$D_{ét}(\mathrm{Bun}_G, \Lambda)$

G/E reductive group, residue field $\mathbb{F}_q$ of char $p>0$.

Λ ring of coeff., $n\Lambda = 0$ n prime to p.

Recall. $D_{ét}(\mathrm{Bun}_G, \Lambda)$ has infinite semi-orthogonal decomposition into

$$D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

§1 Compact Objects.

Recall. An object $A \in C$ triang. cat. is compact if $\mathrm{Hom}_C(A, -)$ commutes with all direct sums.

Fact. If C homotopy category of a stable

∞-category $\mathcal{C}$ with all colimits, then $A \in C$ compact iff $A \in \mathcal{C}$ has the property that $\mathrm{Hom}_{\mathcal{C}}(A, -)$ comm. w/ <u>all</u> colimits.

If $\mathcal{C}$ generated under colimits by its compact objects $\mathcal{C}^{\omega} \subseteq \mathcal{C}$, then

$$\mathrm{Ind}(\mathcal{C}^{\omega}) \xrightarrow{\sim} \mathcal{C} \quad \text{equiv. of } \infty\text{-cat's}$$

<u>Prop'n</u>. $D(G_b(E), \Lambda)$ is compactly generated

compact generators are

$$\text{c-Ind}_K^{G_b(E)} \Lambda \quad , \quad K \subseteq G_b(E) \text{ open pro-}p.$$

<u>Proof</u>. $$\mathrm{Hom}_{G_b(E)}(\text{c-Ind}_K^{G_b(E)} \Lambda, A) = \mathrm{Hom}_K(\Lambda, A)$$
$$= A^K$$

K pro-p $\Rightarrow$ taking K-inv. (= K-cohom.) commutes with all direct sums.

(If A repr. by

$$\cdots \to A_2 \to A_1 \to A_0 \to A_{-1} \to \cdots)$$

A_i smooth K-repr.

$$\left(\Rightarrow A^K \text{ repr. by } \cdots \to A_2^K \to A_1^K \to A_0^K \to A_{-1}^K \to \cdots \right)$$

so $\text{c-Ind}_K^{G_b(E)} \Lambda$ are compact.

If A s.th. $A^K = 0$ $\forall K$, then $A = 0$. $\Rightarrow$ generate. $\square$.

<u>Theorem</u>. $D_{\text{ét}}(\text{Bun}_G, \Lambda)$ is compactly generated. $A \in D_{\text{ét}}(\text{Bun}_G, \Lambda)$ is

compact iff $\forall b \in B(G)$, $\qquad i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$

$$i^{b*} A \in D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

is compact, and $= 0$ for almost all b.

<u>Proof</u>. First, exhibit compact generators.

Fix $b \in B(G)$, $K \subseteq G_b(E)$ open pro-p.

<u>Goal</u>. Show that $\exists\, A_K^b \in D_{ét}(\mathrm{Bun}_G, \Lambda)$

s.th. $R\mathrm{Hom}(A_K^b, B) = (i^{b*} B)^K$

for $B \in D_{ét}(\mathrm{Bun}_G, \Lambda)$.

If it exits, necessarily compact, and these objects generate.

to find A_K^b, use cohom. smooth chart

$$\begin{tikzcd} \pi_b : \mathcal{M}_b \arrow[r] & \mathrm{Bun}_G \\ \tilde{\mathcal{M}}_b \arrow[u, "G_b(E)\text{-torsor}"'] & \end{tikzcd}$$

$\rightsquigarrow \quad f_K : [\tilde{\mathcal{M}}_b / \underline{K}] \longrightarrow \mathrm{Bun}_G$. cohom. smooth.

<u>Claim</u>. $A_K^b = Rf_{K!} Rf_K^! \Lambda$ works.

<u>Proof of Claim</u>.

$$\mathrm{RHom}(A_K^b, B) = \mathrm{RHom}(Rf_{K!} Rf_K^! \Lambda, B)$$

$$\cong \mathrm{RHom}(Rf_K^! \Lambda, Rf_K^! B)$$

$$\overset{f_K \text{ ULA}}{\cong} \mathrm{RHom}(Rf_K^! \Lambda, f_K^* B \otimes Rf_K^! \Lambda)$$

$$\overset{Rf_K^! \Lambda \text{ inv.}}{\cong} R\Gamma([\tilde{\mathcal{M}}_b / K], f_K^* B)$$

$$\underset{\text{local}}{\overset{\tilde{M}_b\text{ strictly}}{\cong}} R\Gamma\left([*/\underline{K}],\ f_K^* B|_{[*/\underline{K}]}\right) \cong \left(i^{b*} B\right)^K.$$

(Claim). $\square$.

For characterization of compact objects, argue by induction on quasicompact open substacks $U \subset \mathrm{Bun}_G$.

Pick some $b \in B(G)$ s.th.

$$\begin{array}{ccl} i^b : \mathrm{Bun}_G^b & \hookrightarrow & U \quad \text{closed.} \\ & & \cup\, j \\ & & V = U \setminus \mathrm{Bun}_G^b, \end{array}$$

know the result for $D_{et}(V, \Lambda)$.

<u>enough:</u> j^* preserves compact objects.

Indeed, if so, then if A compact, also $j^* A$ is compact, and hence $i^{b*} A$

$j^* A \Rightarrow$ $i^{b'*} A$ compact $\forall b' \neq b$ induction
compact

$+$ $i^{b*} A$ compact.

Similarly for converse.

<u>Claim 2</u>. $j^* A_K^b \in D_{\acute{e}t}(V, \Lambda)$ is compact.

<u>Proof of Claim 2</u>.

$$\begin{array}{ccc} f_K : [\widetilde{\mathcal{M}}_b / K] & \longrightarrow & \mathcal{U} \subseteq \mathrm{Bun}_G \\ \cup & & \cup j \\ f_K^\circ : [\widetilde{\mathcal{M}}_b^\circ / \underline{K}] & \longrightarrow & V \end{array}$$

$\widetilde{\mathcal{M}}_b^\circ = \widetilde{\mathcal{M}}_b \setminus *$ spatial diamond of finite dim. trg.

$j^* A_K^b = Rf_{K!}^\circ Rf_K^{\circ !} \Lambda$ by formula for A_K^b.

By similar computation,

$$R\mathrm{Hom}(j^* A_K^b, B) \cong R\Gamma([\tilde{\mathcal{M}}_b^\circ / \underline{K}], f_K^{\circ *} B).$$

$B \in D_{et}(V, \Lambda)$ $\cong R\Gamma(\tilde{\mathcal{M}}_b^\circ, \text{pullback of } B)^K$

But $R\Gamma(\tilde{\mathcal{M}}_b^\circ, -)$ commutes with all direct sums

as $\tilde{\mathcal{M}}_b^\circ$ is a spatial (qcqs!) diamond (of finite dim trg.)

(Claim 2) $\square$.

(Thm) $\square$

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§2 Bernstein-Zelevinsky duality.

A duality on compact objects.

<u>Prop'n</u>. For any $A \in D(G_b(E), \Lambda)^\omega$,

$\exists!$ $A' \in D(G_b(E), \Lambda)^\omega$ s.th.

$$\mathrm{RHom}(A', B) = (A \otimes B)_{G_b(E)}$$

($\overset{?}{\text{derived}}$ homology: left adj. of $D(\Lambda) \to D(G_b(E), \Lambda)$)

for $A = \text{c-Ind}_K^{G_b(E)} \Lambda$, get

$A' = \text{c-Ind}_K^{G_b(E)} \Lambda$; in general

$$A' = \mathrm{RHom}_{G_b(E)}(A, \mathcal{H}(G_b(E))) .$$

Hecke alg. of compactly supp. locally constant functions on $G_b(E)$

The biduality map

$A'' = (A')' \longrightarrow A$ is an isomorphism.

<u>Proof.</u> Yoneda: A' unique if it exists.

existence: enough to take $A = \text{c-Ind}_K^{G_b(E)} \Lambda$.

$$(A \otimes B)_{G_b(E)} = B_K \underset{\uparrow}{\cong} B^K = \mathrm{RHom}(\text{c-Ind}_K^{G_b(E)} \Lambda, B)$$

average map $\square$.

<u>Theorem.</u> For any $A \in D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$,

$\exists ! A' = \mathbb{D}_{BZ}(A) \in D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$ s.th.

$$R\mathrm{Hom}(\mathbb{D}_{BZ}(A), B) = \pi_{\natural}(A \overset{L}{\underset{\Lambda}{\otimes}} B)$$

where $\pi_\natural : D_{ét}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$
$\quad\ \ \ \, \| $
$R\pi_!(- \otimes R\pi^! \Lambda)$ left adj. to π^*, $\pi : \mathrm{Bun}_G \to *$.

+ biduality map $\mathbb{D}_{BZ}(\mathbb{D}_{BZ}(A)) \to A$ is an isomorphism.

If $U \subset \mathrm{Bun}_G$ open substack, then $\mathbb{D}_{BZ}$ respects $D_{ét}(U, \Lambda)^\omega \subseteq D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$.

For $U = \mathrm{Bun}_G^b$, b basic, reduce to usual BZ-duality on $D_{ét}(\mathrm{Bun}_G^b, \Lambda)^\omega$

$$D_{ét}(\mathrm{Bun}_G^b, \Lambda)^\omega \cong D(G_b(E), \Lambda)^\omega.$$

Proof. Check existence for a class of generators.

Take $i^b_! [c\text{-Ind}_K^{G_b(E)} \Lambda]$, $i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$.

Claim $\mathbb{D}_{BZ}\left(i^b_! [c\text{-Ind}_K^{G_b(E)} \Lambda]\right) = A^b_K$.

Proof of Claim.

$$R\mathrm{Hom}(A^b_K, B) = (i^{b*} B)^K = \pi_{\natural}\left(i^b_! [c\text{-Ind}_K^{G_b(E)} \Lambda] \otimes B\right).$$

(up to shifts maybe) □ (Claim).

to prove biduality, need to compute

$$\mathbb{D}_{BZ}(A^b_K) \overset{?}{=} i^b_! [c\text{-Ind}_K^{G_b(E)} \Lambda].$$

OK on Bun_G^b, need to check that after pullback to complement; LHS $= 0$.

$j : U \hookrightarrow \mathrm{Bun}_G$ open subset (= proper generizations of b.)

to see: $\forall\ B \in D_{et}(U,\Lambda)$

$$R\mathrm{Hom}\left(\mathbb{D}_{Bz}(A^b_{\underline{K}}),\ Rj_* B\right) \overset{!}{=} 0.$$

$\parallel$ def'n

$$\pi_{\natural}\left(A^b_{\underline{K}} \otimes Rj_* B\right).$$

$\cong$ formula for $A^b_{\underline{K}}$

$$R\Gamma_c\left([\tilde{\mathcal{M}}_b/\underline{K}],\ \widehat{Rj_* B}\right) = \text{// cohom. of } [\tilde{\mathcal{M}}^{\circ}_b/\underline{K}]$$

pullback of.

with compact supp. towards boundary of $[\tilde{\mathcal{M}}_b/\underline{K}]$, no supp. condition near $[*/\underline{K}] \subseteq [\hat{\mathcal{M}}_b/\underline{K}]$.

$= 0$ by "partial compact support vanishing from last time".

(Thm.) $\square$

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§3 Verdier Duality.

$$A \mapsto R\mathcal{H}om(A, R\pi^! \Lambda) = \mathbb{D}(A)$$

contravariant endofunctor on $D_{et}(\mathrm{Bun}_G, \Lambda)$.

Verdier Duality

On $D_{et}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ is just

smooth duality (up to shift + twist).

Dualizing complex $\cong$ Haar measures.

Thm. For any open immersion $j : U \hookrightarrow V$

of open substacks of Bun_G, $A \in D_{et}(U, \Lambda)$,

$$Rj_* R\mathcal{H}om(A, \mathbb{D}_U) \cong R\mathcal{H}om(j_! A, \mathbb{D}_V) \quad (1)$$

and $$j_! R\mathcal{H}om(A, \mathbb{D}_U) \cong R\mathcal{H}om(Rj_* A, \mathbb{D}_V) \quad (2)$$

Cor. $A \in D_{et}(\mathrm{Bun}_G, \Lambda)$ reflexive i.e.

$A \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(A))$, iff.

$\forall\ b \in B(G)$, $i^{b*} A \in D_{et}(\mathrm{Bun}_G, \Lambda)$

is reflexive, i.e. $(i^{b*}A)^K \in D(\Lambda)$ reflexive for all $K \subseteq G_b(E)$ open pro-p.

Thm $\Rightarrow$ i^{b*} commutes with $\mathbb{D}(\mathbb{D}(-))$.

<u>Proof of Thm.</u> Can assume by induction

$$U = V \setminus \mathrm{Bun}_G^b, \quad b \in B(G).$$

(1) clear.

(2) Clear after j^*, so enough to show it's an isom after $R\mathrm{Hom}(A_K^b, -)$.

As $R\mathrm{Hom}(A_K^b, B) = (i^{b*}B)^K$, the LHS vanishes.

$$\begin{aligned}
\text{RHS} = \ & R\mathrm{Hom}(A_K^b, R\mathcal{H}om(Rj_* A, \Lambda)) \\
&\cong \\
& R\mathrm{Hom}(A_K^b \overset{L}{\underset{\Lambda}{\otimes}} Rj_* A, \pi^* \Lambda) \\
&\cong \\
& R\mathrm{Hom}(\pi_\natural(A_K^b \overset{L}{\underset{\Lambda}{\otimes}} Rj_* A), \Lambda)
\end{aligned}$$

(up to twist)

enough: $\pi_{\natural}\left(A_K^b \overset{L}{\underset{\Lambda}{\otimes}} Rj_* A\right) \overset{!}{=} 0.$

$$\mathbb{D}_{\mathrm{BZ}}(A) = i_!^b\left[c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda\right]$$

$$RHom\left(i_!^b\left[c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda\right], Rj_*A\right)$$

$$\| \quad \leftarrow \quad j^* i_!^b(\ldots) = 0. \quad (\mathrm{Thm}) \ \square.$$

$$0,$$

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§4 ULA sheaves.

Bun_G Artin v-stack $\Rightarrow$ notion of ULA complexes

for $\pi : \mathrm{Bun}_G \to *$.

(being ULA is cohom. smooth local on source).

Prop'n. (consequence of "dualizability" characterization of being ULA).

$A \in D_{\mathrm{\acute{e}t}}(\mathrm{Bun}_G, \Lambda)$ is ULA iff

$$p_1^* RHom(A,\Lambda) \overset{L}{\underset{\Lambda}{\otimes}} p_2^* A \xrightarrow{\sim} RHom(p_1^* A, p_2^* A).$$

$p_1, p_2 : \mathrm{Bun}_G \times \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G.$

<u>Thm.</u> $A \in D_{et}(\mathrm{Bun}_G, \Lambda)$ is ULA iff

$\forall\, b \in B(G)$, $K \subseteq G_b(E)$ open pro-p,

$(i^{b*} A)^K \in D(\Lambda)$ perfect complex.

<u>Proof of Thm</u>.

<u>Lemma</u>. Exterior $\otimes$-product

$$- \boxtimes - : D_{et}(\mathrm{Bun}_G, \Lambda) \otimes_{D(\Lambda)} D_{et}(\mathrm{Bun}_G, \Lambda) \downarrow D_{et}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$$

is an equiv. of ∞-categories.

i.e.: for $A_1, A_2 \in D_{et}(\mathrm{Bun}_G, \Lambda)^\omega$,

also $A_1 \boxtimes A_2 \in D_{et}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$ is compact,

these form compact generators, and

$\forall B_1, B_2 \in D_{ct}(\mathrm{Bun}_G, \Lambda)$

$$\begin{array}{c} RHom(A_1 \boxtimes A_2, B_1 \boxtimes B_2) \\ \uparrow \wr \\ RHom(A_1, B_1) \overset{L}{\underset{\Lambda}{\otimes}} RHom(A_2, B_2). \end{array}$$

Proof of Lemma : use compact generators A_k^b. (Lemma 1)

Proof of Thm. need to figure out when

$$p_1^* RHom(A, \Lambda) \otimes p_2^* A \xrightarrow{\sim} RHom(p_1^* A, p_2^* A).$$

Apply $RHom(A_1 \boxtimes A_2, -)$ for compact A_i,

get:

$$\begin{array}{c} RHom(\pi_{H}(A_1 \overset{L}{\underset{\Lambda}{\otimes}} A), \Lambda) \overset{L}{\underset{\Lambda}{\otimes}} RHom(A_2, A) \\ ? \downarrow \wr \\ RHom(\pi_{H}(A_1 \overset{L}{\underset{\Lambda}{\otimes}} A), RHom(A_2, A)). \end{array}$$

satisfied iff $\pi_{\natural}\left(A_1 \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} A\right) \in D(\Lambda)$ perfect.

Use $A_1 = i_!^b\left[c\text{-Ind}_K^{G_b(E)} \Lambda\right]$ to see that this translates to $\left(i^{b*} A\right)^K \in D(\Lambda)$ perfect. (Thm) □.

<u>$G = SL_2$</u>: $D_{et}(\mathrm{Bun}_G, \Lambda) \cong \mathrm{Ind}\,\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\hat{G}})$.

$$\cdots < \underset{\mathcal{O}(-2)\oplus\mathcal{O}(2)}{\bullet} < \underset{\mathcal{O}(-1)\oplus\mathcal{O}(1)}{\bullet} < \underset{\mathcal{O}^2}{\bullet}$$

$$\cdots \quad E^* \quad E^* \quad SL_2(E)$$

$$H^1(\mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty})), \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell.$$

fails with $\mathbb{Z}[\tfrac{1}{p}]$-coefficients.

$$[*/H] \xrightarrow{\pi} *$$

H locally prop.

$(R\pi^! \Lambda)([*/H']) = \Lambda$- valued Haar measures on H'.

$H' \subseteq H$ open

$$\begin{tikzcd}
D_{et}(\mathrm{Bun}_G, \Lambda) \arrow[r, "T_V"] & D_{et}(\mathrm{Bun}_G, \Lambda) \arrow[d] \\
D(G(E), \Lambda) \arrow[u, hook] \arrow[r, dashed] & D_{et}(G_b(E), \Lambda)
\end{tikzcd}$$

b basic.

ess. Jacquet-Langlands.