Geometric Satake

So far, for G/E as usual, have defined "Artin v-stack" Bun_G on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$D_{\acute{e}t}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z}) \qquad \ell \neq p.$$

Question (Drinfeld) Can one define

$$D(\mathrm{Bun}_G, \mathbb{Z}[1/p]) \text{ s.th.}$$

1)
$$D(\mathrm{Bun}_G, \mathbb{Z}[1/p]) \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}/\ell^n\mathbb{Z} \qquad ?$$
$$\|$$
$$D_{\acute{e}t}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z})$$

2) stratified into pieces

$$D(\mathrm{Bun}_G^b, \mathbb{Z}[1/p]) \cong D(G_b(E), \mathbb{Z}[\tfrac{1}{p}])$$

<u>Partial answer:</u> Such categories exist with

$\mathbb{Z}_\ell$- coefficients, in particular $\overline{\mathbb{Q}}_\ell$.

But unclear whether

$$D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_{\ell'})$$

for isom. $\iota: \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}$.

Can only work canonically with $\mathbb{Z}_\ell$- coeff.,

as category implicitly knows about

Take twist

$$\mathbb{Z}_\ell(1) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \overline{\mathbb{F}}_q^\times),$$

free $\mathbb{Z}_\ell$-module of rk 1.

would need $\mathbb{Z}[1/p]$-structure on $\mathbb{Z}_\ell(1)$'s.

<u>e.g.:</u> Choose an isomorphism

$$\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell, \quad \text{all } \ell \neq p.$$

with this choice, it seems that such a category ought to exist.

Related Fact: For any $\ell \neq p$, have canonical (on Langlands dual side) Artin stack Par_G over $\mathbb{Z}_\ell$ of L-parameters, cont. 1-cocycles.

$$W_E \longrightarrow \hat{G}(A) \qquad A/\mathbb{Z}_\ell$$

$$/\hat{G}\text{-conj.}$$

tame inertia is $\prod_{\ell \neq p} \mathbb{Z}_\ell(1)$, can map nontrivially to $\mathbb{Z}_\ell$-alg.

If fix top. gen. $\tau \in \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$, can form a partially discretized version

$$W_E^\tau \subseteq W_E \text{ of Weil group,}$$

replacing tame inertia by $\mathbb{Z}[1/p]\cdot\tau$.

Then $\{W_E^\tau \to \hat{G}\}/\hat{G}$ defines an Artin stack over $\mathbb{Z}[1/p]$, base changing to all canonical ones over $\mathbb{Z}_\ell$.

(Almost surely depends on τ.)

Reference: Dat - Helm - Kurinczuk - Moss, Zhu.

<u>Question</u> (Drinfeld) Can one make this explicit when $G = SL_2$?

<u>Key Problem:</u>

$$\mathrm{Bun}_G^{b_1} \xrightarrow{i_1} \mathrm{Bun}_G \xleftarrow{i_2} \mathrm{Bun}_G^{b_2}$$

What is $i_2^* R i_{1*} : D(G_{b_1}(E), \Lambda) \to D(G_{b_2}(E), \Lambda)$?

<u>Abstract answer:</u> (follows from last lecture).

$$\begin{tikzcd}
\tilde{\mathcal{M}}_{b_2} \arrow[r] & \mathrm{Bun}_G \\
\tilde{\mathcal{M}}^{b_1}_{b_2} \arrow[u, phantom, "\cup"] \arrow[r, "f"] & \mathrm{Bun}_G^{b_1} \arrow[u, phantom, "\cup"] & \backslash\, [\pi] \\
\tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2} \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd}$$

$\circlearrowleft$ $G_{b_2}(E)$ (acting on $\tilde{\mathcal{M}}^{b_1}_{b_2}$)

functor is given by

$$\pi \longmapsto R\Gamma\left(\tilde{\mathcal{M}}^{b_1}_{b_2}, f^*[\pi]\right).$$

$\circlearrowleft$ $G_{b_1}(E)$ $\qquad$ $\circlearrowleft$ $G_{b_2}(E)$ $\qquad$ (maybe up to shift)

<u>Example</u> $G = SL_2$: 1) $b_1 \hat{=} \mathcal{O}^2$ $\qquad$ $b_2 \hat{=} \mathcal{O}(-1) \oplus \mathcal{O}(1)$.

$\tilde{\mathcal{M}}^{b_1}_{b_2}$ param. inj. $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^2$ with cokernel $\mathcal{O}(1)$, i.e.

$\uparrow$ trivialize

$\tilde{\tilde{\mathcal{M}}}^{b_2}_{b_1}$ $\swarrow$ also the second bundle.

all saturated injections

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}^2.$$

Nonsaturated ones extend to maps

$$\mathcal{O} \hookrightarrow \mathcal{O}^2.$$

$$\Rightarrow \tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2} = \underbrace{\mathcal{BC}(\mathcal{O}(1))^2}_{\cong\ \mathbb{D}^2} \setminus \underbrace{GL_2(E) \times \mathcal{BC}(\mathcal{O}(1))}_{2}.$$

$\mathbb{P}^1(E)$ copies of $\mathbb{D}$, glued at 0

Need to compute

$$R\Gamma\Big(SL_2(E), \pi \otimes \underbrace{R\Gamma\big(\tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2}, \Lambda\big)}_{}\Big).$$

by excision, get 1, St, twists.

(+ Tate twists)

2) $b_1 \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$, $b_2 \cong \mathcal{O}(-2) \oplus \mathcal{O}(2)$.

$\tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2}$ param. saturated injections

$$\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

nonsaturated ones extend to

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1).$$

$$\tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2} = \mathcal{BC}(\mathcal{O}(1)) \times \mathcal{BC}(\mathcal{O}(3)) \setminus \text{image of}$$

$$(xz, yz) \leftarrow\!\shortmid (x, y, z) \qquad \underline{E} \times \mathcal{BC}(\mathcal{O}(2)) \times \mathcal{BC}(\mathcal{O}(1))$$

Can compute $R\Gamma(\tilde{\tilde{\mathcal{M}}}^{b_1}_{b_2}, \Lambda)$ by excision.

We want to extract L-parameters

$$\varphi : W_E \longrightarrow \hat{G} \qquad \text{1-cocycles.}$$

need to make dual group $\hat{G}$ appear.

<u>Idea</u>. This is "spectral information" arising as 'eigenvalues' of <u>Hecke operators</u> acting on $D_{et}(\mathrm{Bun}_G, \Lambda)$.

+ Hecke operators are enumerated by $\mathrm{Rep}\,\hat{G}$.

Hecke operators

(Hecke operators used to be related to elements of Hecke algebra

$$\Lambda[{}_K\backslash G(E)/K] \qquad K \subseteq G(E) \text{ compact open.}$$

The ones I'm talking about now are unrelated to those.

Definition. Let Hecke_G be the small v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$\mathrm{Hecke}_G(S) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, S^\#, f) \;\middle|\; \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_G(S), \\ S^\# \in \mathrm{Div}^1_X(S) \text{ untilt of } S \text{ over } E, \text{ up to Frob.}, \\ S^\# \hookrightarrow X_S \\ f: \mathcal{E}_1|_{X_S \setminus S^\#} \cong \mathcal{E}_2|_{X_S \setminus S^\#}, \\ \text{meromorphic at } S^\# \end{array} \right\}.$$

$\cup|$

$\mathrm{Hecke}_{G,\leq\mu}$

bound the modification.

$$\begin{tikzcd} & \mathrm{Hecke}_{G,\leq\mu} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Div}^1 \end{tikzcd}$$

proper, repr. in spatial diamonds

$\rightsquigarrow$ operators like

$$Rh_{2*}\, h_1^* : \mathcal{D}_{et}(\mathrm{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{et}(\mathrm{Bun}_G \times \mathrm{Div}^1, \Lambda)$$

$\mathcal{D}_{et}(\mathrm{Bun}_G \times \mathrm{Div}^1, \Lambda) \simeq \mathcal{D}_{et}(\mathrm{Bun}_G, \Lambda)^{W_E}$

$\hookleftarrow (\mathrm{Spa}\,\hat{\breve{E}})^\diamond / \underline{W_E}$

W_E-equiv. objects.

(use invariance of $\mathcal{D}_{et}(\mathrm{Bun}_G, \Lambda)$ under base change $(\mathrm{Spa}\,\hat{\breve{E}})^\diamond \to (\mathrm{Spa}\,\overline{\mathbb{F}}_q)^\diamond$)

Slightly better to allow kernels on Hecke_G.

<u>Thm</u>. $\exists$ canonical exact $\overset{\text{monoidal}}{\vee}$ functor

(1st incarnation of geom. Satake)

$$\mathrm{Rep}_\Lambda(\hat{G}) \longrightarrow D_{ét}(\mathrm{Hecke}_G, \Lambda)$$
$$V \longmapsto \mathcal{S}_V.$$

$\rightsquigarrow$ get

$$T_V : Rh_{2*}(h_1^* \otimes \mathcal{S}_V) : D_{ét}(\mathrm{Bun}_G, \Lambda)$$
$$\downarrow$$
$$D_{ét}(\mathrm{Bun}_G, \Lambda)^{W_E}.$$

Hecke operator.

monoidal $\Rightarrow T_W \circ T_V \cong T_{V \otimes W}$.

Statement of geometric Satake

(Mirković-Vilonen, Lusztig, Ginzburg, ...)

<u>Usual setup</u>:

$G / \mathbb{C}$ reductive group.

L^+G positive loop group. inf.-dim'l affine scheme

$\cap$ $L^+G(A) = G(A[[t]])$.

LG loop group ind-scheme.

$LG(A) = G(A((t)))$.

<u>Def'n</u>. Affine Grassmannian

$$\mathrm{Gr}_G = LG/L^+G$$

$L^+G \curvearrowright$

$$A \mapsto \left\{ \begin{array}{l} \mathcal{E} \ G\text{-torsor over } A[[t]], \\ \text{trivialized over } A((t)) \end{array} \right\}.$$

ind-scheme, transition maps closed immersions, each scheme is projective$/\mathbb{C}$.

<u>Def'n</u>. $\mathrm{Sat}_G = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbb{Z})$.

Satake category of L^+G-equiv. perv. sheaves.

$$\mu(t) \longleftarrow\!\shortmid \mu$$

$$L^+G_{(\mathbb{C})} \backslash LG(\mathbb{C}) / L^+G(\mathbb{C}) = X_*^+$$

dominant cocharacters

closure of L^+G-orbit of $\mu(t)$ is

a proj. scheme $Gr_{G, \leq \mu} \subseteq Gr_G$

(affine) Schubert variety.

In particular, for each μ, have

$$IC_\mu = IC_{Gr_{G, \leq\mu}} \in Sat_G.$$

$$\mathrm{im}\left({}^p j_{\mu !} \mathbb{Z}[d_\mu] \to {}^p j_{\mu *} \mathbb{Z}[d_\mu] \right) \qquad j_\mu : Gr_\mu \hookrightarrow Gr_{G, \leq \mu}$$

$$\Delta_\mu \qquad \nabla_\mu. \qquad Gr_\mu = L^+G \cdot \mu.$$

"standard & costandard objects"

$$d_\mu = \dim Gr_\mu = \langle 2\rho, \mu \rangle.$$

With $\mathbb{Q}$-coeff., $\quad {}^p j_{\mu!}\, \mathbb{Q}[d_\mu] \xrightarrow{\sim} {}^p j_{\mu *}\, \mathbb{Q}[d_\mu]$.

$$\begin{tikzcd} {}^p j_{\mu!}\, \mathbb{Q}[d_\mu] \arrow[r, "\sim"] \arrow[dr, "\sim"'] & {}^p j_{\mu *}\, \mathbb{Q}[d_\mu] \\ & IC_{\mu,\mathbb{Q}} \arrow[u, "\sim"'] \end{tikzcd},$$

but not with $\mathbb{Z}$-coeff.

$\rightsquigarrow$ Sat_G structure of "highest weight category"

with weights given by X_*^+,

simple objects $= \{IC_\mu\}$.

With $\mathbb{Q}$-coeff., semisimple.

<u>Definition</u>. Convolution monoidal structure on

Sat_G, $\qquad A \star B = Rm_* \, \pi^*(A \boxtimes B)$

$$
\begin{tikzcd}
L^+G\backslash LG/L^+G \times L^+G\backslash LG/L^+G \\
L^+G\backslash LG \overset{L^+G}{\times} LG/L^+G \arrow[u, "\pi"'] \arrow[d, "m"] \\
L^+G\backslash LG/L^+G.
\end{tikzcd}
$$

<u>Thm</u> (Mirković–Vilonen)

$(\mathrm{Sat}_G, \star) \xrightarrow{\bigoplus H^i(\mathrm{Gr}_G, -)} (\mathrm{Vect}, \otimes)$ is a fibre functor, $(\mathrm{Sat}_G, \star)$ can be upgraded to a symm. mon. structure making $\bigoplus H^i(\mathrm{Gr}_G, -)$ into a symm. mon. functor.

Corresponding Tannaka group is $\hat{G}$, so

$$(\mathrm{Sat}_G, \star) \cong (\mathrm{Rep}\,\hat{G}, \otimes)$$

$\backslash \bigoplus H^i(\mathrm{Gr}_G, -) \;/$

$$\swarrow \text{forget}$$

$$(\mathrm{Vect}, \otimes)$$

with $\mathbb{Q}$- coeff.,

$$IC_\mu \;\hat{=}\; V_\mu$$

with $\mathbb{F}_p$ - coeff.,

$$IC_\mu \;\hat{=}\; L_\mu$$ irr. repr.

$${}^p j_{\mu !}\, \mathbb{F}_p[d_\mu] \;\hat{=}\; \Delta_\mu$$

$${}^p j_{\mu *}\, \mathbb{F}_p[d_\mu] \;\hat{=}\; D_\mu$$

std & costd repr.

highest weight repr. of $\hat{G}$ of weight $\mu \in X_*^+ = X^*_+$ as $X_*(G) = X^*(\hat{G})$.

Want a version for B_{dR}^+ -affine Grassmannian:

<u>Definition</u>. G/E as usual., Div^1.

$$\mathrm{Gr}_G \longrightarrow \mathrm{Div}^1$$

param. $S^\# \in \mathrm{Div}^1(S)$

+ G-torsor $\mathcal{E}$ at completion of X_S at $S^\#$

+ trivialization on $(X_S)^\wedge_{S^\#} \setminus S^\#$.

What does this mean?

Only define it for $S = \mathrm{Spa}(R, R^+)$ affinoid.

Then $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ also affinoid,

$$\theta : W_{\mathcal{O}_E}(R^+)\left[\frac{1}{[\varpi]}\right] \longrightarrow R^\#$$

$\ker \theta = (\xi)$.

$B^+_{dR}(R^\#) = (\ker \theta)$-adic completion of $W_{\mathcal{O}_E}(R^+)\left[\frac{1}{[\varpi]}\right]$

(isom. to completing A at $R^\#$ for

$S^\# \subset \mathrm{Spa}(A, A^+) \subseteq X_S.)$

$$B_{dR}(R^\#) = B_{dR}^+(R^\#)\left[\frac{1}{\xi}\right].$$

analogue of $A[[t]]$ $\rightsquigarrow$ $B_{dR}^+(R^\#) \cong \mathcal{O}\left((X_S)^\wedge_{S^\#}\right)$

analogue of $A((t))$ $\rightsquigarrow$ $B_{dR}(R^\#) \cong \mathcal{O}\left((X_S)^\wedge_{S^\#} \setminus S^\#\right).$

$R^\# \cong A.$

$$Gr_G(S) = \left\{ \begin{array}{l} S^\#, \ G\text{-torsors on} \\ B_{dR}^+(R^\#), \text{ trivialized} \\ \text{over } B_{dR}(R^\#) \end{array} \right\}.$$

$$Gr_G = LG/L^+G, \qquad /\mathrm{Div}^1$$

$$\begin{array}{l} LG : \ R^\# \mapsto G(B_{dR}(R^\#)) \\ \cup\!\!\mid \\ L^+G : \ R^\# \mapsto G(B_{dR}^+(R^\#)) \end{array}$$

<u>Def'n</u>. local Hecke stack

$$\begin{array}{l} \mathcal{H}ck_G = L^+G \backslash Gr_G \\ \big\downarrow \qquad = L^+G \backslash LG / L^+G. \\ \mathrm{Div}^1 . \end{array}$$

We will define

$$\mathrm{Sat}_G(\Lambda) = \mathrm{Perv}^{ULA}(\mathcal{H}ck_G, \Lambda),$$

endowed with convolution monoidal structure,

$\mathrm{Rep}_{W_E}(\Lambda)$ -linear category.

via tensoring with pullback of local systems on Div^1.

("Geometric Satake").

<u>Theorem</u>. $(\mathrm{Sat}_G(\Lambda), \star)$ upgrades naturally

to a symm. monoidal category s.th.

$$\bigoplus H^i(\mathrm{Gr}_{G,\bar{\eta}}, -) : \mathrm{Sat}_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E}(\Lambda)$$

is a fibre functor, corresponding Tannaka dual is given by $\hat{G}$, with W_E-action that can be made explicit.

(a cyclotomic twist of usual action)

$$\Rightarrow \quad \begin{tikzcd} (\mathrm{Sat}_G(\Lambda), \star) \arrow[r, "\cong"] \arrow[d, "\text{cohom.}"'] & (\mathrm{Rep}\ \hat{G}, \otimes) \arrow[dl, "\text{forget}"] \\ (\mathrm{Rep}_{W_E}(\Lambda), \otimes) & \end{tikzcd}$$

internally in $\mathrm{Rep}_{W_E}(\Lambda)$.

<u>Corollary</u>. (Zhu's geometric Satake for Witt vector affine Grassmannian). Fix $G/\mathcal{O}_E$ reductive.

Zhu: $\overline{\mathbb{Q}}_\ell$, Yu: $\mathbb{Z}_\ell$.

$$\mathrm{Gr}_G^{\mathrm{Witt}} : \left(\text{Perfect } \overline{\mathbb{F}}_q\text{-alg. } A\right) \mapsto$$

$$\circlearrowright \quad \left(\begin{array}{l} G\text{-torsor on } W_{\mathcal{O}_E}(A), \\ \text{trivialized on } W_{\mathcal{O}_E}(A)[\tfrac{1}{\pi}] \end{array} \right)$$

$L^+_{\mathrm{Witt}} G : A \mapsto G(W_{\mathcal{O}_E}(A))$.

$$\left(\mathrm{Perv}_{L^+_{\mathrm{Witt}}G}\left(\mathrm{Gr}_G^{\mathrm{Witt}}\right), \star\right)$$
$$\cong$$
$$\left(\mathrm{Rep}\, \hat{G}, \otimes\right).$$

<u>Use:</u> There is a degeneration

$$\begin{tikzcd}
\left(\mathrm{Gr}_G^{\mathrm{Witt}}\right)^{\diamond} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_G^{\mathrm{BD}} \arrow[d] & \mathrm{Gr}_G \times_{\mathrm{Div}^1} \mathrm{Spa}\, \breve{E} \arrow[l] \arrow[d] \\
\mathrm{Spa}\, \overline{\mathbb{F}}_q \arrow[r] & \mathrm{Spa}\, \mathcal{O}_{\breve{E}} & \mathrm{Spa}\, \breve{E}. \arrow[l]
\end{tikzcd}$$

+ use nearby cycles/ formalism of ULA sheaves

to "specialize" perverse sheaves.

Comment: "Symmetric monoidal structure comes from fusion": Let two points on curve collide.

requires a space like

$$\mathrm{Spec}\, \mathbb{Q}_p \times \mathrm{Spec}\, \mathbb{Q}_p$$

when $E = \mathbb{Q}_p$.

But this exists in world of diamonds!

$\mathrm{Div}^1 \times \mathrm{Div}^1$ is a surface.