"'] \arrow[d, equal] \\
\eta = \operatorname{Spec} C \arrow[r] & S = \operatorname{Spec} \mathcal{O}_C. & s = \operatorname{Spec} k \arrow[l]
\end{tikzcd}$$

C complete alg closed rk 1.

$$A \in {}^pD^{\geq 0}(X_\eta, \Lambda) = {}^{p/\eta}D^{\geq 0}(X_\eta, A)$$

$$\underset{\text{formal.}}{\Longrightarrow} \quad Rj_* A \in {}^{p/S}D^{\geq 0}(X, \Lambda)$$

$$\underset{\text{Thm}}{\Longrightarrow} \quad i^* Rj_* A \in {}^pD^{\geq 0}(X_s, \Lambda).$$