Smoothness.

Last time: Λ ring killed by n prime to p.

X small v-stack $\longmapsto D_{ét}(X,\Lambda)$

$(\cong D(X_{ét},\Lambda)$ if X locally spatial + $\dim.\mathrm{trg}\, X < \infty)$

closed symmetric monoidal triang. category, $-\overset{\mathbb{L}}{\underset{\Lambda}{\otimes}}-$, $R\mathrm{Hom}_\Lambda(-,-)$.

$f : Y \to X \rightsquigarrow f^*, Rf_*$.

If f repr. in locally spatial diamonds, compactifiable, (locally) $\dim.\mathrm{trg}\, f < \infty$

$\rightsquigarrow Rf_! : D_{ét}(Y,\Lambda) \longrightarrow D_{ét}(X,\Lambda)$

with a right adjoint $Rf^!$.

Why 'repr. in locally spatial diamonds + $\dim \mathrm{trg}\, f < \infty$'?

<u>Want:</u> $Rf_!$ commutes with all direct sums.

($\Leftrightarrow$ it has a right adjoint.)

If $A_n \in D_{et}^{\leq}(Y,\Lambda)$ conc. in degree 0 $(n \geq 0.)$

then $\bigoplus A_n[n] \cong \prod A_n[n]$ (by left-completeness)

$$Rf_! \left(\bigoplus A_n[n]\right) \cong Rf_! \left(\prod_n A_n[n]\right)$$

$$\cong \prod_n (Rf_! A_n)[n].$$

$$\overset{!}{\cong} \bigoplus_n (Rf_! A_n)[n].$$

this can only be true if $Rf_!$ has finite cohom. dimension.

So <u>need</u> finite cohomological dimension.

this follows from those assumptions.

Key:

<u>Thm</u> (Scheiderer '94). If T any spectral topological space $\Rightarrow$ coh.dim.$(T) \leq \dim(T)$.

Grothendieck,

(Krull)

drastically fails for compact Hausdorff spaces.

(Krull dim $= 0$, but may have ∞ coh. dim.)

e.g. $\prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}$.

so need "locally spatial" assumption to control cohomological dimension.

This satisfies all the properties of a "6-functor-formalism".

in practice, need to understand $Rf^!$.

<u>Definition</u>. Fix $\ell \neq p$. Then

$f : Y \longrightarrow X$ as above (compactif.,
repr. in loc. spat. d.,
loc. dim.trg $f < \infty$)

is ℓ-cohomologically smooth if, after any base change,

$$Rf^! \simeq \mathbb{D}_f \otimes f^* : \quad D_{ét}(X, \mathbb{F}_\ell) \to D_{ét}(Y, \mathbb{F}_\ell).$$

where $\mathbb{D}_f$ is locally isom. to $\mathbb{F}_\ell[n]$, $n \in \mathbb{Z}$.

<u>Remarks.</u> In that case, $\mathbb{D}_f = Rf^! \mathbb{F}_\ell$, formation commutes with any base change.

Conversely, if $Rf^! \mathbb{F}_\ell$ is invertible and its formation commutes with base change along $Y \longrightarrow X$, then f ℓ-cohom. smooth.

<u>Def.</u> $f : Y \to X$ cohom. smooth if ℓ-cohom. smooth $\forall \ell \neq p$.

then also $Rf^! \Lambda$ loc. isom. to $\Lambda[n]$, any Λ.

$$\wr$$

$$Rf^!\mathbb{Z}/n\mathbb{Z} \overset{L}{\underset{\mathbb{Z}/n\mathbb{Z}}{\otimes}} \Lambda .$$

Examples.

1). $\mathbb{B}_C^\diamond \longrightarrow (\operatorname{Spa} C)^\diamond$ is cohom. smooth.

follows from results of Huber.

2) If $f : Y \longrightarrow X$ smooth map of analytic adic spaces $/\mathbb{Z}_p$,

locally étale over a ball

then $f^\diamond : Y^\diamond \longrightarrow X^\diamond$ is cohom. smooth.

3) $(\operatorname{Spa} E)^\diamond \longrightarrow (\operatorname{Spa} \mathbb{F}_q)^\diamond$ is cohom. smooth.

$(\mathrm{Div}^1_q)^d / \Sigma_d$.

$\mathrm{Div}^d \longrightarrow *$ is cohom. smooth.

$(\operatorname{Spa} \mathcal{O}_E)^\diamond \longrightarrow (\operatorname{Spa} \mathbb{F}_q)^\diamond$ cohom. smooth.

4) If $f: Y \longrightarrow X$ cohom. smooth,

G pro-p-group acting freely on Y/X

then $f/\underline{G} : Y/\underline{G} \longrightarrow X$ is still cohom. smooth.

Converse very much false!

ex. $$\underset{\substack{\circlearrowleft \\ G.}}{Y} = \mathrm{Spa}\, C \times \underline{G} \xrightarrow[\text{\underline{not} cohom. smooth.}]{} X = \mathrm{Spa}\, C.$$

$$Y/\underline{G} = \mathrm{Spa}\, C \xrightarrow{\sim} \mathrm{Spa}\, C.$$

5) $$\underset{\text{open Schubert cell}}{\mathrm{Gr}_{G,\mu}} \xrightarrow[\text{cohom. smooth.}]{} (\mathrm{Spa}\, E)^\diamond$$

$$\cong L^+G / \underset{\text{stabilizer of } [\mu] \in \mathrm{Gr}_G.}{(L^+G)_\mu}.$$

but not $\mathrm{Gr}_{G,\le\mu}$ in general!

6) $\mathcal{E}$ vector bundle on X_S,
all fibres have only positive HN slopes,
(>0)
then $\mathcal{BC}(\mathcal{E}) \longrightarrow S$

cohom. smooth.

not true for slope ≥ 0:

$$\mathcal{E} = \mathcal{O}_{X_S}, \quad \mathcal{BC}(\mathcal{E}) = \underline{E},$$

not cohom. smooth.

7) Being cohom smooth can be checked
v-locally on target.
(+ cohom. smooth local on source.)

All these statements can be proved by geometric reductions to balls.

Artin stacks.

Definition. A small v-stack X is Artin

if - $\Delta_X : X \to X \times X$ is repr. in locally spatial diamonds

- $\exists\, f : Y \longrightarrow X$ cohom. smooth surjection. with Y locally spatial diamond.

Schemes ⇝ alg. spaces → Artin stacks.

perf'd spaces → locally spatial diamonds → Artin v-stacks.

For Artin v-stack X, can define

'$X \to *$ cohom. smooth":

means for $f : Y \to X$ as in Def'n,

$Y \to X \to *$ is cohom.-smooth.

Can extend dualizing complex to such X

⇝ notion of dimension by looking at the degree of dualizing cplx.

Thm Bun_G is a cohom. smooth Artin v-stack (of dim 0 in sense $R\pi^!\Lambda$ locally isom. to $\Lambda[0]$).

$\pi : \mathrm{Bun}_G \to *$

Sketch. Show

$$\mathrm{Gr}_{G,\mu} / \underline{G(E)} \longrightarrow \mathrm{Bun}_G$$

fibres are open in $\mathrm{Gr}_{G,\mu^{-1}}$.

cohom. smooth of dim $\langle 2\rho, \mu \rangle$.

and $\mathrm{Gr}_{G,\mu} / \underline{G(E)}$ is also cohom. smooth of dim $\langle 2\rho, \mu \rangle$.

⇒ open image of this map has desired property.

(use: cohom. smooth → open.)

Take U_μ : covers Bun_G. $\square$

In particular, $[*/\underline{G(E)}] \simeq \mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$

is cohom. smooth $/*$.

to study Bun_G, need better smooth atlas.

<u>Example</u>. $G = GL_2$.

$$\underbrace{\boxed{\overset{\mathcal{O}(1/2)}{\bullet} \rightsquigarrow \overset{\mathcal{O}\oplus\mathcal{O}(1)}{\bullet}}}_{\mathcal{U}} \subseteq \mathrm{Bun}_G .$$

Want nice atlas for $\mathcal{U}$.

$$\begin{tikzcd} b \in B(G) \arrow[d, leftrightarrow] \\ \mathcal{O}\oplus\mathcal{O}(1). \end{tikzcd}$$

<u>Def'n</u>.

Let $\mathcal{M}_b$ be the moduli space of

extensions

$$0 \to \mathcal{L} \to \mathcal{E}_e \to \mathcal{L}' \to 0$$

where $\mathcal{L}$ line bundle of degree 0 and $\mathcal{L}'$ line bundle of degree 1.

$$\rightsquigarrow \pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_{GL_2} = \mathrm{Bun}_G.$$

$$(\mathcal{L} \to \mathcal{E}_e \to \mathcal{L}') \longmapsto \mathcal{E}_e.$$

Thm. π_b cohom. smooth.

Structure of $\mathcal{M}_b$:

$$\mathcal{M}_b = \tilde{\mathcal{M}}_b / \underline{E}^\times \times \underline{E}^\times$$

$\tilde{\mathcal{M}}_b$ param. extensions

$$0 \to \mathcal{O} \to \mathcal{E}_e \to \mathcal{O}(1) \to 0.$$

$$\Rightarrow \tilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1]).$$

negative Banach-Colmez spaces

$$\begin{tikzcd} \widetilde{\mathcal{M}}_b^\circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{M}}_b \arrow[d] \arrow[r, phantom, "\supseteq"] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ \mathrm{Bun}_G^{b'} \arrow[r, phantom, "\overset{j}{\subseteq}"] & \mathcal{U} \arrow[r, phantom, "\overset{i}{\supseteq}"] & \mathrm{Bun}_G^{b} = [*/\mathcal{G}_b]. \end{tikzcd}$$

$i^* Rj_*.$

$\mathcal{U} \cap \mathrm{Bun}_G.$

$b' \cong \mathcal{O}(1/2)$

$1 \to \mathcal{BC}(\mathcal{O}(1)) \to \mathcal{G}_b \to \underline{E^*} \times \underline{E^\times} \to 1$

$$\widetilde{\mathcal{M}}_b^\circ = \mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\}$$

$$\cong (\mathrm{Spa}\, k((t)))^\diamond / \underline{SL_1(D_{1/2})}$$

$D_{1/2} / E$ quat. algebra.

(trivialize $\mathcal{E}_b \cong \mathcal{O}(1/2)$,

use $\mathcal{BC}(\mathcal{O}(1/2)) \setminus \{0\}$)

$\wr\wr$

$\mathrm{Spa}\, k((t))$

__In general:__ will define cohom. smooth map

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G .$$

"strict henselization of Bun_G

at $[*/\underline{G_b(E)}] \to \mathrm{Bun}_G^b \to \mathrm{Bun}_G$"

coh. smooth.

$$\begin{tikzcd}
\tilde{\mathcal{M}}_b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[d] \\
* \arrow[r] \arrow[u, dashed, bend left, "\tilde{s}"] & {[*/\underline{G_b(E)}]} \arrow[u, dashed, bend left, "s"]
\end{tikzcd}$$

section = preimage of $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$.

$$\tilde{\mathcal{M}}_b^\circ = \tilde{\mathcal{M}}_b \setminus * \longrightarrow \mathrm{Bun}_G^{>b} \subseteq \mathrm{Bun}_G$$

spatial diamond.

__Note:__ $\tilde{\mathcal{M}}_b \longrightarrow *$ is repr. in loc. spatial diamonds,

but $\tilde{\mathcal{M}}_b$ is __not__ a locally spatial diamond.

example. $\operatorname{Spa} k[\![t]\!] \longrightarrow *$ is repr. in loc. spatial diam.

$$\begin{tikzcd} \operatorname{Spa} k[\![t]\!] \\ \text{functor: } \operatorname{Spa}(R, R^+) \mapsto R^{\circ\circ} \arrow[u] \end{tikzcd}$$

base change to any S is open unit disc/S.

but $\operatorname{Spa} k[\![t]\!]$ not a diamond, as it has non-analytic point $\operatorname{Spa} k \subseteq \operatorname{Spa} k[\![t]\!]$.

$$\operatorname{Spa} k[\![t]\!] \setminus \operatorname{Spa} k = \operatorname{Spa} k(\!(t)\!)$$

is a spatial diamond.

Definition of $\mathcal{M}_b$:

For GL_n, b corresponds to a $\mathbb{Q}$-graded vector bundle $\mathcal{E}_b = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{E}_b^{\lambda}$,

$\mathcal{E}_b^{\lambda}$ ss. of slope λ.

$\mathcal{M}_b$: param. iterated extensions of the $\mathcal{E}^{\lambda}$.

In general:

Definition. Let $\mathcal{M}$ moduli space taking $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to exact $\otimes$-functors from $\mathrm{Rep}_E G$ to exact cat. of increasing $\mathbb{Q}$-filtered vector bundles $\mathcal{E}_S \supseteq \mathcal{E}_S^{\leq \lambda}$ s.th. each $\mathcal{E}_S^{\lambda} = \mathcal{E}^{\leq\lambda} / \bigcup_{\lambda' < \lambda} \mathcal{E}^{\leq \lambda'}$ is sst. of slope λ.

This is "opposite" to HN filtration!

$$\begin{tikzcd}
\mathcal{M} \arrow[r] & \mathrm{Bun}_G \\
(\mathcal{E} \supseteq \mathcal{E}^{\leq\lambda}) \arrow[r, mapsto] & \mathcal{E}
\end{tikzcd}$$

$$\bigsqcup_{b\in B(G)} [*/\underline{G_b(E)}] \longrightarrow \left(\bigoplus \mathcal{E}^{\lambda}\right)$$

exact ⊗-functor w/ values in

isocrystals $\cong$ $\mathbb{Q}$-graded vector bundles, $\mathcal{E}^{\lambda}$ sst. of slope λ.

$$\rightsquigarrow \quad \mathcal{M} = \bigsqcup_{b\in B(G)} \mathcal{M}_b ,$$

$$\mathcal{M}_b \longrightarrow [*/\underline{G_b(E)}] .$$

Theorem. $\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$ are cohom. smooth.

Remark. For $G = GL_n$: Can be proved by direct attack.

no naive approach for general G.

Deduce it from a general "Jacobian criterion".

$$\mathbb{P}^1/\underline{GL_2(E)} \xrightarrow[\text{smooth}]{\text{coh.}} \mathrm{Bun}_{GL_2}.$$

image $= \mathcal{U}$

$\mathcal{O}(1/2),\ \mathcal{O}\oplus\mathcal{O}(1).$

strata $=$

$$\begin{tikzcd}
\underline{\mathbb{P}^1(E)}/\underline{GL_2(E)}, \arrow[dr, hook, "i"] & \\
& \mathbb{P}^1/\underline{GL_2(E)}. \\
\Omega^2/\underline{GL_2(E)}. \arrow[ur, hook', "j"'] &
\end{tikzcd}$$

$$(i^* Rj_* \mathcal{F})_{\bar{x}} = \varinjlim_{\substack{\text{small balls } U \\ \text{around } \mathbb{P}^1(E)}} R\Gamma\underbrace{\big((\mathcal{U}\setminus\mathbb{P}^1(E))/\ldots,\ \mathcal{F}\big)}_{\varprojlim R\Gamma(V/\ldots,\ \mathcal{F})}.$$

$V \subseteq U \setminus \mathbb{P}^1(E)$

qc open

In $\mathcal{M}_b$ - chart,

see $(i^* Rj_* \mathcal{F})_{\bar{x}} = R\Gamma(\tilde{\mathcal{M}}_b^{\circ}, \mathcal{F})$

Spatial, fin.-dim'l diamond.

Can be made very explicit.

$f : Y \to X$ cohom. smooth, qcqs.

$\Rightarrow$ $Rf_!$ preserves constructible sheaves

(as $Rf^!$ preserves direct sums, so $Rf_!$ preserves compact objects)

$\Rightarrow$ $Rf_! \mathbb{F}_\ell$ is constructible,

in part., $\underset{\text{image of } f.}{\underset{\parallel}{\underline{\text{support}}}}$ is (qc) open.

Thm (Drinfeld-Simpson) $\mathrm{Bun}_P^{\text{generic}} \longrightarrow \mathrm{Bun}_G$ is smooth.

$P \subseteq G$ parabolic.

$$\begin{tikzcd}
R\Gamma((\mathrm{Spa}\, k[[t_1, \dots, t_d]])^{\eta}, A) \arrow[d, "\wr"] \\
A_s
\end{tikzcd}$$

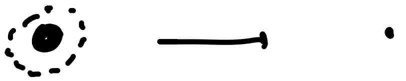
s = closed point.

Similarly for $\tilde{\mathcal{M}}_b$.

$$\begin{tikzcd}
\mathrm{Spa}\,\mathbb{F}_p[t] \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & {[*/\mathrm{Spa}\,\mathbb{F}_p[t]]}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spa}\ C' \arrow[r] & \mathrm{Spa}\ C
\end{tikzcd}$$

usually not compactifiable.



$$\begin{tikzcd}
Y \arrow[r] & Y/\underline{G} \\
\underline{G} \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd}$$

<u>not</u> cohom. smooth.

<u>Example</u>. $\mathcal{E}$ vector bundle on X_S, all fibers have only <u>negative</u> slopes, then

$$\mathcal{BC}(\mathcal{E}[1]) \longrightarrow S$$

is cohom. smooth.