G-bundles on the Fargues-Fontaine curve.

$$E \supseteq \mathcal{O}_E \ni \pi \quad , \mathbb{F}_q \quad , \overline{\mathbb{F}}_q$$

$$\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[\tfrac{1}{\pi}] / E.$$

Also fix a reductive group G/E.

(e.g. $G = GL_n, Sp_{2n}, SL_n, SO_n, E_8, G_2, \ldots$
$U_n, D^\times$ div. alg. D —)

<u>always</u> assumed connected.

General notion of G-bundle / G-torsors.

<u>Prop.</u> Let X scheme $/E$. The following are naturally equivalent:

1) "Geometric G-torsors": Schemes $Y \to X$ with a G-action over X s.th.

-étale / smooth / fppf / fpqc locally on X,

there is a G-equiv. isom. $Y \cong G \times X$.

2) "Cohomological G-torsors": sheaf $\mathcal{F}$ on $X_{ét}$ + action of G s.th. locally on $X_{ét}$, $\mathcal{F} \cong \mathcal{G}$ G-equivariantly.

3) "Tannaka G-torsors": exact $\otimes$-functors

$$\mathrm{Rep}_E G \longrightarrow VB(X)$$
$$\| $$
$$\{\text{vector bundles on } X\}.$$

Examples. 1) $G = GL_n$, these are rk n vector bundles on X.

2) If $G = Sp_{2n}$, these are rk $2n$ vector bundles E/X + perfect alternating form on E.

<u>Sketch.</u> 1) $\to$ 2): Take sections of $Y \to X$.

2) $\to$ 3): $V \in \mathrm{Rep}_E G$, $\mathcal{F}$ cohom. G-torsor,

then $V \times^G \mathcal{F}$ is an $\mathcal{O}_X$-module on X_{et},

locally free of fin rank.

so a vector bundle on X by étale descent.

3) $\to$ 1): Can consider $G \curvearrowright \mathcal{O}(G) \circlearrowleft_G \in \mathrm{Ind}(\mathrm{Rep}_E G)$

with G-action, in fact an algebra object.

Apply exact $\otimes$-functor

$$F : \mathrm{Rep}_E G \longrightarrow VB(X)$$

$$\rightsquigarrow \quad F(\mathcal{O}(G)) \in \mathrm{Alg}(\mathrm{Ind}\, VB(X))$$

$$\downarrow$$

$$\mathrm{Alg}(\mathrm{QCoh}(X))$$

with G-action.

Can take $Y = \underline{\mathrm{Spec}}\ F(\mathcal{O}(G))$. $\square$.

Will tacitly identify these notions.

<u>Remark</u>. A similar discussion to G-torsors on adic spaces.

Most convenient option is often that of exact $\otimes$-functors.

<u>Cor</u>. G-torsors on X are classified by

$$H^1(X_{et}, G).$$

<u>G-Isocrystals</u>. (Kottwitz). $\sigma \circlearrowright \breve{E}$. Frob.

Recall $\mathrm{Isoc}_E = \{ (V, \phi) \mid V$ f.d. $\breve{E}$-v.s. $\phi : V \xrightarrow{\sim} V$ σ-linear $\}$.

<u>Definition</u>. A G-isocrystal is an exact $\otimes$-functor

$$\mathrm{Rep}_E G \longrightarrow \mathrm{Isoc}_E.$$

Proposition. Any G-isocrystal is of the form

$$\begin{aligned} \mathrm{Rep}_E\, G &\longrightarrow \mathrm{Isoc}_E \\ V &\longmapsto (V \otimes_E \breve{E},\ b\sigma) \end{aligned}$$

for some $b \in G(\breve{E})$.

$\rightsquigarrow$ $\{$isom. classes of G-isocrystals$\}$

$\cong$

$G(\breve{E}) / \sigma$-conjugation: $b \sim g^{-1} b \sigma(g)$ for $g \in G(\breve{E})$.

Remark. G-isocrystals

$=$ "G-torsors on $\mathrm{Spec}\, \breve{E} / \sigma^{\mathbb{Z}}$".

Sketch. enough to see that all G-torsors on $\mathrm{Spec}\, \breve{E}$ are trivial.

But <u>Thm</u> (Steinberg) $H^1_{\text{ét}}(\operatorname{Spec} \breve{E}, G) = *$. $\square$.

(use: $\breve{E}$ has cohom. <u>dim. 1</u>.)

<u>Definition</u>. $B(G) = \{G\text{-isocrystals}\}/\cong$

$$\cong G(\breve{E}) / \sigma\text{-conjugation}.$$

Elements are denoted $b \in B(G)$.

(often a choice of repr. in $G(\breve{E})$ is implicit.)

<u>Example</u>. $G = GL_n$, these are just rk n isocrystals.

$\rightsquigarrow$ $B(GL_n) \cong$ Newton polygons of width n.

Kottwitz <u>gives combinatorial description</u> of $B(G)$ <u>for all G</u>:

roughly, Newton polygons with a certain symmetry condition

+ a finite amount of extra data.

Newton point:

Note: For any $(V, \phi) \in \mathrm{Isoc}_E$, V is naturally $\mathbb{Q}$-graded $V = \bigoplus_{\lambda \in \mathbb{Q}} V^\lambda$:

slope decomposition.

$\rightsquigarrow$ map $\mathbb{D} \longrightarrow GL_{\breve{E}}(V)$

(pro-) torus with character group $\mathbb{Q}$

$$\mathbb{D} = \varprojlim_{\times n} \mathbb{G}_m = \mathrm{Spec}\, E[T^{\mathbb{Q}}].$$

$\left(\rightsquigarrow \mathrm{Rep}_E(\mathbb{D}) = \{ \mathbb{Q}\text{-graded } E\text{-v.s} \}. \right)$

If $F: \mathrm{Rep}_E G \longrightarrow \mathrm{Isoc}_E$ exact $\otimes$-functor,

get compatible maps $\mathbb{D} \to GL_{\breve{E}}(F(V))$

for all $V \in \mathrm{Rep}_E G$.

$\hat{=}$ map $\mathbb{D} \longrightarrow G_{\breve{E}}$.

if $F \underset{b}{\cong}$ has underlying functor given by $V \mapsto V \otimes_E \breve{E}$.

$\rightsquigarrow$ get well-defined conj. class of maps

$$\mathbb{D} \longrightarrow G_{\breve{E}}.$$

This can be factored over a torus.

$$\text{If } X^+ \subseteq X = X_*(T) \qquad T \subset B \subset G_{\bar{E}}.$$

$\Gamma = \mathrm{Gal}(\bar{E}/E)$ (is canonically indep't of choice of T).

$\rightsquigarrow$ corresponds to an element

$$\nu(b) \in \left(X^+_{\mathbb{Q}}\right)^\Gamma$$

<u>Example.</u> $G = GL_n \supseteq B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \supseteq T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

$$X = X_*(T) = \mathbb{Z}^n \circlearrowleft \Gamma \text{ trivial}$$
$$\cup\!|$$
$$X^+ = \{(m_1 \geq \ldots \geq m_n)\}. \quad (G \text{ split}).$$

$$\rightsquigarrow \quad X^+_{\mathbb{Q}} = \{(\lambda_1 \geq \ldots \geq \lambda_n)\}.$$
$$\lambda_i \in \mathbb{Q}$$

for rk n isocrystal, this is just recording the slopes.

$V = \bigoplus_{\lambda \in \mathbb{Q}} V^\lambda$, list λ with mult. $\dim V^\lambda$.

For GL_n,
$$\nu : B(G) \to (X^+_{\mathbb{Q}})^\Gamma$$
is injective, but this fails for general G.

<u>Example</u>. Tori. $T \,\hat{=}\, (X = X_*(T_{\bar{E}}) \circlearrowleft \Gamma)$

<u>Proposition</u>. There is a functorial isomorphism.

$$\mathcal{B}(T) \cong X_*(T)_\Gamma$$

$\Vert$ $\qquad\qquad$ ↑ coinvariants.

$T(\breve{E})/\sigma\text{-conj}$

$\Vert$ $\qquad$ abelian group $\qquad$ for tori T/E.

$T(\breve{E})/(\sigma-1)$

Under this isomorphism,

$$\nu: \mathcal{B}(T) \longrightarrow (X_{\mathbb{Q}}^+)^\Gamma = X_{\mathbb{Q}}^\Gamma$$

is given by

$$\text{average} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma : X_*(T)_\Gamma \to X_*(T)_{\mathbb{Q}}^\Gamma .$$

(replace Γ by any finite quotient over which the action factors)

not injective if $X_*(T)_\Gamma$ has torsion.

e.g. $X_*(T) = \mathbb{Z} \circlearrowleft \Gamma$ by ± 1.

<u>Sketch</u>. 1) $T = \mathbb{G}_m$.

$$\begin{array}{ccc} B(T) = \breve{E}^\times/(\sigma-1) & \longrightarrow & \mathbb{Z} = X_*(T) \quad \circlearrowleft \Gamma \text{ trivial.} \\ b & \longmapsto & v(b) \end{array}$$

is isom. (Classification of rk 1 isocrystals!)

2). $T = \mathrm{Res}_{E'/E}\, \mathbb{G}_m$. E'/E finite separable.

$$B(T) = B(E, T)$$

$$\cong B(E', \mathbb{G}_m)$$

"Shapiro": $B(E, \mathrm{Res}_{E'/E} G) \cong B(E', G)$.

$$\cong \mathbb{Z}.$$

$$X_*(T) = \mathrm{Ind}_{\Gamma_{E'}}^{\Gamma_E} \mathbb{Z}.$$

$$X_*(T)_{\Gamma_E} = \mathbb{Z}_{\Gamma_{E'}} = \mathbb{Z}.$$

$\rightsquigarrow$ get such functorial identification for products of induced tori.

3) Resolve by induced tori.

Any T admits a surjection

$$\prod_{i=1}^{n} \mathrm{Res}_{E_i/E} \mathbb{G}_m \twoheadrightarrow T. \qquad \square.$$

Back to general G. Can define

$$\pi_1(G) = \pi_1(G_{\bar{E}}) \qquad \text{f.g. ab. group.}$$

$\Gamma \circlearrowright$

$$= X_*(T)/\text{coroot lattice}.$$

"Borovoi fundamental group".

for $G/\mathbb{C}$, would recover usual π_1.

<u>Proposition</u> - There is a unique functorial

extension "Kottwitz map".

$$\kappa : \quad B(G) \longrightarrow \pi_1(G)_\Gamma$$

extending above map

$$B(T) \xrightarrow{\cong} X_*(T)_\Gamma = \pi_1(T)_\Gamma$$

for tori.

<u>Sketch</u>. 1) Tori ✓.

2) G s.th. G_{der} is simply connected.

Then $\quad 1 \to G_{der} \to G \to D \to 1.$

D torus,

$$\pi_1(G) \xrightarrow{\sim} \pi_1(D).$$

so κ defined by projecting to D.

3) General G: $\exists$ z-extension

$$G' \twoheadrightarrow G$$

(kernel central) s.th. G'_{der} is simply connected.

$$B(G') \twoheadrightarrow B(G)$$

$$\begin{tikzcd}
{} \arrow[d, "\kappa"] & {} \arrow[d, dotted, "\exists"] \\
\pi_1(G')_\Gamma \arrow[r] & \pi_1(G)_\Gamma
\end{tikzcd}$$

$\square$.

<u>Example</u>. $E = \mathbb{F}_q((t))$

$$\begin{tikzcd}
B(G) \arrow[r] & \pi_1(G)_\Gamma . & \text{"top. loops"} \\
G(\breve{\mathbb{F}}_q((t))) . \arrow[u] \arrow[ur] & & \text{"loops in } G\text{"} . \arrow[ull] \arrow[u, mapsto]
\end{tikzcd}$$

<u>Thm</u>. (Kottwitz) For all G,

$$(\nu, \kappa) : B(G) \longrightarrow (X_{*,\mathbb{Q}}^{+})^\Gamma \times \pi_1(G)_\Gamma$$

is injective.

$\rightsquigarrow$ partial order on $B(G)$:

$b \leq b'$ if $\nu(b) \preceq \nu(b')$ in dom. order

$+ \quad \kappa(b) = \kappa(b')$.

minimal elements in this order are called "basic". ("semistable G-torsors").

Prop. $B(G)_{\text{basic}} \xrightarrow[\kappa]{\cong} \pi_1(G)_\Gamma$.

(for tori, all elements basic.)

nonbasic elements can be understood in terms of Levi subgroups. (at least if G quasisplit).

Prop. b basic $\iff$ $\nu(b)$ central.

Prop. For any $b \in B(G)$, can look at σ-centralizer of b

= automorphisms of corr. $\otimes$-functor.

This define a connected reductive

group G_b over E. If b basic, G_b is inner form of G; in general, for G quasisplit, it is an inner form of a Levi subgroup of G.

centralizer of $\nu(b)$.

(Usual notation is J_b.

$b=1. \rightsquigarrow J_1 = G$, would be strange.)

Cf. discussion last time. $\square$

Back to Fargues-Fontaine curve:

$S \in \mathrm{Perf}_{\mathbb{F}_q}. \rightsquigarrow X_S = X_{S,E}.$

Definition. A G-torsor on X_S is an exact $\otimes$-functor

$$\xi : \mathrm{Rep}_E\, G \longrightarrow VB(X_S).$$

Definition. Bun_G is the v-stack on

$$\mathrm{Perf}_{\overline{\mathbb{F}}_q} : S \longmapsto \{ G\text{-bundles on } X_S \}$$

groupoid.

"stack of G-bundles on the Fargues-Fontaine curve".

Thm. (Fargues if $E/\mathbb{Q}_p$, Anschütz in general). If $S = \mathrm{Spa}(C, C^+)$ C complete alg closed, the functor

$$G\text{-Isoc} \longrightarrow \mathrm{Bun}_G(S)$$

$$\| \qquad\qquad\qquad \|$$

$$\begin{array}{ccc} \| & & G\text{-torsors on} \\ G\text{-torsors on } \mathrm{Spa}\,\breve{E}/\sigma^{\mathbb{Z}} & \longleftarrow & Y_S/\phi^{\mathbb{Z}} = X_S. \end{array}$$

induces a bijection on isom. classes

$\rightsquigarrow \quad \mathrm{Bun}_G(S)/\cong \;\; \cong B(G).$

$\rightsquigarrow \quad |\mathrm{Bun}_G| \;\; \cong B(G).$

<u>Sketch</u>. Let $\mathcal{E}$ G-torsor on $X_C := X_S$.

For any $V \in \mathrm{Rep}_E G$,

$\mathcal{E}(V)$ has HN-filtr., so

actually get exact $\otimes$-functor

Use: HN + $\otimes$ compatible, by classification.

trivial if $E/\mathbb{Q}_p$, as then $\mathrm{Rep}_E G$ is semisimple.

$$\begin{tikzcd} \mathrm{Rep}_E G \arrow[r] \arrow[d] & \mathbb{Q}\text{-Fil}\,VB(X_C)^{HN} \arrow[dl] \\ VB(X_C). & \end{tikzcd}$$

$\mathbb{Q}\text{-Fil}\,VB(X_C)^{HN} = \left\{ \begin{array}{l} \mathcal{E} \;\; \mathbb{Q}\text{-filt. VB on } X_C \\ \cup\!\shortmid \\ \mathcal{E}^{\geq\lambda} \;\; \text{s.th. all } \mathcal{E}^{\lambda} \text{ are semistable of } \ldots \end{array} \right\}.$

in char. p, need to use theorem of Haboush. slope λ

$\rightsquigarrow$ can project to

$$\mathbb{Q}\text{-}\mathrm{Gr}\,\mathrm{VB}(X_C)^{HN} \cong \mathrm{Isoc}_E.$$

$\rightsquigarrow$ get candidate G-isocrystal, need to split filtration.

Use $H^1(X_C, \mathcal{O}(\lambda)) = 0$ for $\lambda > 0$.

$\square$.

Cor. $b \in B(G)$ basic

$\iff \mathcal{E}_b \in \mathrm{Bun}_G(X_C)$ semistable in sense of Atiyah-Bott.

Thm. $|\mathrm{Bun}_G| \longrightarrow B(G)$ is continuous, i.e.:

$-\nu : |\mathrm{Bun}_G| \longrightarrow (X^+_{\mathbb{Q}})^{\Gamma}$ is semi-continuous.

- $\kappa : |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$ locally constant.

In fact,

$$\kappa : \pi_0 \mathrm{Bun}_G \xrightarrow{\ \cong\ } \pi_1(G)_\Gamma .$$

Proof: Next time.

Analogue of Thm of Rapoport-Richartz for families of G-isocrystals

$$\begin{tikzcd}
H^1_{\text{ét}}(\mathrm{Spec}\, E, G) \arrow[r, hook] \arrow[d, phantom, "\wr"] \arrow[dr] & B(E,G). \arrow[d, "G."'] \\
G\text{-torsors}/E \arrow[d, phantom, "\wr"] & H^1_{\text{ét}}(X_S, G) \\
\text{exact} \otimes\text{-functors to } E\text{-v.s.} \arrow[d, hook] & \\
\mathrm{Isoc}_E . &
\end{tikzcd}$$