The stack of vector bundles on the curve.

E nonarch local field

$\cup\!\!|$

$\mathcal{O}_E \ni \pi$, $\mathbb{F}_q$, $\overline{\mathbb{F}}_q$.

$S \in \mathrm{Perf}_{\mathbb{F}_q} \rightsquigarrow X_S = X_{S,E}$

relative FF curve.

$n \geq 1$.

<u>Definition</u>. Let Bun_n be the moduli (pre)stack on $\mathrm{Perf}_{\mathbb{F}_q}$,

$$S \longmapsto \{\text{rk } n \text{ vector bundles on } X_S\}.$$

groupoid.

<u>Proposition</u>. 1). Bun_n is a v-stack.

2) On $\mathrm{Perfd} = \{\text{perfectoid spaces } / \mathbb{Z}_p\}$, have v-descent for vector bundles.

Proof. 2). [Berkeley, Lemma 17.1.8].

(Sketch). know analytic descent by Kedlaya-Liu.

have to prove:

If $Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$

$\underbrace{\text{v-cover}}_{\uparrow}$ of affinoid perfectoid spaces,

$(|Y| \to |X| \text{ surj.})$

then (Proj = cat. of fin. proj. modules)

$$\mathrm{Proj}(R) \xrightarrow{\sim} \left\{ \begin{array}{l} N \in \mathrm{Proj}(S) + \alpha : N \hat{\otimes}_R S = S \hat{\otimes}_R N \\ \text{of fin. proj. } S \hat{\otimes}_R S\text{-modules} \\ \text{satisfying cocycle condition} \\ \text{over } S \hat{\otimes}_R S \hat{\otimes}_R S \end{array} \right\}.$$

equiv. of categories.

already know fully faithfulness: have right adj.

$$(N, \alpha) \mapsto \mathrm{eq}\left(N \rightrightarrows N \hat{\otimes}_R S\right).$$

unit of adjunction

$$M \longmapsto eq(M \hat{\otimes}_R S \rightrightarrows M \hat{\otimes}_R S \hat{\otimes}_R S)$$

is an isom. as structure sheaf is a v-sheaf (+ M fin. proj.)

need to see effectivity of descent data.

<u>Step 1.</u> Case $R = K$ perfectoid field.

may assume S (which is a K-Banach algebra)

is topologically countably generated.

(Use: everything commutes with "countably-filtered colimits".)

$\Rightarrow$ S is free as a K-Banach space.

In particular, $- \hat{\otimes}_K S$ exact, conservative.

Let (N, α) descent datum,

$M = eq(N \rightrightarrows N \hat{\otimes}_R S)$. want:

$$M \hat{\otimes}_K S \xrightarrow{\sim} N.$$

But
$$M \hat{\otimes}_K S \cong \mathrm{eq}\left(N \hat{\otimes}_K S \rightrightarrows N \hat{\otimes}_K S \hat{\otimes}_K S \right) \cong N.$$

$$0 \to N \to N \hat{\otimes}_K S \to N \hat{\otimes}_K S \hat{\otimes}_K S \to \cdots$$

always exact.

(as it admits contracting homotopy).

finishes Case 1.

<u>General Case</u>. Back to $R \to S$ general.

Let $x \in X$ any point, with completed residue field $K(x)$ (some perfectoid field).

$\rightsquigarrow$ $K(x) \longrightarrow S \hat{\otimes}_R K(x)$. by base change,

can do descent here.

In particular, given any descent datum

(N, α), $\quad N \hat{\otimes}_S (S \hat{\otimes}_R K(x))$ is finite free,

and admits basis that is invariant under α.

Thus for some rational nbhd $\mathcal{U} \subseteq X$ of x,

$$N \hat{\otimes}_S (S \hat{\otimes}_R \mathcal{O}_X(\mathcal{U})) \quad \text{is finite free,}$$

and admits a basis such that α is given

by a matrix $\equiv 1 \bmod \varpi$, for $\varpi \in R^+$ pseudouniformizer.

enough to descend over $\mathcal{U}$ (by analytic descent),

so wlog $\mathcal{U} = X$, so

$$(N, \alpha) = \left(S^n, \alpha \in GL_n(S \hat{\otimes}_R S)\right).$$

In fact $\alpha \in GL_n(S^+ \hat{\otimes}_{R^+} S^+)$

$\alpha \equiv 1 \bmod \varpi$.

Claim. Can change basis so that α becomes equal to 1.

Prove claim by successive approximation.

Use: $\left(\frac{\alpha-1}{\varpi} \bmod \varpi\right) \in M_n(S^+ \hat{\otimes}_{R^+} S^+/\varpi)$

is an *additive* cocycle.

But $H^1_v(X, \mathcal{O}^+/\varpi) \overset{\text{almost}}{=} 0$.

i.e. killed by ϖ^ε, $\varepsilon > 0$.

$\rightsquigarrow$ Can change basis to ensure

$\alpha \equiv 1 \bmod \varpi^{2-\varepsilon}$, any $\varepsilon > 0$.

Then continue.

$E_\infty \hat{\otimes}_E E_\infty$ not 'uniform'.

1). Bun_n is a v-stack.

$S \mapsto VB(X_S)$.

Use: $X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$

$E_\infty = E(\pi^{1/p^\infty})^\wedge$

is perfectoid, takes v-covers to v-covers.

$\rightsquigarrow$ can descent vector bundles on

$X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ by part 2).

Now descend along E_∞ / E by argument from Case 1. $\square$

$\{E\text{-v.s.}\} \hookrightarrow \{G_E\text{-eq. } \hat{\bar{E}}\text{-v.s.}\}$.

<u>Question</u>. Can one descend perfect complexes?

<u>Remark</u>. In "Proj. of Witt vector affine Grassm",

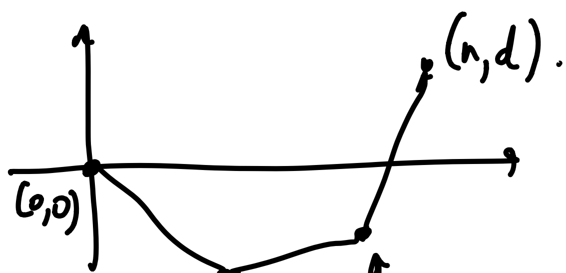
similar v-descent results for VB & perfect

complexes on perfect <u>schemes</u> are proved.

<u>Structure of Bun_n.</u>

<u>Points of Bun_n:</u> By classification of vector bundles,

$|\mathrm{Bun}_n|$ has only countably many points, enumerated by Newton polygons of width n.



break pts have integral coordinates.

let $\mathcal{B}(GL_n)$ = set of such Newton polygons

$= \{n\text{-dim'l isocrystals}\}/\cong$.

$\rightsquigarrow$ bijection $|\mathrm{Bun}_n| = B(GL_n)$.

Say $U \subset |\mathrm{Bun}_n|$ open if it corresponds to an open substack. $\rightsquigarrow$ $|\mathrm{Bun}_n|$ top. space.

equiv., if $X \twoheadrightarrow \mathrm{Bun}_n$ v-cover by perfd space

$$Y \twoheadrightarrow X \times_{\mathrm{Bun}_n} X,$$

then $|\mathrm{Bun}_n| = |X|/|Y|$.

<u>Question</u>. How to describe this on $B(GL_n)$?

Introduce partial order on $B(GL_n)$ by majorization order: $P \geq P'$ if P lies on or above P' with same endpoints.

$\rightsquigarrow$ topology on $B(GL_n)$: $U \subset B(GL_n)$ open if $\forall\, P \in U$, $P' \geq P$, also $P' \in U$.

<u>Thm</u> (Kedlaya-Liu, last time):

$$|\mathrm{Bun}_n| \longrightarrow B(GL_n) \quad \text{continuous.}$$

<u>Thm</u> (Hansen et. al.)

$$|\mathrm{Bun}_n| \longrightarrow B(GL_n) \quad \text{is a homeomorphism.}$$

Remark. For general G, this was announced by Viehmann.

Picture:

$$|\mathrm{Bun}_1| = |\mathrm{Pic}| \cong \mathbb{Z}.$$

discrete

$\vdots$

- $\mathcal{O}(2)$
- $\mathcal{O}(1)$
- $\mathcal{O}$
- $\mathcal{O}(-1)$
- $\mathcal{O}(-2)$

$\vdots$

$|\mathrm{Bun}_2|$

$\vdots$

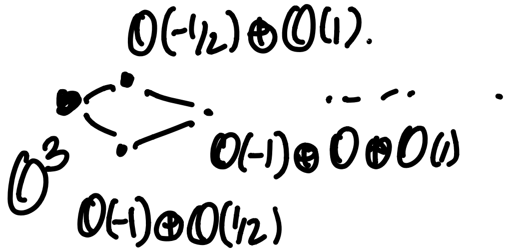
- $d=1$: $\mathcal{O}(1/2)$, $\mathcal{O}\oplus\mathcal{O}(1)$, $\mathcal{O}(-1)\oplus\mathcal{O}(2)$
- $d=0$: $\mathcal{O}^2$, $\mathcal{O}(-1)\oplus\mathcal{O}(1)$, $\mathcal{O}(-2)\oplus\mathcal{O}(2)$, $\mathcal{O}(-3)\oplus\mathcal{O}(3)$
- $d=-1$: $\mathcal{O}(-1/2)$
- $d=-2$: $\mathcal{O}(-1)^2$, $\mathcal{O}(-2)\oplus\mathcal{O}$

$\vdots$

$- \otimes_{\mathcal{O}} \mathcal{O}(1)$ makes the picture 2-periodic.

$|\mathrm{Bun}_3|$

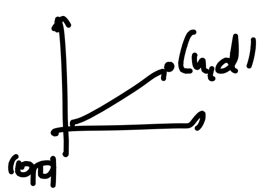
$d=0$



Complicated.

<u>Observation</u>. $\pi_0 \, \mathrm{Bun}_n \xrightarrow{\cong} \mathbb{Z}$ given by degree;

each connected component has a <u>unique</u> semistable point.



$$\mathcal{O}(d/n) = \mathcal{O}(a/b)^c$$

if $d=ac$, $n=bc$, $(a,b)=1$.

For $b \in B(GL_n)$, let

$\mathrm{Bun}_n^b \subseteq \mathrm{Bun}_n$ corresponding locally closed stratum.

Description of Bun_n^b:

1). $b \cong \mathcal{O}^n$.

<u>Note:</u> <u>Cor</u> (of Thm of Kedlaya-Liu)

For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, vector bundles on X_S that are semistable of slope 0 in each fibre are equivalent to $\underline{E}$-local systems via.

$$\mathcal{E}_{\mathcal{E}} \longmapsto \mathcal{B}\mathcal{T}_{\mathcal{E}}(\mathcal{E}_{\mathcal{E}})$$

$$\mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S} \longleftarrow \mathbb{L}$$

(+ pro-étale sheafification)

follows from pro-étale descent + fact that any such $\mathcal{E}_{\mathcal{E}}$ is pro-étale locally trivial.

<u>Cor.</u> The stratum of Bun_n corr. to

$\mathcal{O}^n$ is $[*/\underline{GL_n(E)}]$.

$\uparrow$ classifies rk n $\underline{E}$-local systems.

Note: T any topological space

$\rightsquigarrow$ $\underline{T} : S \longmapsto \mathrm{Cont}(|S|, T)$

is a v-sheaf.

$\rightsquigarrow$ open immersion

$$[*/\underline{GL_n(E)}] \overset{j}{\hookrightarrow} \mathrm{Bun}_n.$$

In particular

$$\{\text{representations of } GL_n(E)\} = \{\text{sheaves on } [*/\underline{GL_n(E)}]\}.$$

$$\downarrow j_!$$

$\{\text{sheaves on } \mathrm{Bun}_n\}$.

2) Semistable points:

For semistable bundle

$$\mathcal{O}(d/n) := \mathcal{O}(a/b)^{\oplus c},$$

$$\underline{\mathrm{Aut}}(\mathcal{O}(d/n)) = \underline{GL_c(D_{a/b})}$$

$D_{a/b} = \mathrm{End}(\mathcal{O}(a/b))$ is the central division algebra over E of Hasse invariant a/b,

$$\rightsquigarrow \mathrm{Bun}_n^b \cong [*/\underline{GL_c(D_{a/b})}].$$

$$\uparrow$$
$$*$$

(E-valued pts of) inner form of GL_n/E.

$\rightsquigarrow \{\text{repr. of } \underline{GL_c(D_{a/b})}\}$.

$\cap$

$$\downarrow j_{b!}$$

$$\{\text{sheaves on } \mathrm{Bun}_n\}.$$

So Bun_n sees representations of $GL_n(E)$ and all its inner forms.

3) Non-semistable points:

<u>example</u>. Bun_2, $b \leftrightarrow \mathcal{O} \oplus \mathcal{O}(1)$.

always $\mathrm{Bun}_n^b = [*/\underline{\mathrm{Aut}}(\mathcal{E}_b)]$

where $\mathcal{E}_b$ = vector bundle corr. to b,

$\underline{\mathrm{Aut}}(\mathcal{E}_b)$ v-sheaf $S \mapsto \mathrm{Aut}(\mathcal{E}_b|_{X_S})$.

$$\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \underline{E}^\times & \mathcal{BC}(\mathcal{O}(1)) \\ 0 & \underline{E}^\times \end{pmatrix}$$

$\mathcal{BC}(\mathcal{O}(1))$ perfectoid open unit disc.

so

$$1 \to \mathcal{BC}(\mathcal{O}(1)) \to \underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) \to \underline{E}^\times \times \underline{E}^\times \to 1.$$

1-dim'l, connected.

in general:

$$\mathcal{E}_b = \bigoplus_{\lambda \in \mathbb{Q}} \underbrace{\mathcal{O}(\lambda)^{n_\lambda}}_{\mathcal{E}_b^\lambda}$$

$\mathcal{BC}(\mathcal{H}om(\mathcal{E}_b^{\lambda_1}, \mathcal{E}_b^{\lambda_2}))$

...

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = \begin{pmatrix} \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_1}) & & & \\ & \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_2}) & & \\ & & \ddots & \\ & & & \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_n}) \end{pmatrix}$$

$\dim \underline{\mathrm{Aut}}(\mathcal{E}_b)$

$= \langle 2\rho, \nu_b \rangle.$

$$1 \to \text{"unipotent"} \to \underline{\mathrm{Aut}}(\mathcal{E}_b) \to \text{locally profinite group} \to 1.$$

extension of positive Banach-Colmez spaces.

locally profinite group $= \mathrm{Aut}(V_b)$

$J_b(E)$ in Kottwitz' notation

$J_b(E)$.

$\mathcal{V}_b$ = isocrystal corresponding to b.

(E-valued pts of) inner form of a Levi subgroup of GL_n.

<u>Note:</u> unipotent groups cannot act on ℓ-adic sheaves.

$\rightsquigarrow$ $\{\ell\text{-adic sheaves on } \mathrm{Bun}_n^b\}$

$$\cong$$

$\{\text{repr's of } J_b(E)\}$.

$\rightsquigarrow$ $\{\ell\text{-adic sheaves on } \mathrm{Bun}_n\}$

built via semi-orth. from pieces

$\{\ell\text{-adic sheaves on } \mathrm{Bun}_n^b\}$.

$$\cong$$

$\{\text{repr's of } J_b(E)\}$.

How do the different strata interact?

<u>Example.</u> $n=2$.

$$(\mathbb{P}^1_E)^\diamond \longrightarrow \mathrm{Bun}_2 :$$

$$\circlearrowleft GL_2(E)$$

Given $S^\#/E$ + $\mathcal{O}^2_{S^\#} \twoheadrightarrow L$ ← line bundle $/S^\#$.

$$\left(\cong \text{ map } S \longrightarrow (\mathbb{P}^1_E)^\diamond \right)$$

Can build

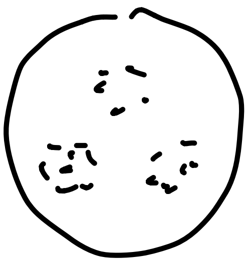
$$\mathcal{E}(L) = \ker\left(\mathcal{O}^2_{X_S} \twoheadrightarrow \mathcal{O}^2_{S^\#} \twoheadrightarrow L \right)$$

$$S^\# \hookrightarrow X_S$$

closed Cartier divisor

giving a rk 2 vector bundle on X_S, so a map $S \longrightarrow \mathrm{Bun}_2$.

Proposition. The image lands in



$$\begin{tikzcd}
\mathrm{Bun}_2^{\mathcal{O}(-1/2)} \arrow[r, phantom, "\sqcup"] & \mathrm{Bun}_2^{\mathcal{O}\oplus\mathcal{O}(-1)} \\
(\mathcal{O}^2)^\diamond \arrow[u] \arrow[r, phantom, "\sqcup"] & \mathbb{P}^1(E) \arrow[u]
\end{tikzcd}$$

$\mathrm{Bun}_2^{\mathcal{O}(-1/2)} \cong [x/D_{1/2}^\times]$

$(\mathcal{O}^2)^\diamond = \mathbb{P}^1_E \setminus \mathbb{P}^1(E)$

giving the above stratification of $\mathbb{P}^1_E$.

Proof. For gen. pt., $\mathcal{E}(L)$ necessarily $\mathrm{rk}\, 2$, $\deg -1$.

$$\begin{tikzcd}
\mathcal{E}(L) \arrow[d, hook] \\
\mathcal{O}^2
\end{tikzcd}$$

$\mathcal{E}(L) \cong \mathcal{O}(-1/2)$ or $\mathcal{O}(-i-1) \oplus \mathcal{O}(i)$, $i \geq 0$.

necessarily $i = 0$.

If $\mathcal{E}(L) \cong \mathcal{O}(-1) \oplus \mathcal{O}$,

$$
\begin{tikzcd}
\mathcal{O} \arrow[r, hook] \arrow[rrrr, bend right, "0."'] & \mathcal{E}_e(L) \arrow[r, hook] & \mathcal{O}^2 \arrow[r, two heads] & \mathcal{O}_{s^*}^{2} \arrow[r, two heads] & L.
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cong E \arrow[r, "\neq 0"] \arrow[rrr, bend right, "0"'] & E^2 \arrow[r] & C^2 \arrow[r, two heads] & L.
\end{tikzcd}
$$

$\Rightarrow$ L must be E-rational, so point of $\mathbb{P}^1(E)$.

Conversely, if the point lies in $\mathbb{P}^1(E)$, then up to $Gl_2(E)$-action may assume it is $[0:1]$, then $\mathcal{E}_e(L) \cong \mathcal{O} \oplus \mathcal{O}(-1)$. $\square$.

$\bar{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{F}_\ell$, $\mathbb{Z}/\ell^n\mathbb{Z}$ $\cancel{\mathbb{F}_p, \mathbb{Z}_p}$.

<u>$\ell \neq p$.</u>