The Fargues-Fontaine curve, I.

On Friday: Lecture starts at 16:30. $(+\varepsilon)$.

Fix some nonarchimedean local field E, residue field $\mathbb{F}_q$ of char. p, $\pi \in \mathcal{O}_E$ uniformizer.

- $E \cong \mathbb{F}_q((t))$ or
- $[E : \mathbb{Q}_p] < \infty$.

Goal. "Make Spec E geometric".

Note: $\cdot$ $(\operatorname{Spec} E)_{\text{Zar}} = *$.

· $(\operatorname{Spec} E)_{ét} = \left\{ \begin{array}{c} \text{finite separable} \\ E\text{-algebras} \end{array} \right\}^{op}$

$= B\,\mathrm{Gal}(\bar{E}/E) = \left\{ \begin{array}{c} \text{finite sets w/} \\ \text{cont. } \mathrm{Gal}(\bar{E}/E) \\ \text{-action} \end{array} \right\}$.

inertia.

$$0 \to I_E \to \mathrm{Gal}(\bar{E}/E) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

$\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} \ni \text{Frob}$, $\text{Frob} = \text{Frob}_q$.

wild inertia, pro-p.

$$0 \to P_E \to I_E \to \prod_{\ell \neq p} \mathbb{Z}_\ell \to 0.$$

$\prod_{\ell \neq p} \mathbb{Z}_\ell = \hat{\mathbb{Z}}^p$

<u>Local Tate Duality</u>. for all torsion $\mathrm{Gal}(\bar{E}/E)$-repr. M (prime to p if $E \cong \mathbb{F}_q((t))$)

the pairing

$$H^i_{et}(\mathrm{Spec}\, E, M) \otimes H^{2-i}_{et}(\mathrm{Spec}\, E, M^*(1))$$

$$\downarrow \cup$$

$$H^2_{et}(\mathrm{Spec}\, E, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

perfect pairing. away from p if $E/\mathbb{F}_q$.

Here $M^* = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

$\mathbb{Q}/\mathbb{Z}(1) =$ Tate twist

$= \bigcup_n \mu_n$.

This looks like Poincaré duality on a <u>compact Riemann surface</u>

want to turn Spec E into something closer to a

Compact Riemann surface.

<u>Example</u>. $E = \mathbb{F}_q((t))$.

Let $\breve{E} = \widehat{\overline{\mathbb{F}}_q}((t)) \circlearrowleft \phi_{\overline{\mathbb{F}}_q}$

Spec $\widehat{\overline{\mathbb{F}}_q}((t))$ "formal punctured open unit disc $/\overline{\mathbb{F}}_q$".

Make more space: Fix $C / \overline{\mathbb{F}}_q$ complete alg. closed nonarchimedean field.
(e.g. $C = \widehat{\overline{\mathbb{F}_q((u))}}$).

Consider

$$\mathrm{Spa}\, C \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q((t)) = \mathbb{D}^*_C \circlearrowleft \phi_C.$$

Spa = adic spectrum

punctured open unit disc
$\{x \mid 0 < |x| < 1\}$ over C

<u>Def'n</u>. The <u>Fargues - Fontaine curve</u> (for E, C) is

$$X_{C,E} := \mathbb{D}^*_C / \varphi_C^{\mathbb{Z}}.$$

an adic space over E.

<u>Thm</u> (FF).

- $H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}) = E$.
- $\mathrm{FEt}(X_{C,E}) = (\mathrm{Spec}\, E)_{\acute{e}t}$.
- $H^i_{\acute{e}t}(X_{C,E}, M) = H^i_{\acute{e}t}(\mathrm{Spec}\, E, M)$.

<u>Some recollections on adic spaces</u>.

Roughly, adic spaces are variants of schemes associated to <u>certain topological rings</u> (eg Banach algebras).

<u>Defn</u>. Let A topological ring.

1). A is <u>adic</u> if there is some ideal $I \subseteq A$ s.th $\{I^n \mid n \geq 0\}$ is a nbhd basis of 0.

Such an I is called <u>ideal of definition</u>.

[Not unique, but for any two I, J, $\exists n$: $I^n \subseteq J$, $J^n \subseteq I$]

2). A is <u>Huber</u> (= f-adic in Huber's papers) if $\exists$ open subring $A_0 \subseteq A$ that is adic with f.g. ideal of def'n.

($A_0 \subseteq A$ "ring of def'n").

Remark. Any such A admits completion $\hat{A}$, contains $\hat{A}_0 \subset \hat{A}$ as open subring, where $\hat{A}_0 = I$-adic completion of A_0.

Most important case:

Defn. A is **Tate** if it contains a topologically nilpotent unit $\varpi \in A$

"pseudouniformizers".

ex. $\mathbb{F}_p((t)) \ni t$ $\quad \mathbb{Q}_p \ni p \quad E \ni \pi$.

any nonarchimedean field, any Huber ring over a nonarch field.

<u>Remark</u>. If K any nonarch field, $\varpi \in K$ pseudouniformizer,

A/K complete Huber ring, then A has natural structure as Banach algebra over K, with

$\{ f \in A \mid \|f\| \leq 1 \} = A_0$.

"unit ball"

Any $A_0 \subseteq A$ ring of defin has ϖ-adic topology.

$$\|\cdot\| : A \longrightarrow \mathbb{R}_{\geq 0}$$

$$a \longmapsto \inf_{\{n \mid \varpi^n a \in A_0\}} 2^n .$$

$$\left\{\begin{matrix}\text{Banach alg}/K \\ \text{w/ cont. maps}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Tate-Huber} \\ \text{rings}/K\end{matrix}\right\}.$$

<u>Def'n</u>. The <u>valuation spectrum</u> of Huber ring A is

$$\mathrm{Cont}\, A := \left\{ \underset{\text{cont. valuation}}{|\cdot| : A \to \Gamma \cup \{0\}} \right\} \Big/ \text{eq.}$$

with topology generated by opens

$$\{ |f| \leq |g| \neq 0 \} \subseteq \mathrm{Cont}\, A$$

for $f, g \in A$.

Here: cont. valuation:

- Γ totally ordered ab. groups (e.g. $\mathbb{R}_{>0}$) or $\boxed{\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}}$ where $1 > \gamma > 1$ for all

$r \in \mathbb{R}_{>0}, r > 1$

- $|\cdot| : A \longrightarrow \Gamma \cup \{0\}$

satisfies $|ab| = |a||b|$ $\quad (0\gamma = \gamma 0 = 0)$

$|a+b| \leq \max(|a|, |b|)$ $\quad (0 \leq \gamma \; \forall \gamma)$

$|0| = 0$

$|1| = 1.$

$\forall \gamma \in \Gamma, \; \{a \mid |a| < \gamma\} \subset A$ open.

Two cont. valuations $|\cdot|_1, |\cdot|_2$ are equivalent if

$$|a|_1 \geq |b|_1 \iff |a|_2 \geq |b|_2.$$

If Γ_i are chosen minimal, then

equiv. to $\exists\, \Gamma_1 \cong \Gamma_2$ s.th.

$$\begin{array}{ccc} & \overset{|\cdot|_1}{\nearrow} & \Gamma_1 \cup \{0\} \\ A & \circlearrowleft & \wr\!\wr \\ & \underset{|\cdot|_2}{\searrow} & \Gamma_2 \cup \{0\} \end{array}$$

<u>**Def'n.**</u> A <u>Huber pair</u> is a pair

(A, A^+) where A is a Huber ring,

$A^+ \subseteq A$ open integrally closed subring of powerbounded elements

$$\left(A^\circ \subseteq A \text{ subring of powerbdd elements} \right)$$

$$A^\circ = \bigcup_{A_0 \subseteq A} A_0$$

<u>**Def'n.**</u> 1) $\mathrm{Spa}(A, A^+) = \{ |\cdot| \mid |A^+| \leq 1 \}$.

$$\mathrm{Spa}(A, A^+) \subseteq \mathrm{Cont}\, A$$

2) $\mathrm{Spa}\, A := \mathrm{Spa}(A, A^\circ)$.

Can endow $\mathrm{Spec}\, A$ with presheaf $\mathcal{O}_{\mathrm{Spec}\, A}$ of Huber rings on

$\cup$ basis of <u>rational subsets</u>

$\mathcal{O}^+_{\operatorname{Spec} A}$

basis of $\operatorname{Spa}(A, A^+)$, given by.

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{ |f_i| \leq |g| \neq 0 \}$$

where $(f_1, \dots, f_n, g)$ generate an open ideal.

$\mathcal{O}_{\operatorname{Spa}(A,A^+)}\left(U\left(\frac{f_1, \dots, f_n}{g}\right)\right) \supseteq \mathcal{O}^+$: minimal choice that contains A^+, all $\frac{f_i}{g}$.

$$\| \quad A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle.$$

"allow conv. series in $\frac{f_i}{g}$'s".

Thm. (Huber, ...) In "all practical cases" $\mathcal{O}_{\operatorname{Spa}(A,A^+)}$ is a sheaf.

But not always!

<u>Remark</u>. Bambozzi–Kremnizer, Clausen–S: Non-sheafyness can be corrected by allowing "derived Huber rings".

Not relevant for this course!

<u>Defn</u>. An <u>adic space</u> is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$.

- X: top. space
- $\mathcal{O}_X$: sheaf of complete top. rings
- $\mathcal{O}_X^+$: subsheaf of $\mathcal{O}_X$

that is locally of form

$$(\mathrm{Spa}\,(A, A^+), \mathcal{O}, \mathcal{O}^+).$$

(formulate correctly analogue of "locally ringed".) C nonarch. field.

<u>Example</u>. $\mathbb{D}_C^* =$ "$\{x \,/\, 0 < |x| < 1\}$".

$\cap$ open

$\mathbb{B}_C$

Tate algebra

$$C\langle T\rangle = \left\{ \sum_{n\geq 0} a_n T^n \,\middle|\, a_n \in C,\ a_n \to 0 \right\}.$$

For all $x \in C$, $|x| \leq 1$,

get map $C\langle T\rangle \longrightarrow C.$

$$\sum a_n T^n \longmapsto \sum a_n x^n.$$

So $\operatorname{Spa} C\langle T\rangle =: \mathbb{B}_C =$ "$\{x \,/\, 0 \leq |x| \leq 1\}$".

"closed unit disc".

$\cup$|

$\mathbb{B}_C^* = \mathbb{B}_C \setminus \{0\}.$ $\{|a| \leq |T| \neq 0\}$

$$\begin{array}{c} \| \\ \bigcup_{\varepsilon>0} \underbrace{A(\varepsilon,1)} \end{array} \subseteq \mathbb{B}_C$$

rational subset

annulus $a \in C$, $|a| = \varepsilon$.

$\{x \mid \varepsilon \le |x| \le 1\}$.

$\varepsilon \in |C|$.

$$A(\varepsilon,1) = \mathrm{Spa}\ A_\varepsilon$$

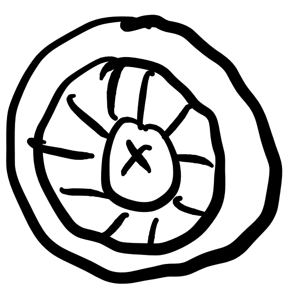
$$A_\varepsilon = \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \;\middle|\; a_n \in C,\ |a_n| \xrightarrow{n\to\infty} 0,\ \varepsilon^n |a_n| \xrightarrow{n\to-\infty} 0 \right\}$$

Similarly

$$\mathbb{D}^*_C = \bigcup_{\substack{\varepsilon>0 \\ r<1}} A(\varepsilon,r).$$

$$A(\varepsilon,r) = \{\varepsilon \le |T| \le r\}.$$

($|T| \neq 0$)



<u>Beware:</u> $\mathbb{D}^*_C \subseteq \mathbb{B}_C$ open, but not equal to

$$\{0 < |T| < 1\} \quad !$$

<u>Problem</u>. There is one point

$$x \in \mathbb{B}_C = \operatorname{Spa} C\langle T\rangle$$

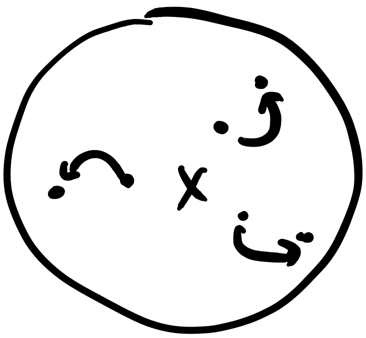
s.th. $r < |T(x)| < 1$

$\forall r \in |C|, r < 1.$

In fact, there is a natural map

$$\begin{array}{ccc} \operatorname{rad}: \mathbb{B}_C & \longrightarrow & [0,1] \\ \cup\! | & & \cup\! | \\ \mathbb{D}^* & & \end{array}$$

$$\mathrm{rad}: \mathbb{D}^*_C \longrightarrow (0,1).$$
$$x \longmapsto |T(x)|$$



$\mathbb{D}^*_C \circlearrowleft \phi_C$.

This satisfies

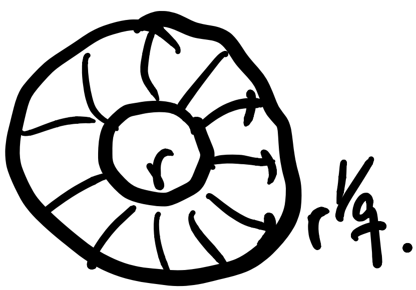
$$\mathrm{rad} \circ \phi_C = \mathrm{rad}^{1/q}.$$

$\Rightarrow$ action of ϕ_C on $\mathbb{D}^*_C$ free, properly discontinuous.

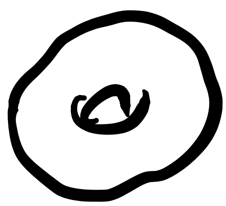
$$\Rightarrow X_{C,E} = \mathbb{D}^*_C / \phi_C^{\mathbb{Z}}$$

(well-defined adic space.

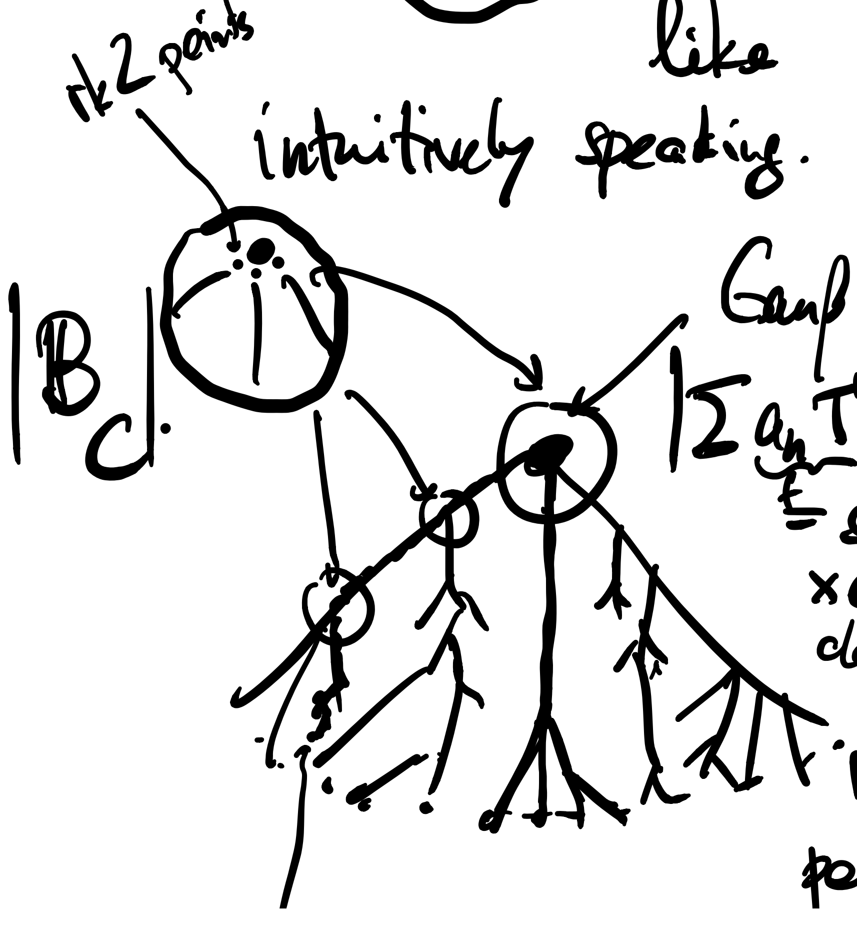
$\mathbb{A}(r, r^{1/q}) \Big/$ identify boundary annuli.



$\phi : \mathbb{A}(r,r) \cong \mathbb{A}(r^{1/q}, r^{1/q})$.



quotients looks like complex torus intuitively speaking.



Gauß point: $\left|\sum a_n T^n\right| = \sup |a_n|$ $= \sup_{x \in \mathbb{B}_{\mathbb{C}} \text{ classical}} |f(x)|$

Classical points:

"dead ends" $\{x \in \mathbb{C} \mid 0 \le |x| \le 1\}$.