Étale Cohomology of Diamonds.

Goal: Set the foundations to define

$$D(\mathrm{Bun}_G, \mathbb{Z}_\ell) \qquad \ell \neq p.$$

$\rightarrow$ ℓ-adic étale sheaves on general small v-stacks.

was developed in 'Étale Cohomology of diamonds'

at least with torsion coefficients prime to p.

today: Brief summary of that paper.

Recall: Perf = perfectoid spaces of char. p.

endow with v-topology.

Small v-sheaf: small colimit of representable sheaves

$\Longleftrightarrow$ a v-sheaf $\mathcal{F}$ s.th.

$\exists$ surjection $X \longrightarrow \mathcal{F}$ for some perfectoid space X.

($\mathcal{F} = X/R$, $R \subseteq X \times X$ equiv. relation, sub-v-sheaf, .)

automatically small.

similarly, "small v-stack".

<u>General idea</u>: Always use descent to strictly totally disconnected spaces.

Let Λ be a ring killed by some integer n prime to p.

<u>Goal</u>: For each X small v-stack, define triangulated Λ-linear category

$D_{ét}(X, \Lambda)$ + 6 functors:

1) If $f : Y \to X$ any map of small
v. stacks,

- $f^* : D_{ét}(X, \Lambda) \to D_{ét}(Y, \Lambda)$

with a right adjoint

- $Rf_* : D_{ét}(Y, \Lambda) \longrightarrow D_{ét}(X, \Lambda)$

2) Any $D_{ét}(X, \Lambda)$ has a symmetric monoidal $\otimes$-product

- $- \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} -$ & f^* preserves $- \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} -$,

& has partial right adjoint

- $R\mathcal{H}om_\Lambda(-,-) : D_{ét}(X,\Lambda)^{op} \times D_{ét}(X,\Lambda) \to D_{ét}(X,\Lambda)$

$$\mathrm{RHom}_{D_{ét}(X,\Lambda)}(A, R\mathcal{H}om_\Lambda(B, C)) \cong \mathrm{RHom}_{D_{ét}(X,\Lambda)}(A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} B, C).$$

3) If $f : Y \to X$ rep. in locally spatial

diamonds, compactifiable, $\dim \operatorname{trg} f < \infty$ (locally),

geometric transcendence dimension

a functor

- $Rf_! : D_{et}(Y, \Lambda) \longrightarrow D_{et}(X, \Lambda)$

satisfying a projection formula

$$Rf_!\left(A \overset{L}{\underset{\Lambda}{\otimes}} f^*B\right) \simeq Rf_! A \overset{L}{\underset{\Lambda}{\otimes}} B$$

$A \in D_{et}(Y, \Lambda)$, $B \in D_{et}(X, \Lambda)$,

and base change, and
(*)
admitting a right adjoint

- $Rf^! : D_{et}(X, \Lambda) \longrightarrow D_{et}(Y, \Lambda)$.

Really, only need three: f^*, $- \overset{L}{\underset{\Lambda}{\otimes}} -$, $Rf_!$

the others as right adjoints.

(*): $\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$ Cartesian diagram, f so that $Rf_!$ defined

$\Rightarrow$ $g^* Rf_! \simeq Rf'_! g'^*$: $D_{ét}(Y, \Lambda) \to D_{ét}(X', \Lambda)$.

Actually, all $D_{ét}(X,\Lambda)$ arise as homotopy categories of stable Λ-linear ∞-categories $\mathcal{D}_{ét}(X,\Lambda)$ and all functors are defined on this level.

<u>Descent:</u> $X \mapsto \mathcal{D}_{ét}(X,\Lambda)$ is a v-sheaf of ∞-categories.

In particular, if $Y \to X$ v-cover,

$$\mathcal{D}_{ét}(X,\Lambda) \xrightarrow{\sim} \varprojlim \Big(\mathcal{D}_{ét}(Y,\Lambda) \rightrightarrows \mathcal{D}_{ét}(Y \times_X Y, \Lambda) \substack{\to\\ \to\\ \to} \mathcal{D}_{ét}(Y \times_X Y \times_X Y, \Lambda) \substack{\to\\ \to\\ \to\\ \to} \dots \Big).$$

Even a hyper-v-sheaf.

$$\dots Y_2 \substack{\to\\ \to\\ \to} Y_1 \rightrightarrows Y_0 \to X$$

hyper-v-cover

$$\rightsquigarrow \quad \mathcal{D}_{ét}(X,\Lambda) \xrightarrow{\sim} \varprojlim_{\Delta} \mathcal{D}_{ét}(Y_\bullet, \Lambda).$$

$\Rightarrow$ enough to define $\mathcal{D}_{ét}(X,\Lambda)$ for X strictly totally disconnected.

<u>Def</u>. $\mathcal{D}_{ét}(X,\Lambda) := \mathcal{D}(X_{ét}, \Lambda)$

derived ∞-category of abelian category of étale Λ-modules on X.

+ symmetric monoidal $-\overset{L}{\underset{\Lambda}{\otimes}}-$

+ pullback functoriality.

From here, we already get well-defined

$\mathcal{D}_{et}(X,\Lambda)$ for any small-v-stack X

+ $-\overset{L}{\underset{\Lambda}{\otimes}}-$ + f^*.

<u>Proposition.</u> f^* admits a right adj. Rf_*,

$-\overset{L}{\underset{\Lambda}{\otimes}}-$ $\rightsquigarrow$ partial right adj. $R\mathcal{H}om_\Lambda(-,-)$.

<u>Sketch:</u> $\mathcal{D}_{et}(X,\Lambda)$ presentable stable ∞-category.

Then apply Lurie's adjoint functor theorem. $\square$.

<u>Question:</u> What is this?

Assume X locally spatial diamond.

$\rightsquigarrow$ has site $X_{ét}$.

($f: Y \to X$ is étale if it is locally separated and for all perf'd spaces $X' \to X$,

$Y \times_X X' \longrightarrow X'$ is repr. by a perf'd space, and étale $/X'$.)

<u>Theorem</u>. There is a natural functor

$$\mathcal{D}(X_{ét}, \Lambda) \longrightarrow \mathcal{D}_{ét}(X, \Lambda),$$

$\stackrel{\text{def}}{=}$ derived ∞-category of abelian cat. of étale Λ-modules on X

induces an equiv.

$$\mathcal{D}^+(X_{ét}, \Lambda) \cong \mathcal{D}^+_{ét}(X, \Lambda),$$

and $\mathcal{D}_{\text{ét}}(X,\Lambda)$ is the left-completion

$$\mathcal{D}_{\text{ét}}(X,\Lambda) \simeq \varprojlim_n \mathcal{D}_{\text{ét}}^{\geq -n}(X,\Lambda) \cong \varprojlim_n \mathcal{D}^{\geq -n}(X_{\text{ét}},\Lambda).$$

In particular, if $\mathcal{D}(X_{\text{ét}}, \Lambda)$ is left-complete, for example under "finite cohomological dimension of $X_{\text{ét}}$", then

$$\mathcal{D}(X_{\text{ét}},\Lambda) \xrightarrow{\cong} \mathcal{D}_{\text{ét}}(X,\Lambda).$$

<u>Sketch</u>. • $\mathcal{D}_{\text{ét}}(X,\Lambda)$ always has natural t-str., is left-complete.

(reduce to str. tot. disc. X, where cohom. dim. $=0$.)
$\Rightarrow$ left-complete.

$\rightsquigarrow$ enough:

$$\begin{tikzcd} \mathcal{D}^{\geq 0}(X_{ét}, \Lambda) \arrow[r, "\cong"] \arrow[dr, hook, "\lambda^*"'] & \mathcal{D}^{\geq 0}_{ét}(X, \Lambda). \arrow[d, phantom, "\cong" rotate=90] \\ & \mathcal{D}^{\geq 0}(\underline{X_v}, \Lambda). \end{tikzcd}$$

↑ maybe with (cardinal cutoff)

$\lambda : X_v \longrightarrow X_{ét}$

<u>Key</u>: λ^* fully faithful; equiv.

for all étale Λ-modules $\mathcal{F}$,

$$\mathcal{F} \xrightarrow{\cong} \lambda_* \lambda^* \mathcal{F}$$

and $\quad R^i \lambda_* \lambda^* \mathcal{F} = 0 \quad$ for $i > 0$.

"invariance of cohom. under passage from étale site to v-site".

<u>Sketch</u>. 1) étale $\rightsquigarrow$ pro-étale:

write pro-étale covers as cofiltered limits of étale covers

$\rightsquigarrow$ on cohomology, get filtered colimits;
filtered colimits are exact.

2) pro-étale $\rightsquigarrow$ v: By pro-étale descent, can assume X str. tot. disc.

If $Y \longrightarrow X$ v-cover, can write

aff'd perf'd $\quad Y = \varprojlim_i Y_i \to X$

where each $Y_i \to X$ is open in some fin. dim'l ball over X:

$X = \mathrm{Spa}(R, R^+)$

$Y = \mathrm{Spa}(S, S^+)$

$\exists \quad R\langle T_i^{1/p^\infty} \mid i \in I \rangle \longrightarrow\!\!\!\rightarrow S$ for some set I.

$$Y = V(f_j \mid j \in J) \subseteq \mathbb{B}_X^I .$$

This is a limit of

$$\{|f_j| \leq |\varepsilon|,\ j \in J' \subseteq J\} \subseteq \mathbb{B}_X^{I'}$$

take these as the Y_i''s above. $\qquad I' \subseteq I$ finite, $J' \subseteq J$ finite.

descent for $Y \to X$ reduces to descent for all $Y_i \to X$, but this has a section ($Y_i \to X$ smooth cover, X str. tot. disc.)

$\Rightarrow$ descent automatic. $\square$.

<u>Proposition</u>. For all small v-stacks X,

$$D_{et}(X, \Lambda) \hookrightarrow D(X_v, \Lambda)$$

+ $A \in D(X_v, \Lambda)$ lies in image

$\Rightarrow$ all $\mathcal{H}^i(A)$ lie in image,

and this can be checked after pullback

to any locally spatial diamond $Y \to X$.

then means: comes from étale Λ-module on Y.

Base Change.

Theorem. (qcqs base change)

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

cartesian diagram of locally spatial diamonds, f qcqs.

$\Rightarrow$ $Rf'_* g'^* \cong g^* Rf_*$:

$$\mathcal{D}^+_{et}(Y, \Lambda) \to \mathcal{D}^+_{et}(X', \Lambda).$$

If f has finite cohom. dimension, even on

$$\mathcal{D}_{et}(Y, \Lambda) \to \mathcal{D}_{et}(X', \Lambda).$$

no properness needed here.

But: for strictly local space

$\mathrm{Spa}(C, C^+)$, do not have base

change wrt.

$$\begin{array}{ccc} \{s\} & \hookrightarrow & \mathrm{Spa}(C, C^+). \\ \| & & \| \\ (X') & & X. \end{array}$$

not an adic space.

Sketch. If $X' \to X$ pro-étale, automatic.

$\Rightarrow$ Can assume X, X' strictly local.

also, using descent on Y, can assume

Y str. tot. disc., even strictly local.

$X = \mathrm{Spa}(C, C^+)$, $X' = \mathrm{Spa}(C', C'^+)$,

$Y = \mathrm{Spa}(\tilde{C}, \tilde{C}^+)$. $\quad R = C' \hat{\otimes}_C \tilde{C} \supseteq R^+$.

$Y' = \mathrm{Spa}(R, R^+)$

Lemma. $H^i(Y', \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & i=0 \\ 0 & i>0. \end{cases}$

This follows from "invariance of cohom under change of alg. closed base field" by Huber.

$\square$.

Theorem (Proper Base Change) If f is addition in proper, also get these kinds of base changes.

This is required to define $Rf_!$ later

Sketch. reduced to proper base change for schemes by similar reductions. $\square$.

$Rf_!$.

If $f : Y \to X$ rep. in spatial diamonds, compactifiable, dim.trg $f < \infty$.

$$\begin{tikzcd} j: Y \arrow[r, hook, "\text{open}"] \arrow[dr] & \overline{Y}^{/X} \arrow[d] \\ & X \end{tikzcd}$$

where $\overline{Y}^{/X}(R, R^+)$
$\qquad\quad \|$
$X(R,R^+) \times_{X(R,R^\circ)} Y(R,R^\circ)$

$$\begin{array}{c} \mathrm{Spa}(R,R^\circ) \\ \downarrow \\ \mathrm{Spa}(R,R^+) \end{array}$$

<u>recall:</u> $Y \to X$ proper iff.

$$\begin{tikzcd} \mathrm{Spa}(R,R^\circ) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spa}(R,R^+) \arrow[r] \arrow[ur, dotted, "\exists!\,?"] & X \end{tikzcd}$$

$\Longleftrightarrow \quad Y \xrightarrow{\sim} \overline{Y}^{/X}$ is an isomorphism

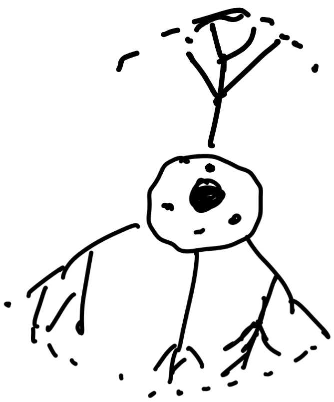
For general $Y \to X$ as above,

$\overline{Y}^{/X}$ is proper, and the initial proper diamond $/X$ with a map from Y.

<u>Def'n</u>. $Rf_! = Rf^{-/X}_* \circ j_! : D_{et}(Y,\Lambda) \to D_{et}(X,\Lambda)$.

($j_!$ = ext. by zero = left adj. of j^*.)

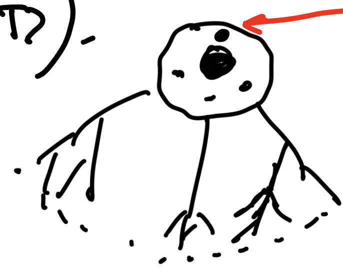
<u>Example.</u> $Y = \mathbb{B}_C \longrightarrow X = \operatorname{Spa} C.$

 $\subseteq$ 

$$\operatorname{Spa}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle) = \mathbb{B}_C \subseteq \mathbb{P}^1_C$$

$$\mathbb{B}_C \subseteq \overline{\mathbb{B}}_C^{/C} = \operatorname{Spa}(C\langle T\rangle, \mathcal{O}_C + \mathfrak{m}_C\langle T\rangle).$$

$\overline{\mathbb{B}}_C^{/C} =$



one extra rk 2 point, where $|T|>1$, but only infinitesimally so.

more generally, can be defined if f is only repr. in locally spatial diamonds

(+ compactifiable, dim.trg $f < \infty$.)

<u>Theorem</u>. $Rf_!$ satisfies base change + proj. formula. + composes.

easy from Proper Base Change.
(↳ comm. of $j_!$ & Rf_* prop f)

<u>Proposition</u>. $Rf_!$ has a right adjoint $Rf^!$.

<u>Verdier duality</u>.

<u>Def'n</u>. $f : Y \longrightarrow X$ (repr. in loc. spat. diam., compactif., dim.trg $f < \infty$)

is (Λ-) <u>cohomologically smooth</u>

if

$$Rf^! A \simeq f^* A \otimes \mathbb{D}_f$$

for some invertible $\mathbb{D}_f \in D_{et}(Y, \Lambda)$

(+ after any base change.)

<u>Nontrivial Task!</u> Find many examples.

next time.

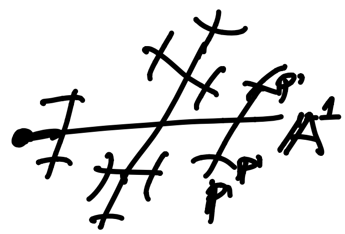
X usual qcqs rigid space / C.

Prop $|X^{ad}| \xrightarrow[(sp_{\mathfrak{X}})_{\mathfrak{X}}]{\cong} \varprojlim_{\mathfrak{X}} |\mathfrak{X}|$

formal model of X

$$|\overline{X}^{/C}| \xrightarrow{\cong} \varprojlim_{\mathfrak{X},\ \mathfrak{X}_s \hookrightarrow \overline{\mathfrak{X}}_s \text{ proper}} |\overline{\mathfrak{X}}_s|.$$

$X = \mathbb{B}_C.$



$$f: \underline{S} \times \mathrm{Spa}\, C \longrightarrow \mathrm{Spa}\, C.$$

$$\underline{S} \times \mathrm{Spa}\, C = \mathrm{Spa}\, \mathrm{Cont}(S, C) \qquad \text{pro-étale}.$$

S profinite.

If S is infinite, f not cohom. smooth.

$\rightsquigarrow$ pro-étale $\not\Rightarrow$ smooth.

$$R\Gamma(S, Rf^! \Lambda) = R\mathrm{Hom}(Rf_! \Lambda, \Lambda)$$

$$\|$$

$$\mathcal{M}(S,\Lambda) \mathbin{/\!/} = R\mathrm{Hom}(\underbrace{Rf_*\Lambda}_{\mathrm{Cont}(S,\Lambda)}, \Lambda)$$

Λ- valued measures on S.

$$(Rf^!\Lambda)_s = \varinjlim_{U \ni s} \mathcal{M}(U,\Lambda).$$

If s not an isolated pt, this is $\neq \Lambda$.

- $[*/\underline{G}] \longrightarrow *$

$\underline{G}$ locally pro-p.

is coh. smooth

(once this notion is extended to stacks.)

$\mathrm{CHaus} \hookrightarrow$ diamonds $/ \mathrm{Spa}\, C$.

$X \longmapsto \underline{X}$

<u>Prop</u>. $\quad D^+_{et}(\underline{X}, \Lambda) \cong D^+(X, \Lambda)$.

bounds Krull dimension

bounds cohom. dimension.

$\dim \mathrm{trg}\, f$: defined in terms of geom. pts.

$\sup(\text{top.tr.dg.})$

$$\begin{array}{ccc} \bar{y} = \mathrm{Spa}\, C(\bar{y}) & & \\ \downarrow & & \text{top.tr.dg.}\big(C(\bar{y})/C(\bar{x})\big) \\ \bar{x} = \mathrm{Spa}\, C(\bar{x}). & & \end{array}$$

$\mathcal{F}$ pro-étale sheaf

$\nu : X_v \longrightarrow X_{proet}$

$\Rightarrow R^i\nu_*\nu^*\mathcal{F} = 0$ for $i>0$? .

Very likely false.

$$\begin{array}{ccc} \mathrm{Spa}(R,R^+) & \overset{i}{\subseteq} & \mathrm{Spa}(R,R'^+) \\ \| & & \| \\ X & & X' \end{array}$$

$\mathcal{F}$ étale sheaf on X'

$$\Rightarrow H^i_{\text{ét}}(X',\mathcal{F}) \xrightarrow{\sim} H^i_{\text{ét}}(X,\mathcal{F}|_X).$$

also, for all $\mathcal{G}$ on X,

$$H^i_{\text{ét}}(X,\mathcal{G}) \cong H^i_{\text{ét}}(X', i_*\mathcal{G}).$$

But this fails completely for cohomology with compact support!

$$X = \mathbb{B}_{\mathbb{C}} \subset X' = \overline{\mathbb{B}}_{\mathbb{C}}.$$

$$\begin{array}{ccc} H^i_c(X, \Lambda) & \longrightarrow & H^i_c(X', \Lambda) \\ \| & & \| \\ \begin{cases} 0 & i \neq 2 \\ \Lambda(-1) & i = 2. \end{cases} & & H^i(X', \Lambda) = \begin{cases} \Lambda & i = 0 \\ 0 & i > 0 \end{cases} \end{array}$$