$D_{ét}(\mathrm{Bun}_G)$

next lecture: Friday, January 8.

(Course runs until Friday, February 12.)

Where are we? G/E reductive group, E nonarch. local field

defined moduli stack Bun_G of G-bundles on Fargues-Fontaine curve

$$S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q} \mapsto \{G\text{-bundles on } X_S\}.$$

Thm 1) Bun_G is an Artin v-stack, cohom. smooth of dimension 0.

2) map $|\mathrm{Bun}_G| \to B(G)$ continuous bijection,

$\rightsquigarrow$ for any $b \in B(G)$, get locally closed stratum $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$,

$$\mathrm{Bun}_G^b = [*/\mathcal{G}_b]$$

$$1 \to \text{"unipotent group diamond"} \to \mathcal{G}_b \to \underline{G_b(E)} \to 1$$

iterated ext'n of positive Banach-Colmez spaces.

$$\rightsquigarrow \quad [*/\underline{G_b(E)}] \to [*/\mathcal{G}_b] = \mathrm{Bun}_G^b$$

$$\mathrm{Bun}_{G_b}^1 \xrightarrow{\cong} [*/\underline{G_b(E)}]$$

cohom. smooth.

Artin v-stack

cohom. smooth

$\Rightarrow \mathrm{Bun}_G^b$ also a cohom. smooth Artin v-stack, of dimension $-\langle 2\rho, \nu_b \rangle$.

2ρ = sum of positive roots.

<u>Corollary</u>. $\pi_0 \operatorname{Bun}_G \xrightarrow[\kappa]{\cong} \pi_1(G)_\Gamma$.

Equivalently, each connected component of Bun_G is the closure of Bun_G^b for a unique basic $b \in B(G)$.

<u>Proof</u>. <u>enough</u>: Any nonempty open substack $U \subseteq \operatorname{Bun}_G$ contains a basic point.

(Then for any $b \in B(G)_{\text{basic}}$, any

$$\phi \neq U \subseteq \underbrace{\kappa^{-1}(\kappa(b))}_{} \underset{\text{open + closed}}{\subseteq} \operatorname{Bun}_G,$$

have $\operatorname{Bun}_G^b \subseteq U \Rightarrow \kappa^{-1}(\kappa(b))$ connected, b unique basic point in it)

Take minimal element $b \in B(G)$ s.th.

$\mathrm{Bun}_G^b \subseteq \mathcal{U}$.

b minimal $\rightsquigarrow$ $\mathrm{Bun}_G^b \subseteq \mathcal{U}$ open.

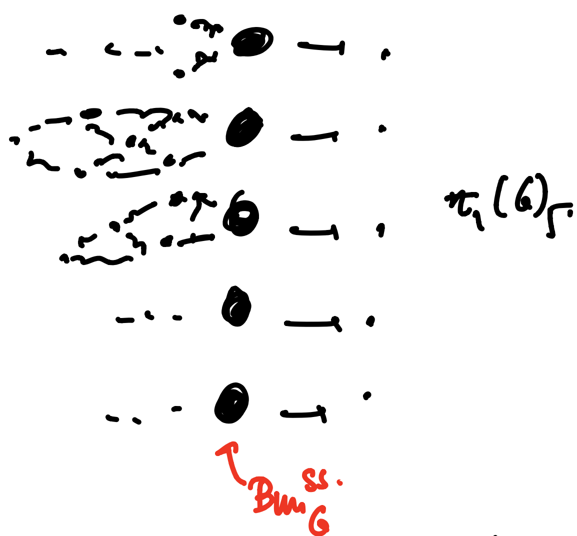
$\mathcal{U}$ coh. smooth of dim. 0 (open in Bun_G!)

open $\cup$

Bun_G^b coh. smooth of dim $-\langle 2\rho, \nu_b\rangle$.

$\Rightarrow \quad 0 = -\langle 2\rho, \nu_b\rangle \Rightarrow \nu_b$ central.

$\Leftrightarrow b$ basic $\square$.



$\mathrm{Bun}_G \longrightarrow \pi_1(G)_\Gamma$

$D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda) \circlearrowleft$ geometric Hecke operators.

$\rightsquigarrow$

L-parameters for Schur irred. objects in $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$.

repr. theory of all $G_b(E)$ $\hookrightarrow$

Proposition. For each $b \in B(G)$,

$$D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda).$$

derived category of ab. category of smooth representations of $G_b(E)$ on Λ-modules.

Sketch. **Step 0.**

$$D_{ét}(*, \Lambda) \cong D(\Lambda)$$

(For Step 0: What is easy is

$$D_{ét}(\mathrm{Spa}\, C, \Lambda) \cong D(\Lambda)$$

as $\mathrm{Spa}\, C$ spatial diamond of fin. coh. dim.

so $D_{ét}(\mathrm{Spa}\, C, \Lambda) \cong D(\underbrace{(\mathrm{Spa}\, C)_{ét}}_{\text{site of finite sets, topos = punctual topos}}, \Lambda) = D(\Lambda)$.

But $* = \mathrm{Spa}\, \overline{\mathbb{F}}_q$ not a diamond

need to analyze via descent along

$$\mathrm{Spa}\, C \to \mathrm{Spa}\, \overline{\mathbb{F}}_q .$$

$\leadsto$ **Prop.** For any small v-stack $X / \overline{\mathbb{F}}_q$, C complete alg. closed.

pullback

$$D_{ét}(X, \Lambda) \to D_{ét}(X \times_{\overline{\mathbb{F}}_q} \mathrm{Spa}\, C, \Lambda)$$

is fully faithful.

$$\Rightarrow \quad \begin{tikzcd} D_{ét}(*,\Lambda) \arrow[r, hook] & D_{ét}(\mathrm{Spa}\, C,\Lambda) \simeq D(\Lambda). \\ D(\Lambda) \arrow[u] \arrow[ur, "\simeq"] \arrow[u, phantom, shift left=6, "\circlearrowleft"] \end{tikzcd}$$

$$\Rightarrow D_{ét}(*,\Lambda) \simeq D(\Lambda).)$$

<u>Step 1</u>. $D_{ét}([*/\underline{G_b(E)}],\Lambda) \simeq D(G_b(E),\Lambda)$

holds for any locally pro-p-group H in place of $G_b(E)$.

also $D_{ét}([\mathrm{Spa}\, C/\underline{G_b(E)}],\Lambda) \simeq D(G_b(E),\Lambda)$.

(for Step 1: <u>idea</u>: use descent along

$$\mathrm{Spa}\, C \longrightarrow [\mathrm{Spa}\, C/\underline{G_b(E)}].$$

<u>better</u>:

$$D_{ét}([\mathrm{Spa}\, C/\underline{G_b(E)}],\Lambda)$$

$$\wr\wr$$

$$D(\underbrace{[\mathrm{Spa}\, C/\underline{G_b(E)}]_{ét}}_{\simeq \text{ site of sets with continuous } G_b(E)\text{-action}},\Lambda)$$

$\wr\wr$

$\mathcal{D}(G_b(E), \Lambda)$: Λ-modules on that site are smooth $G_b(E)$-rep. on Λ-modules.)

Step 2 $\quad D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \cong D_{\acute{e}t}([*/\underline{G_b(E)}], \Lambda)$:

$$\begin{tikzcd}
 & {[*/\underline{G_b(E)}]} & \\
{[*/\underline{G_b(E)}]} \arrow[ur, equal] \arrow[rr] & & \mathrm{Bun}_G^b . \arrow[ul]
\end{tikzcd}$$

coh. smooth,
fibers have trivial cohomology.

$$\begin{tikzcd}
\rightsquigarrow \quad D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \arrow[rr, hook] & \circlearrowleft & D_{\acute{e}t}([*/\underline{G_b(E)}], \Lambda) \\
 & D_{\acute{e}t}([*/\underline{G_b(E)}], \Lambda) \arrow[ul] \arrow[ur, "\simeq"] &
\end{tikzcd}$$

fully faithful.

$\Rightarrow$ Step 2.

Step 1 + Step 2 $\Rightarrow$ Claim. $\square$

Cor (of proof). For any $C/\overline{\mathbb{F}_q}$ complete alg. closed,

$$D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \xrightarrow{\cong} D_{\acute{e}t}(\mathrm{Bun}_G^b \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C, \Lambda)$$

Cor $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda) \cong D_{\acute{e}t}(\mathrm{Bun}_G \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C, \Lambda)$, and admits an infinite semiorthogonal decomposition with pieces

$$D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda).$$

Proof. Bun_G has stratification with pieces $i^b : \mathrm{Bun}_G^b \longrightarrow \mathrm{Bun}_G$.

functors $i^b_!, i^{b*}$ etc. induce semiorth. decomposition, on Bun_G, and $\mathrm{Bun}_G \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C$.

$$\left[\begin{array}{c} Z \overset{i}{\subseteq} X \overset{j}{\supseteq} U \\ \rightsquigarrow \begin{tikzcd} D_{et}(Z) \arrow[r, bend right, "i_*"'] & D_{et}(X) \arrow[l, "i^*"'] \arrow[r, bend right, "j^*"'] & D_{et}(U) \arrow[l, "j_!"'] \end{tikzcd} \end{array}\right]$$

know: $D_{et}(Bun_G, \Lambda) \hookrightarrow D_{et}(Bun_G \times_{\bar{\mathbb{F}}_q} Spa\, C, \Lambda)$,

but ess. image contains all

$i_!^b D_{et}(Bun_G^b \times_{\bar{\mathbb{F}}_q} Spa\, C, \Lambda)$, thus

everything. $\square$.

How strata interact is encoded in those spaces

$\pi_b : \mathcal{M}_b \longrightarrow Bun_G$ from last lecture.

<u>Thm</u> 1) $D_{et}(Bun_G, \Lambda)$ is compactly

generated, and a complex

$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compact

$\Leftrightarrow$ all $(i^b)^* A \in D_{ét}(\mathrm{Bun}_G^b, \Lambda) \simeq D(G_b(E), \Lambda)$

are compact; equiv., lie in thick

triang. subcategory generated by

$$\text{c-Ind}_K^{G_b(E)} \Lambda \qquad K \subseteq G_b(E) \text{ open pro-}p;$$

and almost all are $= 0$.

Compact objects in $D_{ét}(\mathrm{Bun}_G, \Lambda)$

are <u>not</u> Verdier self-dual:

smooth dual of $\text{c-Ind}_K^{G_b(E)} \Lambda$ is uncountably

dimensional.

already in pure repr. theory.

<u>Problem:</u> $\text{c-Ind}_K^{G_b(E)} \Lambda$ are not admissible:

$\dim_\Lambda \left(c\text{-Ind}_K^{G_b(E)} \Lambda \right)^{K'} = \infty$ in general.)

2) On $D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega \subseteq D_{ét}(\mathrm{Bun}_G, \Lambda)$ subcategory of compact objects, have Bernstein–Zelevinsky duality functor

$$\mathbb{D}_{BZ} : \left(D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega \right)^{op} \longrightarrow D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$$

s.th.

$$R\mathrm{Hom}(A, B) \cong \pi_\natural \left(\mathbb{D}_{BZ}(A) \overset{L}{\underset{\Lambda}{\otimes}} B \right)$$

Where $\pi : \mathrm{Bun}_G \to *$ proj.,

$\pi_\natural$ = left adjoint of π^*

(= twist of $R\pi_!$, could also take $R\pi_!$).

$$\mathbb{D}_{BZ}^2 \cong \mathrm{id}.$$

For $b \in B(G)$ basic, restricts to self-duality

on $D_{ét}(\mathrm{Bun}_G^b, \Lambda)^\omega \cong D(G_b(E), \Lambda)^\omega$,

and it restricts to usual Bernstein-Zelevinsky duality there.

$$\mathbb{D}_{BZ}(c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda) \cong c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda.$$

[<u>Recall</u>: If $\mathcal{C}$ triang. cat., then $X \in \mathcal{C}$ <u>compact</u> if $\mathrm{Hom}_{\mathcal{C}}(X,-)$ commutes with all direct sums.]

3) "$D_{ét}(\mathrm{Bun}_G)$-analogue of admissibility":

$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is <u>universally locally acyclic</u> (later) iff $\forall b \in B(G)$

(for $\mathrm{Bun}_G \to *$)

$$(i^b)^* A \quad \in D(G_b(E), \Lambda)$$

are <u>admissible</u> in the sense that for all open pro-p subgroups $K \subseteq G_b(E)$,

$$\left[(i^b)^* A\right]^K \in D(\Lambda)$$

is perfect (repr. by finite complex of finite proj. Λ-modules).

4) The class of sheaves in 3) is stable under Verdier duality

$$\mathbb{D}_{\mathrm{Bun}_G}(A) = R\mathcal{H}om(A, R\pi^! \Lambda),$$

and satisfy Verdier biduality:

$$A \xrightarrow{\cong} \mathbb{D}_{\mathrm{Bun}_G}(\mathbb{D}_{\mathrm{Bun}_G}(A)).$$

(This restricts to smooth duality on strata.
This is a general property of universally
locally acyclic sheaves.)

<u>Remark:</u> Ideally, would like to have
notion of "constructible complexes" on
Bun_G; these should be the compact
objects, and they should be universally
locally acyclic for $\mathrm{Bun}_G \to *$.

<u>This does not work!</u>

Theorem is best replacement, but note

compact $\not\Rightarrow$ univ. loc. acyclic
(ULA)

compact $\not\subseteq$ ULA

Both are seen on representations:

- $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ compact, but not admissible ($=$ULA).

- $\bigoplus_{i=1}^{\infty} \pi_i$, where π_i supercusp irr. repr. of $G_b(E)$, with growing conductor

$$\left(\Rightarrow \left(\bigoplus_{i=1}^{\infty} \pi_i\right)^K = \bigoplus_{i=1}^{N(K)} \pi_i^K, \quad N(K) < \infty\right)$$

admissible, but not compact.

<u>Warning</u>. There is also a notion of constr. complexes on (locally) spatial diamonds, by descent on small v-stacks.

(generated by $j_! \Lambda$, $j: U \to X$ qcqs étale map.)

But this is yet different, and almost no

$A \in D_{et}(\mathrm{Bun}_G, \Lambda)$ is constructible

in that sense. (all are local systems)

<u>Example.</u> $i: \mathrm{Spa}\, C \hookrightarrow X = \mathbb{D}_C \overset{j}{\hookleftarrow} U = \mathbb{D}^*_C$ closed unit disc

closed inclusion of origin.

Then $i_* \Lambda$ is <u>not</u> constructible.

<u>Problem:</u> $0 \to j_! \Lambda \to \Lambda \to i_* \Lambda \to 0$,

but j <u>not</u> quasicompact, i.e.

$\mathbb{D}^*_C$ <u>not</u> quasicompact.

In fact, constr. sheaves on rigid-analytic variety X are locally constant in an open nbhd of any classical point.

<u>Upshot:</u> Notions of "finitely generated" and of "admissible" representations, together with Bernstein-Zelevinsky duality and smooth duality, generalize to $D_{ét}(\mathrm{Bun}_G, \Lambda)$.

<u>Remark about coefficients:</u> So far, only allowed Λ s.th. $n\Lambda = 0$ for some n prime to p.

Ideally, want $\Lambda = \overline{\mathbb{Q}}_\ell$.

But passage from $\mathbb{Z}/\ell^n\mathbb{Z}$-coeff to $\mathbb{Z}_\ell$-coeff is more tricky than usual.

Can define

$$\mathcal{D}_{ét}(\mathrm{Bun}_G, \mathbb{Z}_\ell) := \varprojlim_n \mathcal{D}_{ét}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z}).$$

But this is related to representations on ℓ-adically complete $\mathbb{Z}_\ell$-modules.

$$\mathcal{D}_{ét}(*, \mathbb{Z}_\ell) = \varprojlim_n \mathcal{D}(\mathbb{Z}/\ell^n\mathbb{Z})$$

But want representations on discrete $\mathbb{Z}_\ell$-v.s.!

Usual trick: Use

$$\mathrm{Ind}\left(\varprojlim_n \mathcal{D}_{ét}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z})^{\omega}\right)$$

Then $\mathrm{Ind}\left(\varprojlim_n \mathcal{D}\,(\mathbb{Z}/\ell^n\mathbb{Z})^{\omega}\right)$

$$\shortparallel$$

$\mathcal{D}(\mathbb{Z}_\ell)$ derived ∞-category of $\mathbb{Z}_\ell$-modules.

(finite free $\mathbb{Z}_\ell$-modules are ℓ-adically complete.)

But this does not work here, as compact objects are not admissible.

Using idea of solid modules (w/ Dustin Clausen), we were able to define a version of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ for any $\mathbb{Z}_\ell$-algebra Λ, for which all assertions in this lecture hold true.

$$\begin{tikzcd} \mathbb{P}^1_E \setminus \mathbb{P}^1(E) = \underline{\Omega}^2 \arrow[r] \arrow[d] & \mathrm{Bun}_G^{\mathcal{O}(1/2)} \arrow[d, hook, "\text{open}"] \\ \mathbb{P}^1_E \arrow[r, "BL"', "\text{map.}"] & \mathrm{Bun}_G \end{tikzcd}$$

not q.c.   not q.c.

$f : Y \longrightarrow X$ cohom. smooth + "fibres contractible"

f^* fully faithful

$\Updownarrow$

$Rf^!$ $\rightsquigarrow$

$\Updownarrow$

$\forall A : A \xrightarrow{\sim} Rf_! Rf^! A.$

Can be checked fibrewise $\rightsquigarrow$ reduce to

A = constant sheaf.

Assume G split.

$$\forall\ V \in \mathrm{Sat}_G = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$$

$$\cong \mathrm{Rep}\,\hat{G}$$

geometric Satake.

$$\begin{tikzcd} & \mathrm{Hecke}_G \arrow[dl, "h_1"'] \arrow[dr, "h_2"] \arrow[r, "q"] & L^+G\backslash \mathrm{Gr}_G \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Div}_X^2 \end{tikzcd}$$

$$T_V = Rh_{2!}(h_1^* A \otimes q^* V):$$

$$D_{et}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{et}(\mathrm{Bun}_G \times \mathrm{Div}^2, \Lambda).$$

Hecke operators.

<u>Prop</u>. $D_{et}(\mathrm{Bun}_G \times \mathrm{Div}^2, \Lambda) \supseteq$

$$D_{et}(\mathrm{Bun}_G, \Lambda)^{W_E}$$

W_E - equiv. objects.

uses invariance of $D_{ét}(\mathrm{Bun}_G, \Lambda)$ under change of alg closed point:

$$\mathrm{Div}^1 = \left(\mathrm{Spa}\, \hat{\breve{E}}\right)^{\diamond} / \underline{W_E}.$$

$\rightsquigarrow$ excursion operators:

$$V_1, \dots, V_n \in \mathrm{Sat}_G \simeq \mathrm{Rep}\, \hat{G},$$

$$\alpha : \mathbb{1} \longrightarrow V_1 \otimes \dots \otimes V_n$$

$$\beta : V_1 \otimes \dots \otimes V_n \longrightarrow \mathbb{1}$$

$$\gamma_1, \dots, \gamma_n \in W_E.$$

excursion data.

$\rightsquigarrow \quad \forall A \in D_{ét}(\mathrm{Bun}_G, \Lambda):$

$$\begin{tikzcd}
A = T_{\mathbb{1}}(A) \arrow[r, "T(\alpha)"] \arrow[d] & T_{V_1\otimes\cdots\otimes V_n}(A) \arrow[d, "(\gamma_1,\ldots,\gamma_n)"] \\
A = T_{\mathbb{1}}(A) & T_{V_1\otimes\cdots\otimes V_n}(A) \arrow[l, "T(\beta)"']
\end{tikzcd}$$

W_E^n-equiv obj.

If $\mathrm{End}(A) = \overline{\mathbb{Q}_\ell}$, get scalars $\in \overline{\mathbb{Q}_\ell}$.

<u>Lemma</u> (V. Lafforgue)

$\exists !$ ss cont. $\rho_A : W_E \to \hat{G}(\overline{\mathbb{Q}_\ell})$
(up to conj.)
s.th. $\forall$ excursion data, this scalar is

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\alpha"] & V_1\otimes\cdots\otimes V_n \arrow[r, "(\rho_A(\gamma_1),\ldots,\rho_A(\gamma_n))"] & V_1\otimes\cdots\otimes V_n \arrow[d, "\beta"] \\
& & \mathbb{1}.
\end{tikzcd}$$