Geometrization of the local Langlands correspondence

(joint with Laurent Fargues)

Let me try this pen.

Setup. • E nonarchimedean local field

- either $E \cong \mathbb{F}_q((t))$
- or E finite extension of $\mathbb{Q}_p$.

($E = \mathbb{Q}_p$)

Notation: $\mathbb{F}_q$ residue field, $\pi \in \mathcal{O}_E$ uniformizer.

• G/E reductive group.

e.g. $G = GL_n, Sp_{2n}, SO_{2n}, U_n, \ldots, E_8, G_2$.

interested in representation theory of locally profinite group $G(E)$.

<u>Recall</u>. <u>Definition</u> Let Γ locally profinite group. Let L field. A <u>smooth</u> repr. ← "locally constant" of Γ over L is an L-vector space V + map $\Gamma \longrightarrow GL(V)$ s.th. $\forall v \in V \quad Stab(v) \subseteq \Gamma$ is an open subgroup.

<u>Examples</u>. 1) Let $K \subseteq \Gamma$ open compact subgroup, $K \twoheadrightarrow \overline{K}$ finite quotient,

$$\rho : \overline{K} \to GL_n(L) = GL(V_0) \text{ repr. of } \overline{K}$$

$$\rightsquigarrow \quad c\text{-}Ind_K^\Gamma \rho = \left\{ f : \Gamma \to V_0 \;\middle|\; \begin{array}{l} f \text{ has compact support,} \\ f(\gamma k) = f(\gamma) k \\ \forall \gamma \in \Gamma, k \in K \end{array} \right\}.$$

$\circlearrowleft$
Γ is smooth repr.

These repr. form compact projective generators (at least if $\mathrm{Char}\, L = 0$, or $\# K \in L^\times$).

ex. $K = GL_n(\mathcal{O}_E) \subseteq \Gamma = GL_n(E)$

$$\begin{array}{c} \downarrow \\ \tilde{} \\ \bar{K} = GL_n(\mathbb{F}_q) \end{array}$$

ρ supercusp. repr. of $GL_n(\mathbb{F}_q)$.

$\rightsquigarrow$ (up to center) $c\text{-Ind}_K^\Gamma \rho$ is "irreducible supercuspidal repr."

2). If $P \subseteq G$ parabolic with Levi L

(V_0, φ_L) smooth repr. of $L(E)$ then

$$\mathrm{Ind}_{P(E)}^{G(E)} V_0 = \left\{ f : G(E) \to V_0 \,\middle|\, \begin{array}{l} f(gp) = f(g) \cdot \bar{p} \\ \forall g \in G(E), p \in P(E) \end{array} \right\}$$

$$\begin{array}{r} p \\ \downarrow \\ \bar{p} \in L(E). \end{array}$$

smooth repr. of $G(E)$.

In some vague sense, all irred. smooth repr. of $G(E)$ are built ~~fro~~ via parabolic induction from supercuspidal representations of Levi subgroups.

3). If $\mathbb{E}$ ~~[illegible]~~ global field

$\mathbb{G}$ reductive $/\mathbb{E}$

s.th. (G, E) arises as localization of $(\mathbb{G}, \mathbb{E})$,

space of automorphic forms

$$\mathcal{A}\left(\mathbb{G}(\mathbb{E}) \backslash \mathbb{G}(\mathbb{A}_{\mathbb{E}}), \mathbb{C}\right) \circlearrowleft \mathbb{G}(\mathbb{A}_{\mathbb{E}})$$

$$\cup\!\!\mid$$

$$G(E).$$

is a smooth repr. of $G(E)$.

In some sense, study of ~~[illegible]~~ category of smooth representations of $G(E)$ is local analogue of study of space of automorphic forms.

<u>Conjecture</u> (Langlands). (Assume for simplicity that G is split)

$L = \mathbb{C}$.

There is a natural map

$$\mathrm{Irr}(G(E))/\cong \longrightarrow \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))/_{\mathrm{conj}}\, \hat{G}(\mathbb{C})$$

"L-parameters"

$$\pi \longmapsto \varphi_\pi$$

where

- $\hat{G}$ = Langlands dual group (see below).
- W_E = Weil group of E

$$\begin{tikzcd} W_E \arrow[r, two heads] \arrow[d, hook, "\text{dense}"] & \mathbb{Z} \arrow[d, hook] \\ \mathrm{Gal}(\bar{E}/E) \arrow[r, two heads] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} \ni \mathrm{Frob}: x \mapsto x^q \end{tikzcd}$$

s.th. .....

+ description of fibers of

$$\pi \longmapsto \varphi_\pi.$$

<u>Examples:</u> $\widehat{GL_n} = GL_n$

$\widehat{SL_n} = PGL_n$.

$\widehat{Sp_{2n}} = SO_{2n+1}$ $\quad$ $\widehat{SO_{2n}} = SO_{2n}$.

...

<u>Examples of local Langlands</u>

1) $G = \mathbb{G}_m$.

$G(E) = E^\times$ abelian.

$\mathrm{Irr}(E^\times) = \{ \chi : E^\times \longrightarrow \mathbb{C}^\times \}$ characters.

$\hat{G} = \mathbb{G}_m$.

$\mathrm{Hom}(W_E, \hat{G}(\mathbb{C})) = \mathrm{Hom}(W_E, \mathbb{C}^\times)$

$= \mathrm{Hom}(W_E^{ab}, \mathbb{C}^\times)$

Local Class Field Theory:

$W_E^{ab} \cong E^\times \cong \mathcal{O}_E^\times \times \mathbb{Z}$ $\quad$ get desired bijection.

$\left(\text{cf. } \mathrm{Gal}(\bar{E}/E)^{ab} \cong \widehat{E^\times} \cong \mathcal{O}_E^\times \times \hat{\mathbb{Z}} \right)$

2). $G = GL_n$. Then $\hat{G} = GL_n$.

<u>Thm</u> (Laumon–Rapoport–Stuhler '90 $E = \mathbb{F}_q((t))$
Harris–Taylor; Henniart '01 $E/\mathbb{Q}_p$).
$L = \mathbb{C}$.

$$\begin{array}{ccc}
\left\{\begin{array}{l}\text{supercuspidal irred. repr.}\\ \text{of } GL_n(E)\end{array}\right\}\Big/\cong & \xleftrightarrow{\ \cong\ } & \left\{\begin{array}{l}\text{irred.}\\ n\text{-dim'l}\\ \text{repr. } W_E \to GL_n(\mathbb{C})\end{array}\right\}\Big/\cong \\
\cap\!| & & \cap\!| \\
\mathrm{Irr}(GL_n(E))/\cong & \longleftrightarrow & \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))/\cong
\end{array}$$

$\Downarrow$ Bernstein–Zelevinsky 70's

<u>Example</u>. E'/E degree n extension

$\chi' : W_{E'} \to W_{E'}^{ab} \cong E'^{\times} \longrightarrow \mathbb{C}^{\times}$ character, "generic".

$\rightsquigarrow \varphi = \mathrm{Ind}_{W_{E'}}^{W_E} \chi'$ irred. n-dim'l repr. of W_E.

$\rightsquigarrow$ "automorphic induction".

supercusp. repr. $\pi_{\varphi} = \pi_{\chi'}$ of $GL_n(E)$.

Recent improvements (Hellmann, Zhu, Ben-Zvi-Chen-Helm-Nadler) conjecturally describe whole category $\mathrm{Rep}(G(E))$ in terms of $\underbrace{\mathrm{Hom}(W_E, \hat{G})/\hat{G}}_{\text{Artin stack.}}$

<u>Question:</u> 1) How does W_E relate to $\mathrm{Rep}(G(E))$?

2) Where does $\hat{G}$ come from?

$\hat{G}$: split groups over any field are classified by root data.

$$(X, \Phi, X^*, \Phi^\vee)$$

char. lattice, roots, cochar. lattice, coroots.

Also $(X^*, \Phi^\vee, X, \Phi)$ is a root datum. $\rightsquigarrow$ dual group $\hat{G}$.

Goal of course:

- ~~Des~~ Give construction of map

$$\underset{\text{irr. repr.}}{\pi} \longmapsto \underset{\text{L-param.}}{\varphi_\pi}$$

that works uniformly for any reductive group G,

and is purely local.

- In doing so, "explain" where W_E and $\hat{G}$ "come from".

- Formulate a form of local Langlands correspondence as equivalence of categories, and (essentially) construct a functor in one direction.

- Extend everything from char. 0 coefficients to coeff. Λ, $p \in \Lambda^\times$.

<u>Idea</u>. Develop the <u>geometric Langlands program</u> on the <u>Fargues–Fontaine curve</u>, using <u>geometry of perfectoid spaces/ diamonds</u>.

<u>Basic References</u>: - Berkeley Lectures on p-adic geometry

- Lecture Notes for a Montreal "Barbados" workshop.

Big Picture

Contemplation of "space Spec E".

$$\mathrm{Spec}\, E$$

coherent

vector bundles, E-vector spaces, G-torsors

étale

$$\pi_1^{\acute{e}t}(\mathrm{Spec}\, E) = \mathrm{Gal}(\bar{E}/E).$$

$$\underbrace{[*/G]}_{\text{quotient stack, classifying } G\text{-torsors}}(\mathrm{Spec}\, E) = \coprod_{\alpha \in H^1_{\acute{e}t}(E,\, G)} [*/G_\alpha(E)] \supseteq [*/G(E)].$$

$H^1_{\acute{e}t}(E, G)$: G-torsors up to isom.

G_α inner form of G.

$$\rightsquigarrow \quad \mathrm{Rep}(G(E)) = \mathrm{Shv}([*/G(E)]).$$

$$\cap$$

$$\mathrm{Shv}([*/G](\mathrm{Spec}\, E)).$$

Bernstein, Vogan.

want to change $\mathrm{Gal}(\overline{E}/E)$ into W_E.

$\Rightarrow$ For scheme $X/\mathbb{F}_q$, replace X by $X_{\overline{\mathbb{F}}_q} \circlearrowleft \mathrm{Frob}$.

$$\begin{array}{ccc} \pi_1(X_{\overline{\mathbb{F}}_q}/\mathrm{Frob}) & = & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \times \mathbb{Z} \\ \cap & & \cap \\ \pi_1(X) & = & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \times \hat{\mathbb{Z}}. \end{array}$$

This suggests replacing

$\mathrm{Spec}\ \mathbb{F}_q((t))$ by $\mathrm{Spec}\ \overline{\mathbb{F}}_q((t))/\mathrm{Frob}$.

$\mathrm{Spec}\ E$ by $\mathrm{Spec}\ \breve{E}/\mathrm{Frob}$.

completion of max'l unram. extension.

$$\mathrm{Spec}\ \breve{E} / \mathrm{Frob}.$$

coherent

étale

$\mathrm{Isoc}_E = \{ \breve{E}$-vector spaces V + Frob-lin. isom. $\phi : V \cong V \}$.

$\pi_1 = W_E$.

$\cup\!|$

E-v.s.

$\mathbb{R}$. Dieudonné-Manin.

$$\bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^\lambda$$

isocrystals "pure of slope λ".

$\mathrm{Isoc}_E^\lambda \cong D_\lambda$-vector spaces,

where D_λ / E central division algebra of invariant $\lambda \in \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$.

G-torsors in Isoc_E.

(Kottwitz).

$$G\text{-}\mathrm{Isoc} \cong \bigsqcup_{b \in B(E, G)} [* / G_b(E)]$$

G_b inner form of Levi subgroup of G.

$$H^1(E, G) \hookrightarrow B(E, G)$$

$$\alpha \longmapsto b, \quad G_b = G_\alpha.$$

Kottwitz, Kaletha: profitable to consider all G_b

need to be more geometric:
want a geometric stack of
G - isocrystals.

<u>Several ways to define a stack of
G - isocrystals:</u>

0). Isoc_E is E-linear category, so
for any E-algebra A, can consider
G-torsors in $\mathrm{Isoc}_E \otimes_E A$.

$\rightsquigarrow$ Artin stack $/E$,

$$\| \\ \bigsqcup_{b \in B(E,G)} [*/G_b]$$

as algebraic group.

$\rightsquigarrow$ alg repr. of alg group G_b, not
desired category.

~~Bettes~~

1). Replace $\overline{\mathbb{F}}_q$ by any (perfect) $\overline{\mathbb{F}}_q$-algebra R. (discrete).

$$\operatorname{Spec} \overline{\mathbb{F}}_q((t)) / \mathrm{Frob} \rightsquigarrow \operatorname{Spec} R((t)) / \mathrm{Frob}.$$

$$\operatorname{Spec} \breve{E} / \mathrm{Frob} \rightsquigarrow \operatorname{Spec} \left(W(R) \otimes_{W(\overline{\mathbb{F}}_q)} E \right) / \mathrm{Frob}.$$

$$\breve{E} = W(\overline{\mathbb{F}}_q) \otimes_{W(\mathbb{F}_q)} E$$

define stack on perfect $\overline{\mathbb{F}}_q$-alg.:

$$\begin{aligned} R \longmapsto \{ & G\text{-torsors on } \operatorname{Spec} R((t)) / \mathrm{Frob} \\ G\text{-Isoc}: \quad & \text{resp. } \operatorname{Spec} \left(W(R) \otimes_{W(\mathbb{F}_q)} E \right) / \mathrm{Frob} \} \end{aligned}$$

Thm (Rapoport–Richartz, Caraiani–S., Ivanov, Anschütz).

G-Isoc is stack for v- (arc-) topology. ($\to$ fpqc...).

For any $b \in B(E,G)$, get locally closed substack

$$G\text{-}\mathrm{Isoc}^b \subseteq G\text{-}\mathrm{Isoc}.$$

$G\text{-}\mathrm{Isoc}^b \cong [*/G_b(E)]$ — locus where isom. to b.

$G\text{-}\mathrm{Isoc} = LG/_{\sigma\text{-conj}} LG.$

Xiao–Zhu, Hemo-Z. — Gaitsgory, Genestier–V. Lafforgue, Zhu.

They define

$$\boxed{D(G\text{-}\mathrm{Isoc}, \overline{\mathbb{Q}}_\ell)},$$

has semi-orthog. decomposition into pieces

$$D(G\text{-}\mathrm{Isoc}^b, \overline{\mathbb{Q}}_\ell) \simeq D(G_b(E), \overline{\mathbb{Q}}_\ell).$$

(derived cat. of smooth repr.)

$\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

How to get a relation to W_E & $\hat{G}$?

<u>Answer</u>: Hecke operators.

These are related to <u>modifications</u> of G-torsors, i.e.

$\mathcal{E}_1, \mathcal{E}_2$ G-torsors $\big/ \left(\mathrm{Spec}\, R(A) / \mathrm{Frob}\right)$.

+ isom. $\mathcal{E}_1 \cong \mathcal{E}_2$ away from some divisor $D \subseteq \mathrm{Spec}\, R(A)$.

$$\begin{tikzcd}
D \arrow[r, phantom, "\subseteq"] \arrow[dr, "\sim"'] & \mathrm{Spec}\, R(A) \arrow[d] \\
& \mathrm{Spec}\, R \arrow[u, bend right, dotted]
\end{tikzcd}$$

This <u>requires sections</u>!

$\Rightarrow$ need to take R a Banach algebra.

2). To any perfectoid affinoid alg. (R, R^+) $/\overline{\mathbb{F}_q}$,

Can associate

<u>Fargues-Fontaine curve</u>

$$\shortparallel$$

$$\mathbb{D}^*_{\mathrm{Spa}(R,R^+)} / \mathrm{Frob}.$$

(resp. similar object in mixed char.)

$\sim$ moduli space of G-torsors on the Fargues-Fontaine curve.

Bun_G. $\qquad D(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$

$\left(\begin{array}{c} \wr \ \underline{\text{Conj}} \\ D(G\text{-}\mathrm{Loc}, \overline{\mathbb{Q}_\ell}) \end{array}\right).$

$$\mathrm{Hecke}_{G,\leq\mu} = \{ \mathcal{E}_1 + \mathcal{E}_2 + \text{isom. away from } D \}$$

$$\begin{tikzcd}
 & \mathrm{Hecke}_{G,\leq\mu} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \boxed{\mathrm{Spa}\, \breve{E}/\varphi}
\end{tikzcd}$$

param. sections.

$$T_\mu = p_{2!}\, p_1^* : \boxed{D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)} \rightleftarrows \boxed{\mathrm{Rep}(G(E))}$$

$$\downarrow$$

$$D(\mathrm{Bun}_G \times \mathrm{Spa}\, \breve{E}/\varphi, \overline{\mathbb{Q}}_\ell)$$

$$\|$$

$$D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{\boxed{W_E}}$$

Geom. Satake

enumerated by $\boxed{\mathrm{Rep}(\hat{G})}$.

This categorical structure is <u>precisely</u> what is needed to define L-parameters.