The Jacobian criterion

$E \supset \mathcal{O}_E$, π, $\mathbb{F}_q$ as usual.

Thm. S perfectoid space $/\mathbb{F}_q$,

$$Z \longrightarrow X_S = X_{S,E}$$ smooth map of (sousperfectoid) adic spaces.

s.th. Z quasiprojective

($\exists$ Zariski closed embedding
$$Z \hookrightarrow U \underset{\text{open}}{\subseteq} \mathbb{P}^n_{X_S}),$$

letting

$$\mathcal{M}_Z^{\text{smooth}} \subseteq \mathcal{M}_Z = \left\{ \text{sections of } \begin{array}{c} Z \\ \downarrow \\ X_S \end{array} \right\}.$$

$$T/S \mapsto \left\{ \begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[r] \arrow[ur, dashed, "s"] & X_S \end{tikzcd} \right\}.$$

be the open subspace where

$s^* T_{Z/X_S}$ has everywhere positive Harder-Narasimhan slopes, the map $\mathcal{M}_Z^{\text{smooth}} \longrightarrow S$ is

ℓ-cohom. smooth $\forall \ell \neq p$.

(Recall: $\mathcal{M}_Z \to S$ is repr. in loc. spat. diamonds

compactifiable, locally finite dim.trg.)

$\Rightarrow$ same for $\mathcal{M}_Z^{\text{smooth}}$

Strategy: 1) formal smoothness of $\begin{array}{c}\mathcal{M}_Z^{\text{smooth}} \\ \downarrow \\ S\end{array}$.

2) formal smoothness + "geometric finite-dim'l"

$\Downarrow$

$\mathbb{F}_\ell$ f-ULA.

3) $Rf^! \mathbb{F}_\ell$ is invertible.

$\Big\}\Leftrightarrow f$ ℓ-cohom. smooth.

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1) "formal smoothness": $T_0 \subseteq T$ Zariski

closed immersion of aff'd perf'oid spaces,

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & \mathcal{M}_Z^{\text{smooth}} \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

$\Rightarrow \exists\, T' \to T$ étale containing T_0 in image,

$$\begin{tikzcd} T_0 \times_T T' \arrow[r] \arrow[d, hook] & \mathcal{M}_Z^{\text{smooth}} \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

2) <u>Proposition</u>. There is a cohom. smooth + formally smooth surj. map

$$T_0 \longrightarrow \mathcal{M}_Z$$

(repr. in loc. spat. diamonds, compactifiable, locally finite dim.trg.)

s.th. T_0 is a perfectoid space that locally admits a Zariski closed embedding into fin-dim'l perfectoid ball$/S$.

Proof. $\mathcal{M}_Z \subset \mathcal{M}_{\mathbb{P}^n_S}$ locally Zariski closed (in suitable sense)

so reduce to $Z = \mathbb{P}^n_S$.

$$\mathcal{M}_{\mathbb{P}^n_S} \underset{\text{open}}{\subset} \bigcup_{d \geq 0} \left(\mathcal{BC}(\mathcal{O}(d)^{n+1})_S^{\setminus 0} / \underline{E}^\times\right)$$

$\rightsquigarrow$ enough for $\mathcal{BC}(\mathcal{O}(d)^{n+1})_S / \underline{E}^\times$.

This can be done explicitly. $\square$

Corollary. $\mathbb{F}_\ell$ is ULA for $\mathcal{M}_Z^{\text{smooth}} \to S$.

Proof. enough for $T_0 \to S$ instead; and

locally, so enough:

Lemma. Let $T_0 \to S$ be a

map of aff'd perf'd spaces s.th.

1) T_0 formally smooth $/S$

2) $T_0 \hookrightarrow \mathbb{B}^n_S$ Zariski closed in some fin.-dim'l perf'oid ball.

Then $\mathbb{F}_\ell$ is ULA for $T_0 \longrightarrow S$.

<u>Proof</u>.

$$\begin{tikzcd}
T_0 \times_T T' \arrow[r] \arrow[d, hook] & T_0 \arrow[r, equal] \arrow[d, hook] & T_0 \arrow[d] \\
T' \arrow[r, "\text{étale}"] \arrow[urr, dashed] & T = \mathbb{B}^n_S \arrow[r] & S .
\end{tikzcd}$$

$$\begin{tikzcd}
T_0 \times_T T' \arrow[r, "\text{étale}"] \arrow[d, hook] & T_0 \\
T' \arrow[u, dotted, bend right] \arrow[ur] \arrow[d, "\text{étale}"] \arrow[dr] & \\
\mathbb{B}^n_S \arrow[r] & S
\end{tikzcd}$$

Shrinking T', can even find a retraction $T' \longrightarrow T_0 \times_T T'$.

$$\Longrightarrow \quad \begin{tikzcd} & T_0 \times_T T' \arrow[dl, "\text{étale}"'] \arrow[d, "\text{ULA}"] \\ T_0 \arrow[r, "\text{ULA}"'] & S \end{tikzcd} \qquad \Longleftarrow \qquad \text{retract of} \quad \begin{tikzcd} T' \arrow[d, "\text{étale}"] \arrow[dd, bend right=60, "\text{ULA}"'] \\ \mathbb{B}^A_S \arrow[d] \\ S \end{tikzcd}.$$

ULA is étale local (even cohom. smooth)

being ULA is stable under retracts: follows either directly from def'n, or from categorical characterization. □.

3) $Rf^! \mathbb{F}_\ell$ is invertible, i.e. locally isom. to $\mathbb{F}_\ell[n]$.

<u>Fact</u>: If A f-ULA for $f: X \to S$, then $\mathbb{D}_{X/S}(A)$ is again f-ULA <u>and</u> its formation commutes with any base change

$$S' \longrightarrow S.$$

Being invertible can be checked v-locally, so can be checked after pullback along v-cover.

$\Rightarrow$ Passing to universal section of

$$\mathcal{M}_2^{\text{smooth}} \longrightarrow S,$$

enough to prove that for a section

$$s: \quad S \longrightarrow \mathcal{M}_2^{\text{smooth}}$$

$$(\hat{=} \quad s: X_S \longrightarrow Z),$$

the pullback

$$s^* Rf^! \mathbb{F}_\ell \quad \in D_{\acute{e}t}(S, \mathbb{F}_\ell)$$

is invertible.

Now can use deformation to the normal cone:

$$\begin{tikzcd} Z \arrow[d] \\ X_S \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{Z} \arrow[d, "\text{smooth}"] \\ X_S \times \mathbb{A}^1 \arrow[u, bend left, "\tilde{s}"] \end{tikzcd} \circlearrowleft \mathbb{G}_m$$

s.th. $\cdot\ \tilde{Z} \times_{\mathbb{A}^1} \{1\} = Z$

$\cdot\ \tilde{Z} \times_{\mathbb{A}^1} \{0\} =$ normal cone of s in Z

$=$ geometric VB corr. to

$s^* T_{Z/X_S}$.

$$\begin{tikzcd} \underline{E}^\times \arrow[d, phantom, "\circlearrowright"] & & \mathbb{G}_m \arrow[d, phantom, "\circlearrowright"] \\ X_{S\times\underline{E}} & \subseteq & X_S \times \mathbb{A}^1 \\ & X_S \times \underline{E} \arrow[ul, phantom, "\parallel"] & \\ \tilde{Z}' \arrow[uu] & \subseteq & \tilde{Z} \arrow[uu] \end{tikzcd}$$

Get $\tilde{Z}' \xrightarrow{\text{smooth}} X_{S\times\underline{E}}$, fibre over $S\times\{1\}$ is Z,

$\circlearrowright$ $\underline{E}^\times$

fibre over $S \times \{0\}$ is $s^* T_{Z/X_S}$.

$\tilde{Z}'$ still quasiprojective, so all previous results apply.

$$\tilde{f} : \mathcal{M}^{\text{smooth}}_{\tilde{Z}'} \longrightarrow S \times \underline{E}.$$

$$\tilde{s} \circlearrowright \underline{E}^\times$$

$R\tilde{f}^! \mathbb{F}_\ell$ $\tilde{f}$-ULA

+ $R\tilde{f}^! \mathbb{F}_\ell |_{S \times \{1\}} = Rf^! \mathbb{F}_\ell$

+ $R\tilde{f}^! \mathbb{F}_\ell |_{S \times \{0\}}$ = dualizing complex for $\mathcal{BC}(s^* T_{Z/X_S})$, invertible.

$\overset{!}{\Longrightarrow}$ $s^* Rf^! \mathbb{F}_\ell$ is invertible.

$\square$.

Applications to $D_{ét}(\mathrm{Bun}_G, \Lambda)$

Recall charts for Bun_G:

<u>Def'n</u>. Let $\mathcal{M}$ be the moduli space of $\mathbb{Q}$-filtered G-bundles, i.e. exact $\otimes$-functors

$$\mathrm{Rep}_E G \xrightarrow{\rho} \mathbb{Q}\text{-Fil}\,\mathrm{Bun}_{X_S} \quad (\text{increasing filt.})$$

s.th. for all $V \in \mathrm{Rep}_E G$,

$$\rho(V)^{\lambda} := \rho(V)^{\le\lambda} \Big/ \bigcup_{\lambda'<\lambda} \rho(V)^{\le\lambda'}$$

semistable of slope λ. "opposite HN filtration".

$$\text{Then } \bigsqcup_{b\in B(G)} \mathcal{M}_b = \mathcal{M} \longrightarrow \mathrm{Bun}_G$$

$$\mathcal{M} \longrightarrow \bigsqcup_{b\in B(G)} [*/\underline{G_b(E)}].$$

forget filtr.

pass to assoc. graded bundle

$V \mapsto \bigoplus_\lambda \rho(V)^\lambda \otimes \mathcal{O}_{\mathrm{Gr}\,\mathrm{Bun}^{HN}_{\times S}}$
Isoc_S.

Thm. $\mathcal{M} \longrightarrow \mathrm{Bun}_G$ is cohomologically smooth.

Example. $G = GL_2$, $b \cong \mathcal{O} \oplus \mathcal{O}(1)$.

Then $\mathcal{M}_b$ param. extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

$\deg \mathcal{L} = 0$, $\deg \mathcal{L}' = 1$.

$G = GL_n$: similar successive extensions.

Thm is a consequence of Jacobian criterion.

(Take any $S \longrightarrow \mathrm{Bun}_G \cong \mathcal{E}/X_S$
G-bundle,

take Z = moduli space of Q-filtrations on $\mathcal{E}_b$.

Then $\mathcal{M} \subseteq \mathcal{M}_Z$, actually lies in $\mathcal{M}_Z^{\text{smooth}}$ by condition on slopes.)

Now fix $b \in B(G)$, consider

$$\begin{array}{ccc} \pi_b : \mathcal{M}_b & \longrightarrow & \mathrm{Bun}_G \\ \cong & & \\ \mathcal{M} & & \end{array}$$

"chart for Bun_G near Bun_G^b".

<u>Structure of $\mathcal{M}_b$:</u>

- $$\begin{array}{c} \mathcal{M}_b = [\tilde{\mathcal{M}}_b / \underline{G_b(E)}] \\ \downarrow \\ [* / \underline{G_b(E)}]. \end{array}$$

In $\tilde{\mathcal{M}}_b$, graded bundle is trivialized.

e.g. $\{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\}$.

- "base point" $* \in \widetilde{\mathcal{M}}_b$ corresponding to split extension.

$$\rightsquigarrow \begin{tikzcd}
{[*/\underline{G_b}(E)]} \arrow[r, phantom, "\subseteq"] \arrow[ddr, "\simeq"] \arrow[dddd] & \mathcal{M}_b \arrow[dd] \arrow[dddd, bend left] \\
& \\
& {[*/\underset{\sim}{G_b(E)}]} \\
& \\
\mathrm{Bun}_G^b \arrow[r, phantom, "\subseteq"] & \mathrm{Bun}_G
\end{tikzcd}$$

$\square$

(If $\mathcal{E} \in \mathrm{Bun}_G^b$, then HN filtr. of $\mathcal{E}$ gives splitting of given Q-filtr.)

- $\widetilde{\mathcal{M}}_b \longrightarrow *$ repr. in loc. spat. diamonds, cohom. smooth, successive extension of negative Banach-Colmez spaces of dim $= \langle 2\rho, \nu_b \rangle$.

$\left(\{ 0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0 \} = \mathcal{BC}(\mathcal{O}(-1)[1]) \right)$

- $\widetilde{\mathcal{M}}_b \setminus * = \widetilde{\mathcal{M}}_b^{\circ}$ is a spatial diamond.

($\Rightarrow$ qcqs!, but not qcqs $/*$.)

In GL_2-examples, it is

$$\left(\operatorname{Spa} \widehat{\mathbb{F}_q((t))}^{1/p^\infty} \right) \Big/ \underline{SL_1(D)}$$

D/E quaternion algebra.

aff'd perf'oid

profinite.

(On $\widetilde{\mathcal{M}}_b^{\circ}$, $\mathcal{E} \cong \mathcal{O}(1/2)$). Picking such iso,

$\mathcal{O} \hookrightarrow \mathcal{O}(1/2)$ gives section of

$$\mathcal{B}\mathcal{C}(\mathcal{O}(1/2)) \setminus \{0\} . = \operatorname{Spa} \overline{\mathbb{F}}_q((t^{1/p^\infty}))$$

In GL_2-example:

$$\tilde{\mathcal{M}}_b^\circ \subseteq \tilde{\mathcal{M}}_b = \tilde{\mathcal{M}}_b^\circ \cup * .$$

$$\tilde{\mathcal{M}}_b^\circ \cong \operatorname{Spa} \overline{\mathbb{F}}_q((t^{1/p^\infty}))$$

formal punctured open unit disc.

The point $*$ sits near $|t| = 1$, not near $|t| = 0$

after base change to $\mathbb{C}$:

$\tilde{\mathcal{M}}_b^\circ$

• $\tilde{\mathcal{M}}_b$ "strictly local":

<u>Thm.</u> For any $A \in D_{\acute{e}t}(\tilde{\mathcal{M}}_b, \Lambda)$, the restriction

$$R\Gamma(\tilde{\mathcal{M}}_b, A) \longrightarrow R\Gamma(*, A)$$

is an isomorphism.

<u>Sketch.</u> Cone of this map is

$$R\Gamma_{\partial - c}(\tilde{\mathcal{M}}_b^{\circ}, A).$$

compact support towards $*$
no support condition towards boundary of $\tilde{\mathcal{M}}_b$.

Special case of:

Let $\underset{(=\tilde{\mathcal{M}}_b^{\circ})}{X}$ spatial diamond $/\mathbb{F}_q$ $\dim \operatorname{trg} < \infty$.

$X \longrightarrow *$ partially proper

(i.e. $X(R, R^+) = X(R, R^\circ)$.)

Then for any $C/\overline{\mathbb{F}}_q$,

X_C has "two ends".

<u>examples</u> $X = \mathrm{Spa}(R, R^+)$ aff'd perf'oid

$C = \widehat{\overline{\mathbb{F}_q((t))}}$,

$X_C \xrightarrow[\text{profinite}]{} X \times_{\mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q((t)) =$ punctured open unit disc $/X$.

one boundary = origin

other boundary = "boundary of unit disc".

$\Rightarrow$ Can define 'partial compactly supp. cohomology'

$R\Gamma_{\partial-c}(X_C, A)$ $A \in D_{et}(X, \Lambda)$.

<u>Thm</u> $R\Gamma_{\partial-c}(X_C, A) = 0$.

<u>Sketch</u>. reduce to $X = \mathrm{Spa}\, \overline{\mathbb{F}}_q((t^{1/p^\infty}))$.

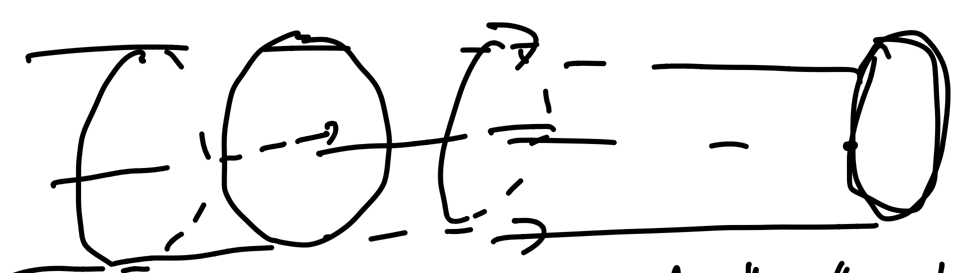
(use proper base change + "correspondence")

+ $A = \Lambda$ + compute.

<u>Picture:</u> Say M topological manifold

+ free action $\mathbb{R} \circlearrowright M$ "flow"

s.th. $\overline{M} = M/\mathbb{R}$ is compact.



two boundaries   "source of flow" + "sink of flow"

$+$ for all $A \in D(M/\mathbb{R}, \mathbb{Z})$,

$$R\Gamma_{\partial - c}(M, A) = 0.$$

(Proof: 'Flow contracts everything.')

How is this analogous?

Roughly: $C = \mathbb{F}_q((t^{\mathbb{R}})) \circlearrowleft \mathbb{R}_{>0} \overset{\exp}{\cong} \mathbb{R}$.
rescaling.

$$X_C \circlearrowleft \mathbb{R}. \qquad X_C/\mathbb{R} \text{ qcqs.}$$

$$\begin{array}{ccc} X_C & & X_C/\mathbb{R} \\ \rotatebox{90}{$\lessdot$}\!\!\| & & \downarrow \\ M & & X. \end{array}$$

If X cohom smooth $/*$,

Cor. X_C as above satisfies 'odd-dim'l Poincaré duality'.

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & \\
R\Gamma_c(X_C) \arrow[r] & R\Gamma_{\partial C_1}(X_C) \oplus R\Gamma_{\partial C_2}(X_C) \arrow[d] \\
 & R\Gamma(X_C)
\end{tikzcd}$$

$$\Rightarrow R\Gamma_c(X_C) \simeq R\Gamma(X_C)[-1].$$

<u>Question</u>. If X Zariski closed in fin.-dim'l perfectoid ball$/C$, is $\dim X = \dim \mathrm{trg}\, X$?

+ if X ℓ-cohom smooth, does $\mathbb{D}_X$ sit in degree $2\dim X$?

$$\begin{tikzcd} S' \arrow[r, "v\text{-cover}", "g"'] \arrow[dr, "h"'] & \mathcal{M}_Z^{\text{smooth}} \arrow[d, "f"] \\ & S \end{tikzcd}$$

$$Z' = Z \times_S S'$$

enough: $g^* Rf^! \mathbb{F}_\ell$ invertible.

$$\begin{tikzcd} \mathcal{M}_{Z'}^{\text{smooth}} \arrow[r, "\tilde{h}"] \arrow[d, "f'"] & \mathcal{M}_Z^{\text{smooth}} \arrow[d, "f"] \\ S' \arrow[u, bend left, "s"] \arrow[r, "h"] & S \end{tikzcd}$$

$$g = \tilde{h} s \quad \Rightarrow \quad g^* Rf^! \mathbb{F}_\ell$$

$$= s^* \tilde{h}^* Rf^! \mathbb{F}_\ell$$

$$= s^* Rf'^! \mathbb{F}_\ell .$$

formation of $Rf^! \mathbb{F}_\ell$ comm. w/ base change ($\Leftarrow \mathbb{F}_\ell$ is ULA)