B_{dR}^+-affine Grassmannian

$E \supseteq \mathcal{O}_E \ni \pi$, $\quad \mathbb{F}_q$, $\bar{\mathbb{F}}_q$, $\breve{E}$

G/E reductive group.

G-Isoc "isocrystals with G-structure"

$B(G) = G\text{-Isoc}/\cong.$ countable set

$= G(\breve{E})/\sigma\text{-conjugation } b \sim g^{-1}b\sigma(g).$

$\nu : B(G) \longrightarrow (X^+_{\mathbb{Q}})^{\Gamma} \quad \Gamma = \mathrm{Gal}(\bar{E}/E).$

$\kappa : B(G) \longrightarrow \pi_1(G)_{\Gamma}.$

$(\nu,\kappa) : B(G) \hookrightarrow (X^+_{\mathbb{Q}})^{\Gamma} \times \pi_1(G)_{\Gamma}.$

Def'n. Bun_G v-stack on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$,

$$S \longmapsto \{ G\text{-bundles on } X_S \}.$$

Note: $G\text{-Isoc} \longrightarrow \mathrm{Bun}_G(S)$ for any S

as $X_S = Y_S/\phi^{\mathbb{Z}} \longrightarrow \mathrm{Spa}\, \breve{E}/\sigma^{\mathbb{Z}}$.

Anschütz: $G\text{-Isoc} \xrightarrow{\sim} \varprojlim_S \mathrm{Bun}_G(S)$.

Thm (Fargues $E/\mathbb{Q}_p$, Anschütz general E). If $S = \mathrm{Spa}(C, \mathcal{O})$

C complete alg. closed, then

$$B(G) = G\text{-Isoc}/\cong \xrightarrow[\text{bij.}]{\cong} \mathrm{Bun}_G(S).$$

$$\rightsquigarrow \quad |\mathrm{Bun}_G| = B(G).$$

Thm 1) $\nu : |\mathrm{Bun}_G| \longrightarrow (X_*^+{}_{\mathbb{Q}})^{\Gamma}$

semicontinuous.

2) $\kappa : \quad |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$

locally constant, induces a bijection

$\pi_0 \mathrm{Bun}_G \xrightarrow{\cong} \pi_1(G)_\Gamma$.

<u>Remark</u>. A complete determination of $|\mathrm{Bun}_G|$ was obtained by Hansen for $G = Gl_n$

Hamann some classical groups

Viehmann for general G.

<u>Proof</u>. 1). know this for Gl_n.

Rapoport-Richartz: can reduce to Gl_n via some embedding $G \hookrightarrow Gl_n$.

for 2), need a lemma:

<u>Lemma</u>. Let $G' \twoheadrightarrow G$ extension by a central torus. Then

$$\mathrm{Bun}_{G'} \longrightarrow \mathrm{Bun}_G$$

is a surjective map of v-stacks.

Assume this Lemma.

Go through construction of κ:

- If G product of induced tori, then $\pi_1(G)_\Gamma$ torsion-free,

$$\begin{tikzcd} & (X^+_{\mathbb{Q}})^\Gamma \arrow[dr] & \\ B(G) \arrow[ur, "\nu"] \arrow[dr, "\kappa"'] & \circlearrowleft & (\pi_1(G)_{\mathbb{Q}})^\Gamma \\ & \pi_1(G)_\Gamma \arrow[ur, "\text{average.}"'] & \end{tikzcd}$$

so continuity follows from 1).

- For general torus T, can find surjection

$$\tilde{T} \longrightarrow T \quad \text{with kernel a torus.}$$

product of induced tori.

Lemma $\rightsquigarrow$ $\mathrm{Bun}_{\tilde{T}} \longrightarrow \mathrm{Bun}_T$ surj. map of v-stacks,

in particular

$$\begin{tikzcd} |\mathrm{Bun}_{\tilde{T}}| \arrow[r] \arrow[d, "\kappa_{\tilde{T}}"] & |\mathrm{Bun}_T| \arrow[d, "\kappa_T"] \\ \pi_1(\tilde{T})_\Gamma \arrow[r] & \pi_1(T)_\Gamma \end{tikzcd}$$

quotient map.

continuous, thus κ_T is continuous.

- G with G_{der} simply connected

$$1 \to G_{der} \to G \to T \to 1$$

$$\pi_1(G) \xrightarrow{\sim} \pi_1(T)$$

thus

$$\begin{tikzcd} |\mathrm{Bun}_G| \arrow[r] \arrow[d, "\kappa_G"] & |\mathrm{Bun}_T| \arrow[d, "\kappa_T"] \\ \pi_1(G)_\Gamma \arrow[r, equal] & \pi_1(T)_\Gamma \end{tikzcd}$$

κ_T cont.

$\Downarrow$

κ_G cont.

- General G: Take

$$G' \to G$$

z-extension s.th. G'_{der} simply connected.

Lemma shows $\kappa_{G'}$ cont $\Rightarrow$ κ_G cont.

$\square$.

$$\begin{tikzcd} \mathrm{Bun}_{SL_n} \arrow[r] \arrow[d, "\kappa"] & \mathrm{Bun}_{PGL_n} \arrow[d, "\kappa"] \\ 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \end{tikzcd}$$

not surj.

Corollary. (slight strengthening of Rapoport - Richartz).

If S perfect scheme $/\overline{\mathbb{F}}_q$, $\mathcal{E}$ G-isocrystal over S. Then

$$\kappa: |S| \longrightarrow \pi_1(G)_\Gamma$$

is locally constant.

Sketch. $S = \mathrm{Spec}\ R$.

For any perfectoid space $S' / \overline{\mathbb{F}}_q$ with a map $S' \to \mathrm{Spa}(R, R^+)$, get map

$$S' \longrightarrow \mathrm{Bun}_G.$$

Here, κ locally constant, this is enough to deduce the result. $\square$

<u>Lemma</u>. $G' \to G$ extension by a central torus

$\Rightarrow$ $\mathrm{Bun}_{G'} \longrightarrow \mathrm{Bun}_G$ surjective map of v-stacks.

We will deduce this from Beauville-Laszlo uniformization:

$$\mathrm{Gr}_G^{B_{dR}^+} \longrightarrow \mathrm{Bun}_G \quad \text{surjective of v-stacks}$$

$+ \quad \mathrm{Gr}_{G'}^{B_{dR}^+} \longrightarrow \mathrm{Gr}_G^{B_{dR}^+} \quad$ —a— .

B_{dR}^+-affine Grassmannian.

<u>B_{dR}^+</u>: Let R be any perfectoid ring / $\mathcal{O}_E$.
$\varpi \in R^\flat$ pseudouniformiser.

$\rightsquigarrow$

$$\theta : W_{\mathcal{O}_E}(R^{\flat\circ}) \twoheadrightarrow R^\circ$$

$$\ker(\theta) = (\xi) \qquad \xi = \pi + [\varpi]\cdot a$$
$$a \in W_{\mathcal{O}_E}(R^{\flat\circ}).$$

$$\theta : W_{\mathcal{O}_E}(R^{\flat\circ})\left[\frac{1}{[\varpi]}\right] \twoheadrightarrow R = R^\circ\left[\frac{1}{\varpi^\sharp}\right].$$

<u>Def'n</u>. $B_{dR}^+(R)$ = ξ-adic completion of

$$W_{\mathcal{O}_E}(R^{\flat\circ})\left[\frac{1}{[\varpi]}\right].$$

<u>Note</u>: – This is a "1-parameter deformation" of R:

$$B_{dR}^{+}(R)/(\xi) = R$$

– $B_{dR}^{+}(R) \twoheadrightarrow R$ universal pro-infinitesimal thickening in <u>solid</u> $\mathcal{O}_E$-algebras.

↑ Condensed Mathematics

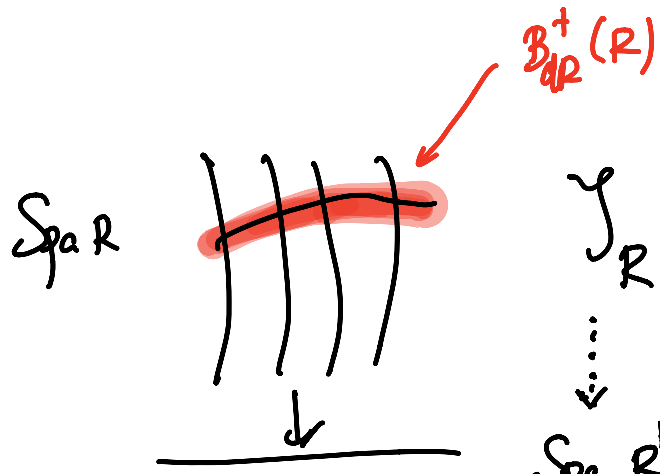
– if $R/\mathbb{F}_q$, then

$$B_{dR}^{+}(R) = W_{\mathcal{O}_E}(R).$$

– if $R = C_{/E}$ complete alg closed,

$$B_{dR}^{+}(R) \simeq C[\![\xi]\!] \quad \text{as } \underline{\text{abstract}} \text{ } E\text{-alg}.$$

– <u>name</u>: it arises as "ring of p-adic de Rham periods" in work of Fontaine.



$$\begin{tikzcd} \mathcal{Y}_R = \operatorname{Spa} W_{\mathcal{O}_E}(R^{\flat\circ}) \setminus V([\varpi]). \arrow[d, dotted] \\ \operatorname{Spa} R^\flat \end{tikzcd}$$

<u>Definition</u>. $Gr_G^{B^+_{dR}}$ is the étale sheafification of functor on

$\mathrm{Perf}_{/(\operatorname{Spa} E)^\diamond} \cong \mathrm{Perfd}_{/E}$ taking

$\operatorname{Spa}(R, R^+) / E$ to

$$G(B_{dR}(R)) / G(B^+_{dR}(R))$$

where $B_{dR}(R) = B^+_{dR}(R)\left[\frac{1}{\xi}\right]$.

equiv., this is classifying G- bundles on

Spec $B_{dR}^+(R)$ with a trivialization over

Spec $B_{dR}(R)$.

<u>Note</u>: This is a p-adic version of usual affine Grassmannian

$$R/\mathbb{C} \longmapsto G(R((t)))/G(R[[t]]).$$

<u>Note</u>: If $E = \mathbb{F}_q((t))$, this is literally

$$R/\mathbb{F}_q((t)) \longmapsto G(R((t-\zeta)))/G(R[[t-\zeta]])$$

$W_{O_E}(R^\circ) \ni t, S$. $\quad \zeta = t - S$.

$\|$

$R^\circ[[t]]$.

so get usual affine Grassmannian!

Prop'n. $\mathrm{Gr}_G^{B_{dR}^+}(C) = \bigsqcup_{\mu \in X^+} G(B_{dR}^+(C)) \cdot [\mu(\xi)]$

complete alg closed.

Cartan decomposition.

$\mu(\xi) \in G(B_{dR}(C))$.

Ex. $G = GL_n$:

$\mu \hat{=} (a_1 \geq \dots \geq a_n)$.

$$\begin{pmatrix} \xi^{a_1} & & 0 \\ & \ddots & \\ 0 & & \xi^{a_n} \end{pmatrix}.$$

Proof. $B_{dR}^+(C) \cong C[[\xi]]$ abstractly, so can use usual Cartan decomposition. $\square$

$$G(\mathbb{C}[[t]]) \backslash G(\mathbb{C}((t))) / G(\mathbb{C}[[t]]) \cong X^+.$$

Assume G split for simplicity.

Def'n. For $\mu \in X^+$, let

$$\mathrm{Gr}_{G, \leq \mu}^{B_{dR}^+} \subseteq \mathrm{Gr}_G^{B_{dR}^+}$$

(In general, this happens after finite étale base change.)

$(+\ \mathrm{Gr}_{G_{E'}} = \mathrm{Gr}_G \times_E E')$

"Schubert variety" subfunctor of all

$S \longrightarrow Gr_G^{B_{dR}^+}$ s.th. at all

geometric points, the point lies in

$G(B_{dR}^+)$-orbit of $\mu'(\xi)$, where

$\mu' \leq \mu$.

($\mu - \mu'$ sum of positive coroots).

<u>Thm</u> $Gr_{G,\leq\mu}^{B_{dR}^+} \longrightarrow (\mathrm{Spa}\, E)^\diamond$ proper

and $Gr_{G,\leq\mu}^{B_{dR}^+}$ spatial diamonds;

$$Gr_G^{B_{dR}^+} = \bigcup_\mu Gr_{G,\leq\mu}^{B_{dR}^+},$$

transition maps are closed immersions.

If μ minuscule,

$$\mathrm{Gr}^{B_{dR}^+}_{G,\leq\mu} \cong (G/P_\mu)^\diamond$$

Proof uses a variant of Artin's recognition for algebraic spaces.

<u>Note</u>: for $E/\mathbb{F}_q((t))$, all $\mathrm{Gr}^{B_{dR}^+}_{G,\leq\mu}$ come from proj. varieties $/E$. But for $E/\mathbb{Q}_p$, get actual diamonds

<u>Cor</u>. $G' \longrightarrow G$ z-extension

$$\Rightarrow \quad \mathrm{Gr}^{B_{dR}^+}_{G'} \longrightarrow \mathrm{Gr}^{B_{dR}^+}_{G} \text{ is a v-cover.}$$

<u>Proof</u>. enough (wlog G', G split)

$$\mathrm{Gr}^{B_{dR}^+}_{G',\leq\mu'} \longrightarrow \mathrm{Gr}^{B_{dR}^+}_{G,\leq\mu} \text{ is a v-cover}$$

(Use: $X'^+ \twoheadrightarrow X^+$). for all $\mu' \mapsto \mu$.

These are spatial diamonds in particular qcqs.

so surjectivity can be checked on geometric points.

Now it follows from Cartan decomposition.

$$+ \quad G'(B_{dR}^{+}(C)) \twoheadrightarrow G(B_{dR}^{+}(C)) \quad \square$$

Beauville-Laszlo Uniformization.

Def'n. (G general again) Beauville-Laszlo unif. map.

$$\mathrm{Gr}_G^{B_{dR}^{+}} \longrightarrow \mathrm{Bun}_G$$ is following map:

$S = \mathrm{Spa}(R, R^{+}) \;/ E$.

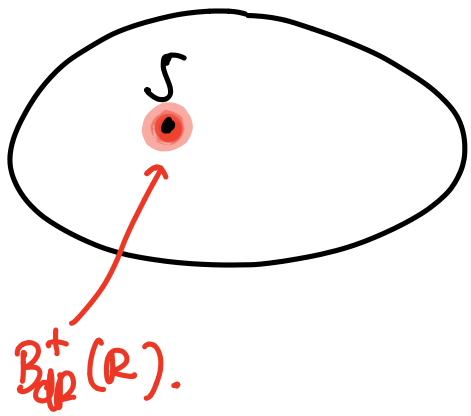
G-torsor $\mathcal{E}_0$ over $B_{dR}^{+}(R)$,

trivialized over $B_{dR}(R)$.

Can glue trivial G-torsor on

$X_{S^b}^{alg} \setminus \mathrm{Spec}(R)$ with $\mathcal{E}_{0}\,/\,B_{dR}^{+}(R)$

along given identification over $B_{dR}(R)$.



$X^{\mathrm{alg}}_{S^b}$.

Lemma. If X scheme $/E$,
(Beauville-Laszlo)
$Z \subseteq X$ Cartier divisor, Z affine.

Then G-torsor $/X$ are equiv. to

$$\left\{ \begin{array}{l} G\text{-torsor } / X \setminus Z \\ + \; G\text{-torsor } / \hat{X}_Z \\ + \; \text{identification } / \hat{X}_Z \setminus Z \end{array} \right.$$

Example. $G = GL_2$, restricting to

$$Gr^{B_{dR}^+}_{GL_2, \leq \mu} \cong \left(\mathbb{P}^1_E\right)^{\diamond},$$ recover example from before.

$\mu = (1,0)$

Theorem. $\mathrm{Gr}_G^{B_{dR}^+} \longrightarrow \mathrm{Bun}_G$ is surjective map of v-stacks.

Remark. Analogue of result of Drinfeld–Simpson.

There it's true only if G semisimple.

Here, works for all reductive G.

Key: Picard group of punctured curve is trivial.

Sketch. On geometric points, due to Fargues / Anschütz, using classification of G-bundles.

In general,

Let $S \longrightarrow \mathrm{Bun}_G$, $\mathcal{E}/X_S$

S strictly totally disconnected.

At geom. point $s \in S$, can lift to $\mathrm{Gr}_G^{B_{dR}^+}$, so get modif. $\mathcal{E}_s'$ of $\mathcal{E}_s$.

Pick a modif. $\mathcal{E}'$ of $\mathcal{E}$ recovering

$\mathcal{E}'_s$ at s.

<u>enough:</u> $\mathcal{E}'$ is trivial in a neighborhood of s.

<u>remains to see:</u> for G-bundle $\mathcal{E}'/X_{S'}$

the locus where $\mathcal{E}'$ is trivial is open in S.

Witt vector affine Grassm. $\mathcal{G}/\mathcal{O}_E$.

R perf. $\mathbb{F}_q$-alg.

$\mapsto$ (Zhu, Bhatt-S.)

$G(W_{\mathcal{O}_E}(R)[\frac{1}{\pi}])/G(W_{\mathcal{O}_E}(R))$.

is repr. by ind(perfectly proj. $\mathbb{F}_q$-scheme).

Can also define

$$\mathrm{Gr}_{\mathcal{G}}^{B_{dR}^+} \longrightarrow (\mathrm{Spa}\, \mathcal{O}_E)^{\diamond}.$$

v-sheaf, defined exactly as above,
ind-(proper, rel. repr. in spat. diamonds $/(\mathrm{Spa}\, \mathcal{O}_E)^{\diamond}$).

This is a degeneration from

$$\mathrm{Gr}_{G}^{B_{dR}^+} \quad \text{to} \quad \left(\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Witt}}\right)^{\diamond}.$$