- Geometric Satake
- Jacobian Criterion

The Jacobian criterion.

Setup. S perfectoid space $/\mathbb{F}_q$ of char. p.

$$\begin{array}{l} Z \\ \downarrow \\ X_S \end{array}$$

Z: smooth adic space.

locally étale over a fin.-dim'l ball $/X_S$.

X_S: Fargues - Fontaine curve (E implicit)

$\leadsto$ space of sections:

Definition. $\mathcal{M}_Z : \{\text{perf'd spaces } T/S\} \to \text{Sets}:$

$$T \mapsto \left\{ \begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[r] \arrow[ur, dashed] & X_S \end{tikzcd} \right\}$$

Example. 1) If Z_0 / E smooth, can take

$$Z = Z_0 \times X_S,$$

then

$$\mathcal{M}_Z(T) = \mathrm{Map}(X_T, Z_0).$$

"Gromov - Witten like space".

2) If $Z = \mathcal{E}$ geometric vector bundle $/X_S$,

then $\mathcal{M}_Z = \mathcal{BC}(\mathcal{E})$.

so $\mathcal{M}_Z$ in general is a

"non linear Banach-Colmez space".

3) If $\mathcal{E}$ G-torsor on X_S,

$P \subseteq G$ parabolic, then

$$\begin{array}{ll} Z = \mathcal{E}_G/P & \\ \downarrow & \text{geom. fibres} \cong G/P \\ X_S & \quad\text{flag varieties} \end{array}$$

$\mathcal{M}_Z(T)$ = reductions of $\mathcal{E}|_{X_T}$ to $P \subseteq G$.

this case will be used for the charts

$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$.

Want to understand geometry of $\mathcal{M}_Z$.

<u>Proposition</u>. If Z/X_S is quasiprojective ($\exists\ Z \underset{\text{Zar. closed}}{\hookrightarrow} U \underset{\text{open}}{\subseteq} \mathbb{P}^n_{X_S}$), then $\mathcal{M}_Z$ is repr. in locally spatial diamonds,

$\mathcal{M}_Z \to S$ compactifiable, of locally finite dim.trg.

<u>Conjecture</u>. This is true for all smooth Z/X_S.

<u>Sketch</u>. reduce to $Z = \mathbb{P}^n_{X_S}$. Then can be explicit:

$$\mathcal{M}_{\mathbb{P}^n}(T) = \mathrm{Map}(X_T, \mathbb{P}^n)$$

$$= \left\{ (\mathcal{L}, s_0, \dots, s_n) \,\middle|\, \begin{array}{l} \mathcal{L} \text{ line bundle on } X_T, \\ s_0, \dots, s_n \in H^0(X_T, \mathcal{L}) \text{ generating } \mathcal{L} \end{array} \right\} / \cong$$

$$= \bigsqcup_{d \geq 0} \mathcal{M}_{\mathbb{P}^n, \deg = d}$$

$$\uparrow \underline{E}^\times\text{-torsor : param. isom. } \mathcal{L} \cong \mathcal{O}(d).$$

$$\widetilde{\mathcal{M}}_{\mathbb{P}^n, \deg = d} \underset{\text{open}}{\subseteq} \mathcal{BC}(\mathcal{O}(d))^{n+1}$$

$$\Rightarrow \mathcal{M}_{\mathbb{P}^n} = \bigsqcup_{d \geq 0} \underbrace{\left(\text{open subset of } \mathcal{BC}(\mathcal{O}(d))^{n+1}\right) / \underline{E}^\times}_{\text{locally spatial diamonds}}.$$

$\uparrow$

not of fin. dim., of fin. dim.

only locally. $\square$.

<u>Remark</u>. See that $\mathcal{M}_{\mathbb{P}^n}$ is "almost" linear.

This is a phenomenon for $G = Gl_n$: spaces in Example 3) are "essentially" linear.

But not for other groups! For classical groups, get "essentially quadratic" spaces, like.

$$\left\{ (x,y,z) \;\middle|\; \begin{array}{l} x,y,z \in H^0(\mathcal{O}(1)), \\ x^2+y^2+z^2 = 0 \in H^0(\mathcal{O}(2)) \end{array} \right\}.$$

$s^*T_{Z/X_S} : 0 \to s^*T_{Z/X_S} \to \mathcal{O}(1)^3 \xrightarrow{(2x,2y,2z)} \mathcal{O}(2) \to 0.$ either $\cong \mathcal{O}(1/2)$ or $\mathcal{O} \oplus \mathcal{O}(1)$.

$\rightsquigarrow$ For $G = Gl_n$, can prove "Jacobian criterion" by hand, but not for other groups.

<u>Goal</u>. Find large open subset $\mathcal{M}_Z^{sm} \subseteq \mathcal{M}_Z$

s.th. $\mathcal{M}_Z^{sm} \longrightarrow S$ is cohomologically smooth.

<u>"Jacobian criterion"</u>. If $s : X_T \longrightarrow Z \,/X_S$ section, get $s^* T_{Z/X_S} \in VB(X_T)$.

tangent bundle: VB on Z.

Classically, deformations of s

$$\widehat{=} \; H^0(X_T, s^* T_{Z/X_S}).$$

obstructions

$$\widehat{=} \; H^1(X_T, s^* T_{Z/X_S}).$$

<u>Idea:</u> If $H^1(X_T, s^* T_{Z/X_S})$ vanishes, then s should define a smooth point of $\mathcal{M}_Z$.

Definition. $\mathcal{M}_Z^{sm} \subseteq \mathcal{M}_Z$ is the open subfunctor of all $s: X_T \to Z/X_S$ s.th. $s^* T_{Z/X_S}$ has everywhere only positive (>0) Harder-Narasimhan slopes.

Theorem. $\mathcal{M}_Z^{sm} \xrightarrow{f} S$ is cohomologically smooth,

$$(Rf^! \Lambda)_s \cong (R(f_s^{lin})^! \Lambda)$$

$$s \in \mathcal{M}_Z^{sm}(C) \qquad (R(f_s^{lin})^! \Lambda) \cong \Lambda(d)[2d]$$

$$f_s^{lin}: \mathcal{BC}(s^* T_{Z/X_S}) \longrightarrow \operatorname{Spa} C.$$

$$0 \in \mathcal{BC}(s^* T_{Z/X_S}) \qquad d = \deg(s^* T_{Z/X_S}).$$

idea: $\mathcal{M}_Z^{sm} \ni s \quad \underset{\text{near } s}{\overset{\sim}{\text{inf. loc.}}} \quad \mathcal{BC}(s^* T_{Z/X_S}) \ni 0.$

Application. Recall

$$\mathcal{M}_{LT,\infty}^{\diamond} \cong \mathcal{M}_{Dr,\infty}^{\diamond} / (\mathrm{Spa}\, \breve{E})^{\diamond}$$ space of maps

$$\mathcal{O}_{X_S}^{n} \hookrightarrow \mathcal{O}_{X_S}(1/n)$$ s.th. cokernel supp. at ∞, i.e. given untilt.

this is given by some

$$\mathcal{M}_{Z}.$$

Ivanov-Weinstein:

Jacobian criterion $\Rightarrow$ connected comp. of $\mathcal{M}_{LT,\infty} \setminus$ points with extra endomorphisms is cohomologically smooth.

e.g. $n=2$: complement of CM points is cohomologically smooth.

In particular, cohomology of qc open subsets w/o CM points stabilizes in the tower.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\mathcal{M}_{LT,2} & \supseteq & U_2 \\
\downarrow & & \downarrow \\
\mathcal{M}_{LT,1} & \supseteq & U_1 \quad \text{w/o CM pts} \\
\downarrow & & \text{qc open} \\
\mathcal{M}_{LT,0} & &
\end{array}
$$

$$H^j(U_1, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^j(U_2, \overline{\mathbb{Q}}_\ell) \hookrightarrow \cdots$$

transition maps are eventually isom!

first "empirically" observed by Weinstein.

How to prove Jacobian criterion?

Naive idea: try to find direct geometric relation to $\mathcal{BC}(s^* T_{Z/X_S})$.

this seems very hard.

actual method has three steps:

1) Definition of "formal smoothness" for maps of diamonds.

2) Definition of "universal local acyclicity".

$$\begin{array}{c} X \\ \downarrow f \\ S \end{array} \qquad A \in D_{et}(X, \Lambda)$$

A "f-ULA": some kind of "flatness".

- f cohom. smooth $\Longleftrightarrow$ Λ f-ULA $+ Rf^! \Lambda$ is invertible.

- formal smoothness + "geometric" fin.-dim. $\Longrightarrow$ Λ f-ULA.

3) deformation to the normal cone.

$$\begin{array}{ccc} \mathcal{M}_Z^{sm} & \overset{\text{degeneration}}{\dashrightarrow} & \mathcal{BC}(s^* T_Z / X_S) \\ \cup & & \cup \\ S & & 0. \end{array}$$

+ dualizing complex must be constant.

needs an argument, uses ULAness.

Formal Smoothness.

<u>idea</u>. replace infinitesimal neighborhoods by small actual neighborhoods.

<u>Theorem</u>. If
$$\begin{array}{ccc} S_0 & \longrightarrow & Y \\ \text{Zar.closed}\ \cap & & \downarrow \text{ smooth.} \\ S & \longrightarrow & X \end{array}$$
diagram of affinoid adic spaces,

then

$\exists\ U \subset S$ open containing S_0, lift

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ U \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

<u>Key</u>: $\varinjlim_{U \supset S_0} \mathcal{O}(U)$ henselian along I

$I = \ker(\ldots \to \mathcal{O}(S_0))$.

<u>Definition</u>. Let $f : Y \to X$ map of small v-stacks.

Then f is <u>formally smooth</u> if for all Zar. closed imm. of aff'd perf'd spaces $S_0 \subseteq S$,

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, hook] & Y \arrow[d, "f"] \\ S \arrow[r] & X \end{tikzcd}$$

$\exists$ étale map $S' \to S$ whose image contains S_0, and a lift

$$\begin{tikzcd} S_0 \times_S S^1 \arrow[r] \arrow[d, phantom, "\cap" description] & Y \arrow[d, "f"] \\ S^1 \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}.$$

<u>Claim</u>. This is related to "absolute neighborhood retracts" (ANR):

Y compact Hausdorff is ANR if for any closed immersion $Y \hookrightarrow Z$, $Z \cong \prod_I [0,1]$, $\exists$ $U \subseteq Z$ open containing Y, retraction $U \to Y$.

Here, assume

$$\begin{array}{c} Y \text{ affd perf'd space} \\ \downarrow \\ X = \mathrm{Spa}\, C. \end{array}$$

Can embed $Y \hookrightarrow S = \mathbb{B}_C^I$

$$Y = S_0 \subseteq S$$

$Y = \mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(C\langle X_i\rangle, \mathcal{O}_C\langle X_i\rangle)$.

$$\mathcal{O}_C\langle X_i \mid i \in I\rangle \twoheadrightarrow R^+$$

$$\begin{tikzcd}
S_0 \arrow[r, "="] \arrow[d, phantom, "\cap"] & Y \arrow[d] \\
S \arrow[r] & \mathrm{Spa}\, C
\end{tikzcd}$$

ess., condition says $\exists\ U \subseteq S$ cont. S_0,

retraction $S \to S_0 = Y$.

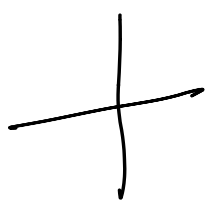
Y formally smooth $\overset{\text{étale locally}}{\Longleftrightarrow}$ retract of a space étale over a possibly inf.-dim'l ball.

Y formally smooth + "geom. fin.-dim'l" $\overset{\text{étale locally}}{\Longleftrightarrow}$ retract of a space étale over a fin.-dim'l.

ball.

Question. Assume Y aff'd perf'd space / $\mathrm{Spa}\, C$ that is a retract of a space étale over a fin-dim'l ball. Is Y cohomologically smooth?

Analogue fails for compact Hausdorff spaces:

 is ANR, not (cohom) smooth

But analogue is true for schemes
(implies fin. pres. + formal smoothness $\Rightarrow$ smooth.)

Theorem. $\mathcal{M}_Z^{sm} \longrightarrow S$ is formally smooth.

Sketch. to see: If $S_0 \subseteq S$ Zar. closed imm. of aff'd perf'd. spaces

$$\begin{tikzcd} X_{s_0} \arrow[r, "s_0"] \arrow[d, phantom, sloped, "\subset"] & Z \arrow[dl] \\ X_S & \end{tikzcd}$$

$s_0^* T_{Z/X_S}$ has only pos. HN slopes

then $\exists\ U \to S$ étale, image contains s_0,

s.th. s_0 lifts to $X_U \to Z$.

<u>idea</u>:

$$\begin{array}{ccccc} X_S & = & Y_{S,[1,q]} & / & Y_{S,[q,q]} \cong Y_{S,[1,1]}. \\ \uparrow & & \uparrow & & \\ Z & = & Z_{[1,q]} & / & Z_{[q,q]} \overset{\varphi_Z}{\cong} Z_{[1,1]}. \end{array}$$

Can arrange $Z_{[1,q]}$ affinoid, a small ball.

all information is in the iso φ_Z:

this preserves zero section over S_0,

can arrange that it is "very close" to linear (by localizing further).

Then do some "Banach fixed point like" argument to produce φ_Z - invariant sections of

$$
\begin{tikzcd}
 & Z_{S,[1,q]} \arrow[d] \\
Y_{S_0,[1,q]} \arrow[ur] \subseteq & Y_{S,[1,q]}
\end{tikzcd}
$$

using quantitative version of vanishing of $H^1(S_0^* T_{Z/X_S})$.