Bun_G.

E .. $\mathbb{F}_q$ ... $\overline{\mathbb{F}}_q$ ... $\breve{E}$

G/E reductive group.

$\rightsquigarrow$ Bun_G v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$S \mapsto \{G\text{-bundles on } X_S\}.$$

<u>Thm</u> $|\mathrm{Bun}_G| \longrightarrow B(G)$

set of G-isocrystals

bijective, continuous.

One piece missing in proof:

<u>Thm</u>. Let $\mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ substack

of all G-bundles $\mathcal{E}_G/X_S$ s.th.

at all $\mathrm{Spa}(C, C^+) \longrightarrow S$

(C complete alg. closed, $C^+ \subseteq C$ val. subring),

$\mathcal{E}_G|_{X_{\mathrm{Spa}(C,C^+)}}$ trivial.

Then $\mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ is an open substack

and $\mathrm{Bun}_G^1 \cong [*/\underline{G(E)}]$.

<u>Note</u>: • We already know this for $G = GL_n$.

• Cannot use continuity of $|\mathrm{Bun}_G| \longrightarrow B(G)$ yet.

<u>Proof</u>. Know: ν semicontinuous,

$\mathrm{Bun}_G^1 \subseteq \{\text{locus where } \nu = 0\}$.

Can reduce to G-bundles

$\mathcal{E}_G / X_S$ s.th. $\nu \equiv 0$.

In that case, for all repr

$\rho: G \longrightarrow GL(V)$ $(\rho \in \mathrm{Rep}_E G)$

$$\rho_* \mathcal{E} \in VB(X_S) \text{ everywhere semistable of slope } 0.$$

evaluation of $\mathcal{E}$ as exact $\otimes$-functor at ρ.

As such, it is equiv. to an $\underline{E}$-local system on S.

(<u>Recall</u>: $\{\underline{E}\text{-local systems on } S\} \cong \{\mathcal{E} \in VB(X_S) \mid \mathcal{E} \text{ everywhere semistable of slope } 0\}$)

By pro-étale localization, can assume S strictly totally disconnected; let

$$A = \mathrm{Cont}(|S|, E) = \mathrm{Cont}(\pi_0 S, E).$$

Then

$$\{\underline{E}\text{-local systems on } S\} \cong \mathrm{Proj}(A) \text{ fin. proj. } A\text{-mod.}$$

$$\mathbb{L} \longmapsto \mathbb{L}(S)$$

Thus, $\mathcal{E}$ defines exact $\otimes$-functor

$$\mathrm{Rep}_E G \longrightarrow \{VB(X_S) \text{ everywhere semistable of slope } 0\}$$

$$\begin{gathered} \| 2 \\ \mathrm{Proj}(A), \end{gathered}$$

equivalently, a G-torsor on $\mathrm{Spec}\, A$.

<u>enough to see:</u> If this G-torsor $\mathcal{F}$ on $\mathrm{Spec}\, A$ is trivial at $\mathrm{Spec}\, E \xhookrightarrow{g} \mathrm{Spec}\, A$, then

corr. to $s \in S$

it is trivial after pullback

$$\mathrm{Spec}\, \mathrm{Cont}(U, E) \subseteq \mathrm{Spec}\, \underbrace{\mathrm{Cont}(|S|, E)}_{= A}.$$

for some open + closed $\underset{s \in}{U} \subseteq |S|$.

This follows from two facts:

1) The local ring

$$\varinjlim_{U \ni s} \mathrm{Cont}(U, E) \text{ is henselian along } \ker(\text{evaluation at } s) \twoheadrightarrow E.$$

(for example, as local rings of analytic adic spaces like

2) If $(B, I)_{/E}$ henselian pair, (Spa A are always henselian.)

then $H^1_{et}(\operatorname{Spec} B, G) \hookrightarrow H^1_{et}(\operatorname{Spec} B/I, G)$,

i.e. any G-torsor over B that splits over B/I also splits over B. $\square$.

Digression on local Shimura Varieties

"Local Shimura Varieties" (cf. Rapoport-Viehmann) are a p-adic analogue of Shimura Varieties, ("local") and are related to Shimura Varieties in uniformization results.

Čerednik '70s. Rapoport-Zink '80s.
Drinfeld same year.

consider "PEL-type" Shimura varieties;

polarization, endomorphism, level.

in that case, relevant local Shimura varieties are moduli spaces of p-divisible groups with PEL-structure

"Rapoport-Zink spaces".

But there should be general local Shimura varieties!

These can be constructed using this machinery. (cf. Berkeley Lectures.)

<u>Local Shimura Data</u>: Usually $E = \mathbb{Q}_p$.
(Can also allow general E.)

triple $(G, [b], \{\mu\})$:

- G / E reductive group.
- $\{\mu : \mathbb{G}_m \to G_{\overline{E}}\}$ conjugacy class of <u>minuscule</u> cocharacters

· $[b] \in B(G)$

In order for assoc. local Shimura variety to be nonempty, need to ask

$$[b] \in B(G, \mu) \subseteq B(G).$$

finite subset, given by explicit combinatorial criterion.

Local Shimura Variety:

tower

$$\left(\mathcal{M}_{(G,b,\mu),K}\right)_{K \subseteq G(E) \text{ compact open}}$$

of smooth rigid-analytic varieties $/\breve{E}$

(+ Weil descent)

equipped with compatibly étale period maps

$$\pi_K : \mathcal{M}_{(G,b,\mu),K} \longrightarrow \mathcal{F}\ell_\mu \quad /\breve{E}$$

nonempty geom. fibres $\cong G(E)/K$.

param. parabolic subgroups of G conj. class determined by μ.

Example. (Drinfeld case). $G = D^\times$ D/E quat. algebra.

$\mu : \mathbb{G}_m \to G_{\bar{E}} \cong Gl_2$.

$t \mapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix}$.

up to signs.

b basic, slope $1/2$.

(unique element of $B(G,\mu)$.)

$\rightsquigarrow (\mathcal{M}_K)_{K \subseteq D^\times} \longrightarrow \Omega^2 = \mathbb{P}^1 \setminus \mathbb{P}^1(E) \subseteq \mathcal{F}l_\mu = \mathbb{P}^1$

Drinfeld covers of Drinfeld upper half-space.

similar example for $D^\times_{1/n}$, $n \geq 2$.

Example (Lubin-Tate case) $G = GL_n$

$\mu : \mathbb{G}_m \to G : t \mapsto \begin{pmatrix} t & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$.

b basic slope $1/n$.

up to signs

$$(\mathcal{M}_K)_{K \subseteq GL_n(E)} \longrightarrow \mathcal{F}\ell_\mu = (\mathbb{P}^{n-1})^{ad}/\breve{E}.$$

parametrize def's of 1-dim'l ht n π-div. $\mathcal{O}_E$-module + level structure.

Gross - Hopkins.

surj. étale map, fibres $GL_n(E)/K$.

So $(\mathbb{P}^{n-1})^{ad}$ admits nontrivial infinite degree étale covering spaces!

$$K = GL_n(\mathcal{O}_E) \subseteq GL_n(E)$$

$$\Rightarrow \quad \mathcal{M}_K \cong \coprod_{\mathbb{Z}} (n-1)\text{-dim'l open unit disc}$$

$$= \coprod_{\mathbb{Z}} \left(\mathrm{Spa}\, W_E(\mathbb{F}_q)[[u_1, \dots, u_{n-1}]]\right)_{\breve{E}}.$$

$\vdots$

Construction of local Shimura Varieties:

Want: open subset $\mathcal{F}\ell_\mu^{adm} \subseteq \mathcal{F}\ell_\mu$.

'admissible locus'

+ $\underline{G}(E)$ - local system $\mathbb{L}$ on $\mathcal{F}\ell_\mu^{adm}$.

Then $\mathcal{M}_K$ can be defined to parametrize reductions of $\mathbb{L}$ to K; equiv, considering $\mathbb{L}$ as $\underline{G(E)}$-torsor

$$\mathbb{L} \longrightarrow \mathcal{F}\ell_\mu^{adm}$$

$$\mathcal{M}_K = \mathbb{L}/\underline{K} \longrightarrow \mathcal{F}\ell_\mu^{adm}.$$

automatically étale.

As $\mathcal{F}\ell_\mu^{adm}$ smooth rigid-analytic variety, also $\mathcal{M}_K$ is a smooth rigid-analytic variety.

Recall: $$\mathrm{Gr}_{G,\leq\mu} \subseteq \mathrm{Gr}_G := \mathrm{Gr}_G^{B_{dR}^+} \longrightarrow \mathrm{Bun}_G$$

$$\mathcal{F}\ell_\mu^\diamond \cong \mathrm{Gr}_{G,\leq\mu}$$

modify $\mathcal{E}_b$.

μ minuscule (If μ not minuscule get similar story replacing $\mathcal{F}\ell_\mu$ by $Gr_{G, \leq \mu}$.)

$$\mathcal{F}\ell_\mu^\diamond \longrightarrow \mathrm{Bun}_G.$$

<u>Prop'n.</u> image meets $\mathrm{Bun}_G^1 \iff b \in B(G,\mu)$.

up to signs

(Appendix of Rapoport to "p-adic cohom. of LT tower").

equiv., $\mathcal{E}_b$ modification of triv. G-torsor of type μ^{-1}.

i.e. in image of analogous map

$$\mathcal{F}\ell_{\mu^{-1}} \longrightarrow \mathrm{Bun}_G.$$

$\uparrow$ modify $\mathcal{E}_1$ $\square$.

$$\begin{array}{ccc} \mathcal{F}\ell_\mu^\diamond & \longrightarrow & \mathrm{Bun}_G \\ \cup\ \text{open} & \square & \cup\ \text{open} \\ (\mathcal{F}\ell_\mu^{adm})^\diamond & \longrightarrow & \mathrm{Bun}_G^1 \\ & & \text{\rotatebox{90}{$\cong$}} \end{array}$$

$$[*/\underline{G}(E)].$$

Get the desired data. $\uparrow$ stack classifying $\underline{G}(E)$-torsors.

$$\mathcal{F}\ell_\mu^{adm} \subseteq \mathcal{F}\ell$$

$+$ $\underline{G}(E)$-torsor on $\mathcal{F}\ell_\mu^{adm}$.

<u>Cor</u> $\varprojlim_K \mathcal{M}_K^\diamond$ parametrizes modifications

$$\mathcal{E}_b \dashrightarrow \mathcal{E}_1 \text{ of type } \mu,$$

more precisely, for all $S \in \mathrm{Perf}/(\mathrm{Spa}\, \breve{E})^\diamond$,

$$\left(\varprojlim_K \mathcal{M}_K^\diamond\right)(S) = \left\{ \text{isom. } \begin{array}{c} \mathcal{E}_b|_{X_S \setminus S^\#} \\ \wr\wr \\ \mathcal{E}_1|_{X_S \setminus S^\#} \end{array}, \text{ modification "is of type } \mu\text{" at all geom. points.} \right\}.$$

Example. Lubin-Tate case

$$\varprojlim_{K \subseteq GL_n(E)} \mathcal{M}_{LT,K}^{\diamond} \cong \left\{ \mathcal{O}_{X_S}^n \hookrightarrow \mathcal{O}_{X_S}(1/n), \text{ cokernel supp. at } S^\# \right\}.$$

(nec. a line bundle on $S^\#$)

$$\cong$$

$$\varprojlim_{K' \subseteq D_{1/n}^\times} \mathcal{M}_{Dr,K'}^{\diamond}$$

Drinfeld case.

isomorphism of Lubin-Tate and Drinfeld towers.

works for all local Shimura varieties with b basic.

relates (G, b, μ) and (G_b, b^{-1}, μ^{-1}).

General points of Bun_G:

1) Semistable points:

Thm $\mathrm{Bun}_G^{ss} \subseteq \mathrm{Bun}_G$ (semistable locus) is open, and

$$\mathrm{Bun}_G^{ss} = \coprod_{b \in B(G)_{basic}} \underbrace{[*/\underline{G_b(E)}]}_{\substack{\shortparallel \\ \mathrm{Bun}_G^b}}.$$

$\leftarrow$ locus where $\mathcal{E} \cong \mathcal{E}_b$ at geom. points.

Proof. – open: semicontinuity of ν.

– decomposition into Bun_G^b: local constancy of κ

$+ \quad \kappa : B(G)_{basic} \xrightarrow{\cong} \pi_1(G)_\Gamma$.

remains: $\mathrm{Bun}_G^b \cong [*/\underline{G_b(E)}]$.

But $\mathcal{E}_b$ G-torsor on X_S,

and $\underline{\mathrm{Aut}}_{X_S}(\mathcal{E}_b) = G_b \times_E X_S$. for basic b.

$\rightsquigarrow \quad \{G\text{-torsors on } X_S\} \cong \{G_b\text{-torsors on } X_S\}$

$$\mathcal{E}_0 \longmapsto \underline{\mathrm{Isom}}(\mathcal{E}_0, \mathcal{E}_b).$$

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = G_b\text{-torsor}.$$

as basic b induce isomorphisms

$$\begin{array}{ccc} \mathrm{Bun}_G & \cong & \mathrm{Bun}_{G_b}. \\ \cup\!\!\mid & \circlearrowleft & \cup\!\!\mid \\ \mathrm{Bun}_G^b & \cong & \mathrm{Bun}_{G_b}^1 = [*/\underline{G_b(E)}]. \end{array}$$

$\square$.

2) Non-semistable b.

<u>Thm</u>. $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$ locally closed,

$\mathrm{Bun}_G^b \cong [*/\mathcal{G}_b]$ where $\mathcal{G}_b$ is a group v-sheaf,

$$1 \to \mathcal{G}_b^\circ \longrightarrow \mathcal{G}_b \longrightarrow \underline{G_b(E)} \to 1$$

extension of positive Banach-Colmez spaces, of dimension $\langle 2\rho, \nu(b) \rangle$.

Can we replace G by a reductive group over Fargues-Fontaine curve?

Better: "reductive groups in Isoc_E". (Anschütz)

$$\begin{tikzcd}
 & \mathrm{Gr}_{G,\mu} \arrow[d, "\pi_\mu"] \\
S^\diamond \arrow[ur, dotted] \arrow[r] & \mathcal{F}\ell_\mu^\diamond
\end{tikzcd}$$

μ general

Bialynicki-Birula fibres are iterated affine spaces.

rigid-analytic variety, smooth for simplicity.

$(\breve{E})^\diamond$.

$\exists \quad 0 \to (\mathbb{A}^1)^\diamond \to X \to (\mathbb{A}^1)^\diamond \to 0$

not a rigid-analytic variety!

<u>Thm</u> $\mathrm{Gr}_{G,\mu}(S^\diamond) \hookrightarrow \mathcal{F}\ell_\mu^\diamond(S^\diamond) = \mathcal{F}\ell_\mu(S)$,

(S., p-adic HT for rigid-anal. var., Kedlaya)

Fontaine

Faltings.

image = those maps that satisfy Griffiths transversality.