Universal local acyclicity (ULA).

$$\begin{array}{ccc} D_{\text{ét}}(\mathrm{Bun}_G) & \xleftarrow{\text{charts}} & \text{Jacobian criterion} \\ \uparrow \text{notion of "admissibility"} & & \nearrow \text{for proof} \\ \text{ULA sheaves} & & \\ \downarrow & & \\ \text{geometric Satake} & & \end{array}$$

<u>Recall</u>. Let $f: X \to S$ finite type, separated map of noetherian schemes, $A \in D_c^b(X_{\text{ét}}, \Lambda)$, where $n\Lambda = 0$, $n \in \mathcal{O}_S^\times$. Then

A is <u>f-locally acyclic</u> if

for all geometric points
$$\begin{array}{ccc} \bar{x} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{t} \rightsquigarrow \bar{s} & \longrightarrow & S \end{array}$$

the map

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} \bar{t}, A)$$

is an isomorphism.

i.e. :

$$\begin{array}{c} X_{\bar{x}} \\ \downarrow \\ S_{\bar{s}} \end{array}$$

cohom. of all geom. fibres agrees.

"étale analogue of

$\mathcal{F}/X$ (quasi) coherent "

asking $\mathcal{F}$ "flat / S".

<u>Thm</u> (Gabber) A f-locally acyclic
(see Lu-Zheng, Duality & Nearby Cycles over general Bases)
$\Rightarrow$ for any base change

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array},$$

also $\tilde{g}^* A$ is f'-locally acyclic.

requires noetherian base. In general, following notion is better:

<u>Definition</u>. A is f-universally locally acyclic if for any base change as above, $\tilde{g}^* A$ is f'-locally acyclic.

<u>Examples</u>. 1). If f is smooth, then Λ (or any locally constant sheaf) is ULA.

2) If $f = id: X = S \to S$, then A is id-ULA $\Leftrightarrow$ A is locally constant.

3)
$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$
h proper

B / Y g-ULA

$A = Rh_* B$ is f-ULA.

2)+3): If f is proper, A f-ULA

$\Rightarrow$ $Rf_* A$ is locally constant.

4): A f-ULA

$\Rightarrow$ A-twisted version of Poincaré duality:

$$\mathbb{D}_{X/S}(A) \otimes f^* B \xrightarrow{\sim} R\mathcal{H}om(A, Rf^! B).$$

$$\forall B \in D^b_c(S_{et}, \Lambda).$$

where $\mathbb{D}_{X/S}(A) = R\mathcal{H}om(A, Rf^! \Lambda)$

relative Verdier dual.

($A = \Lambda$: get

$$Rf^! \Lambda \otimes f^* B \xrightarrow{\sim} Rf^! B$$

if Λ f-ULA, e.g. if f smooth.)

5) A f-ULA $\Rightarrow$ Verdier biduality:

(Lu-Zheng 2020) $\mathbb{D}_{X/S}(A)$ also f-ULA and

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

In fact, they characterize ULA sheaves as dualizable objects in a certain symmetric monoidal category.

6) If S geometric point, all

$$A \in D^b_c(X_{ét}, \Lambda) \quad \text{are ULA.}$$

__Want:__ Variant for diamonds.

__important point:__ have good analogue

$D_{ét}(X, \Lambda)$ of full unbounded derived category,

but "constructibility" is a subtle notion.

$$i : \operatorname{Spa} C \hookrightarrow \mathcal{B}_C$$

$i_* \Lambda$ is __not__ constructible (but should be ULA/$\operatorname{Spa} C$)

Proposition. If X spatial diamond of finite cohom. dimension (uniformly on $X_{\text{ét}}$), then

$D_{\text{ét}}(X, \Lambda)$ is compactly generated, compact objects are exactly the constructible complexes: locally constant after passing to a constructible stratification.

↑ in boolean alg. gen. by $\underline{qc}$ open subsets.

Example. $j : T_C \hookrightarrow B_C$.

$T_C = \{T \mid |T| = 1\}$, $B_C = \{T \mid |T| \leq 1\}$.

$j_! \Lambda$ is constructible.

Definition. Let $f : X \to S$ map of locally spatial diamonds (compactifiable, of locally finite dim.trg)

$\rightsquigarrow Rf_!$ defined ,

$A \in D_{ét}(X, \Lambda)$.

1) A is f-locally acyclic if

a) $\forall$ geom. pts.

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[d, maps to] & X \arrow[d] \\ \bar{t} \rightsquigarrow \bar{s} \arrow[r] & S \end{tikzcd},$$

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$$

is an isomorphism.

b) for all étale $j : U \to X$ s.th.

$f \circ j : U \to S$ is qcqs,

$R(f \circ j)_!(A|_U) \in D_{ét}(S, \Lambda)$ is

constructible.

(i.e., is constr. after any pullback $S' \to S$, S' spatial diamond as in Prop'n.)

2) A f-univ. loc. acyclic if any base change is loc. acyclic.

<u>Remarks</u>. • For schemes, b) is automatic, all information is in a).

• For diamonds, (almost) the opposite is true: a) is almost automatic, but b) powerful.

• Analogue of Gabber's theorem fails:

$S = \mathrm{Spa}\, C$, X cohom. smooth $/S$

$\Rightarrow$ any constr. A is f-loc. acyclic, but only locally constant A are f-univ. loc. acyclic.

<u>Condition a)</u>: $X_{\bar{x}}$ is repr. by $\mathrm{Spa}(C, C^+)$

C complete alg. closed field, $C^+ \subseteq C$ valuation subring.

$$\begin{array}{ll} |X_{\bar{x}}| & : \text{ totally ordered chain of points.} \\ \big\downarrow & \\ |S_{\bar{s}}| & : \text{ similar.} \\ \quad \cup\, |S_{\bar{t}}| & \text{ subset.} \end{array}$$

$$\Longrightarrow \quad X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}} = X_{\bar{y}} \quad \text{for some}$$

$$\begin{tikzcd} \bar{y} \arrow[r, squiggly] \arrow[d] & \bar{x} \arrow[d] \\ \bar{t} \arrow[r, squiggly] & \bar{s} \end{tikzcd}.$$

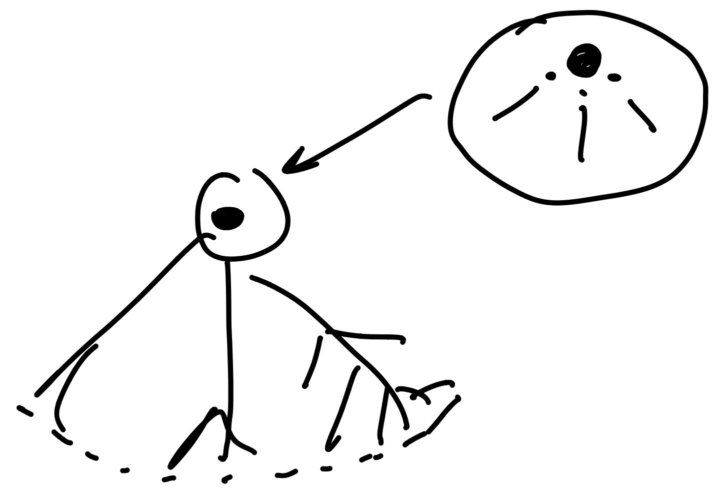
Condition is just

$$A_{\bar{x}} \xrightarrow{\sim} A_{\bar{y}} = R\Gamma(X_{\bar{y}}, A)$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A).$$

"a) universally" $\Longleftrightarrow$ A is <u>overconvergent</u> i.e.

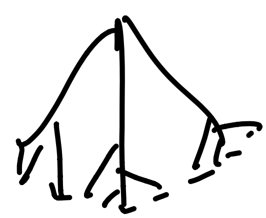
for all spec. $\bar{y} \rightsquigarrow \bar{x}$, $A_{\bar{x}} \xrightarrow{\sim} A_{\bar{y}}$ is an isomorphism.

$|\mathcal{B}_C|$:



$\downarrow$

$|\mathcal{B}_C|^{\text{hausdorff}}$ = Berkovich space : compact Hausdorff.



overconv. sheaves on $|\mathcal{B}_C|$

$\cong$ sheaves on $|\mathcal{B}_C|^{\text{hausdorff}}$.

(similar statement for étale sheaves).

overconv. étale sheaves $\cong$ étale sheaves on Berkovich space.

when this makes sense.

Properties: 1) If f cohom. smooth,
A locally constant
$\to$ A f-ULA:

overconv. is clear.

enough to see: If f qcqs + cohom. smooth

$\to$ $Rf_!$ preserves constr. complexes.

$$Rf_! : D_{et}(X,\Lambda) \rightleftarrows D_{et}(S,\Lambda) : Rf^!$$

know: constr. complexes = compact objects.

$R\mathrm{Hom}(A,-)$ comm. w/ all direct sums.

Lemma: $F: \mathcal{C} \rightleftarrows \mathcal{D} : G$
adj. functors of compactly gen. triang. categories. Then F
preserves compact objects $\Leftrightarrow$ G comm. w/ all

direct sums.

<u>Proof</u>. G preserves direct sums, $A \in \mathcal{C}$ compact

$$\Rightarrow \mathrm{Hom}_{\mathcal{D}}\Big(F(A), \bigoplus_{i\in I} B_i\Big) \cong \mathrm{Hom}_{\mathcal{C}}\Big(A, G\big(\bigoplus_{i\in I} B_i\big)\Big)$$

$$\cong \mathrm{Hom}_{\mathcal{C}}\Big(A, \bigoplus_{i\in I} G(B_i)\Big) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A, G(B_i))$$

$$\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{D}}(F(A), B_i).$$

$\square$.

But f cohom. smooth $\Rightarrow$ $Rf^! \cong f^* \otimes Rf^!\Lambda$

commutes with direct sums.

2) If $f = \mathrm{id}: X = S \to S$, then A f-ULA

$\Leftrightarrow$ A is locally constant with perfect fibres.

(locally constant sheaf on complex of finite proj. Λ modules.)

b) $\Rightarrow$ A constructible ?

a) $\Rightarrow$ A overconvergent $\}$ $\Rightarrow$ locally constant.

3) Proper pushforwards preserve ULA ness.

follows from proper base change.

2) +3) : If f proper, A f-ULA

$\Rightarrow$ $Rf_* A$ is locally constant constructible.

4) twisted version of Poincaré duality:

A f-ULA $\Rightarrow$

$$\mathbb{D}_{X/S}(A) \overset{L}{\underset{\Lambda}{\otimes}} f^* B \xrightarrow{\sim} R\mathcal{H}om_\Lambda(A, Rf^! B).$$

for all $B \in D_{et}(S, \Lambda)$.

Note: This implies b): It implies that

$R\mathcal{H}om_\Lambda(A, Rf^! -) : D_{et}(S, \Lambda) \to D_{et}(X, \Lambda)$

commutes with all direct sums; thus by Lemma, its left adjoint preserves constructibility.

$Rf_!(A \overset{L}{\underset{\Lambda}{\otimes}} -)$. Apply to $j_! \Lambda$.

5) Verdier biduality: If A f-ULA, then $\mathbb{D}_{X/S}(A)$ is f-ULA, and

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

(cf. below).

6) $S = \mathrm{Spa}\, C$, $X = X_0^\diamond$ for some alg. variety X_0/C.

Then for any $A_0 \in D_c^b(X_{0,\mathrm{et}}, \Lambda)$, its analytification $A \in D_{\mathrm{et}}(X, \Lambda)$

is ULA.

$\rightsquigarrow$ $i: \mathrm{Spa}\, C \hookrightarrow \mathbb{A}^1_C$

$i_* \Lambda$ is ULA.

($\Rightarrow$ same for $\mathrm{Spa}\, C \hookrightarrow \mathbb{B}_C$.)

<u>About 5): Biduality</u>

<u>Two proofs:</u> Both use 'dualizability' in 2-categories.

<u>Lu-Zheng approach.</u> Fix base S.

Consider $\overset{\text{symm. mon.}}{2\text{-category}}$ LZ_S:

- objects: (X, A) $X \to S$ as above

$A \in D_{et}(X, \Lambda)$

- morphisms $(X, A) \longrightarrow (Y, B)$: coham. corr.

$$\begin{tikzcd} & Z \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

+ map $c_1^* A \longrightarrow Rc_2^! B$.

- symm. monoidal structure:

$$(X, A) \otimes (Y, B) = (X \times_S Y, A \boxtimes B).$$

closed for this symm. mon. structure!

$$\mathrm{Hom}_{LZ_S}((X,A), (Y,B)) = (X \times_S Y, R\mathcal{H}om(p_1^* A, Rp_2^! B)).$$

<u>Thm</u>. TFAE:

1). A is $(X \to S)$-ULA.

2) (X, A) is dualizable in LZ_S.

3) $(X, A) \otimes (X, A)^\vee \xrightarrow{\sim} \mathrm{Hom}_{LZ_S}((X,A), (X,A))$, i.e.

$$p_1^* \mathbb{D}_{X/S}(A) \overset{L}{\underset{\Lambda}{\otimes}} p_2^* A \xrightarrow{\sim} R\mathcal{H}om_\Lambda(p_1^* A, Rp_2^! A)$$

for
$$\begin{tikzcd} X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & S \end{tikzcd}.$$

(an instance of A-twisted Poincaré duality).

In that case, dual $(X,A)^\vee = (X, \mathbb{D}_{X/S}(A))$,

thus $\mathbb{D}_{X/S}(A)$ is $(X\to S)$-ULA, and

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

<u>Cor</u> 1) Λ is f-ULA iff

$$p_1^* \mathbb{D}_{X/S} \xrightarrow{\sim} \mathbb{D}_{X\times_S X/X} = Rp_2^! \Lambda.$$

2) f is cohom. smooth (wrt. Λ)

$\Leftrightarrow$ $\mathbb{D}_{X/S}$ is inv., and $p_1^* \mathbb{D}_{X/S} \xrightarrow{\sim} Rp_2^! \Lambda$.

Second proof. Define a 2-category $\mathcal{C}_S$.

- objects: $X \to S$ as above.
- categories of morphisms: $\mathrm{Fun}_{\mathcal{C}_S}(X,Y) = D_{et}(X \times_S Y, \Lambda)$.

- composition = convolution

$X, Y, Z \to S$

$$\begin{tikzcd} & X \times_S Y \times_S Z \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X \times_S Y & X \times_S Z & Y \times_S Z \end{tikzcd}$$

$A \in \mathrm{Fun}_{\mathcal{C}_S}(X,Y) \qquad B \in \mathrm{Fun}_{\mathcal{C}_S}(Y,Z)$

$\Rightarrow \quad A * B := R\pi_{13!}(\pi_{12}^* A \overset{L}{\underset{\Lambda}{\otimes}} \pi_{23}^* B)$.

proper base change $\Rightarrow$ associativity.

$\mathrm{id}_X = \Delta_{X/S!}\Lambda$.

maps to 2-category: - obj = $X \to S$ as above

- morphisms = functors $X \to Y$.

$$D_{ét}(X, \Lambda) \to D_{ét}(Y, \Lambda)$$

by using sheaves as kernels:

$$D \mapsto R\pi_{2!}(A \overset{L}{\underset{\Lambda}{\otimes}} \pi_1^* D).$$

<u>Recall:</u> In any 2-category, have notion of adjointness: $f: X \to Y$ left adj. of $g: Y \to X$ if there are $\alpha: id_X \to gf$,

$$\beta: fg \to id_Y$$

s.th.

$$f \xrightarrow{f\alpha} fgf \xrightarrow{\beta f} f \quad \text{and}$$

$$g \xrightarrow{\alpha g} gfg \xrightarrow{g\beta} g \quad \text{are the identity.}$$

<u>Theorem.</u> TFAE:

1) $A \in D_{ct}(X,\Lambda)$ is ULA

2) $A \in \mathrm{Fun}_{\mathcal{C}_S}(X,S)$ is a left adjoint.

In that case, the right adj. is

$$\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S,X).$$

3) $p_1^* \mathbb{D}_{X/S}(A) \overset{L}{\underset{\Lambda}{\otimes}} p_2^* A \xrightarrow{\sim} R\mathcal{H}om_\Lambda(p_1^* A, Rp_2^! A)$.

ULA Base Change:

$$\begin{tikzcd} X' \arrow[r, "\tilde{g}"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Assume Λ is g-ULA

$\Rightarrow \quad g^* Rf_* \cong Rf'_* \tilde{g}^*$.

$$\begin{tikzcd} \widetilde{Gr}_G^I \arrow[r, "h"] \arrow[dr] & Gr_G^I \arrow[d] \\ & (Div_X^1)^I \end{tikzcd}$$

convolution BD Grassm. (locally isom. to a product)

Beilinson-Drinfeld Grassm.

h ind proper.

$(A_i)_{i \in I}$ ULA on $Gr_G \to Div^1$

$\rightsquigarrow \widetilde{\boxtimes}_{i \in I} A_i$ ULA on $\widetilde{Gr}_G^I$

$Rh_*(\widetilde{\boxtimes}_{i \in I} A_i)$ ULA on Gr_G^I.

"fusion product".