

# Geometrization of the local Langlands correspondence

(joint with Laurent Fargues)

Let me try this pen.

Setup. •  $E$  nonarchimedean local field  
- either  $E \cong \mathbb{F}_q((t))$   
- or  $E$  finite extension of  $\mathbb{Q}_p$ .  
( $E = \mathbb{Q}_p$ )

Notation:  $\mathbb{F}_q$  residue field,  $\pi \in \mathcal{O}_E$   
uniformizer.

•  $G/E$  reductive group.

eg.  $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}, \mathrm{U}_{n,1}, \mathrm{E}_8, \mathrm{G}_2$ .

interested in representation theory of  
locally profinite group  $G(E)$ .

Recall. Definition Let  $\Gamma$  locally profinite group. let  $L$  field. A smooth repr. of  $\Gamma$  on  $L$  is an  $L$ -vector space  $V$  <sup>"locally constant"</sup>

+ map  $\Gamma \rightarrow GL(V)$

s.t.  $\forall v \in V$   $Stab(v) \subseteq \Gamma$

is an open subgroup.

Examples. 1) let  $K \subseteq \Gamma$  open compact subgroup,  $K \twoheadrightarrow \bar{K}$  finite quotient,

$\rho: \bar{K} \rightarrow GL_n(L) = GL(V_0)$  repr. of  $\bar{K}$

$\leadsto$   $c\text{-ind}_K^\Gamma \rho = \left\{ f: \Gamma \rightarrow V_0 \mid \begin{array}{l} f \text{ has compact support,} \\ f(yk) = f(y)k \\ \forall y \in \Gamma, k \in K \end{array} \right\}$

$\Gamma$  is smooth repr.

These repr. form compact projective generators (at least if  $\text{Char } L = 0$ , or  $\#K \in L^\times$ .)

ex.  $K = GL_n(O_E) \subseteq \Gamma = GL_n(E)$

$$\downarrow$$

$$\bar{K} = GL_n(\bar{F}_q)$$

$\rho$  supercuspidal repr. of  $GL_n(\bar{F}_q)$ .

$\rightsquigarrow$  (up to center)  $c\text{-ind}_K^\Gamma \rho$  is  
"irreducible supercuspidal repr."

2). If  $P \subseteq G$  parabolic with Levi  $L$   
 $(V_0, \rho_L)$  smooth repr. of  $L(E)$  then

$$\text{ind}_{P(E)}^{G(E)} V_0 = \left\{ f: G(E) \rightarrow V_0 \left( \begin{array}{l} f(y_p) = f(y) \bar{p} \\ \forall y \in G(E), p \in P(E) \end{array} \right) \right\}$$

smooth repr. of  $G(E)$ .

$$\downarrow$$

$$\bar{p} \in L(E).$$

In some vague sense, all irred. smooth  
repr. of  $G(E)$  are built ~~for~~ via parabolic  
induction from supercuspidal representations  
of Levi subgroups.

3). If  $E$  ~~is~~ global field  
 $G$  reductive /  $E$

sth.  $(G, E)$  arises as localization  
of  $(G, E)$ ,

space of automorphic forms

$$\mathcal{L}(G(E), G(A_E), \mathcal{B}) \cong \bigcup_i G(A_E)$$

is a smooth repr. of  $G(E)$ .

In some sense, study of ~~the~~ category  
of smooth representations of  $G(E)$  is  
local analogue of study of space  
of automorphic forms.

Conjecture (Langlands). (Assume for simplicity that  $G$  is split.)  
 $L = \mathbb{C}$ .

There is a natural map

$$\mathrm{Irr}(G(E)) / \cong \longrightarrow \mathrm{Hom}(W_E, \widehat{G}(\mathbb{C}))$$

"L-parameters"

$\cong \widehat{G}(\mathbb{C})$

where  $\pi \longmapsto \mathcal{Y}\pi$

- $\widehat{G}$  = Langlands dual group (see below).

- $W_E$  = Weil group of  $E$

$$\begin{array}{c} \text{Idene} \uparrow \\ \searrow \\ \mathbb{Z} \end{array}$$

$$\mathrm{Gal}(\overline{E}/E) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\mathbb{Z}} \ni \text{Frobenius } x \mapsto x^q$$

sth. ....

+ description of fibers of

$$\pi \longmapsto \mathcal{Y}\pi.$$

Examples:

$$\widehat{GL}_n = GL_n$$

$$\widehat{SL}_n = PGL_n.$$

$$\widehat{Sp}_n = SO_{2n+1} \quad \widehat{SO}_n = SO_{2n}.$$

.....

## Examples of local Langlands

1)  $G = G_m.$

$$G(E) = E^\times \text{ abelian.}$$

$$\text{Irr}(E^\times) = \{ \chi: E^\times \rightarrow \mathbb{C}^\times \} \text{ characters.}$$

$$\widehat{G} = G_m.$$

$$\text{Hom}(W_E, \mathbb{Q}_m \widehat{G}(\mathbb{C})) = \text{Hom}(W_E, \mathbb{C}^\times)$$

$$= \text{Hom}(W_E^{\text{ab}}, \mathbb{C}^\times)$$

Local Class Field Theory:

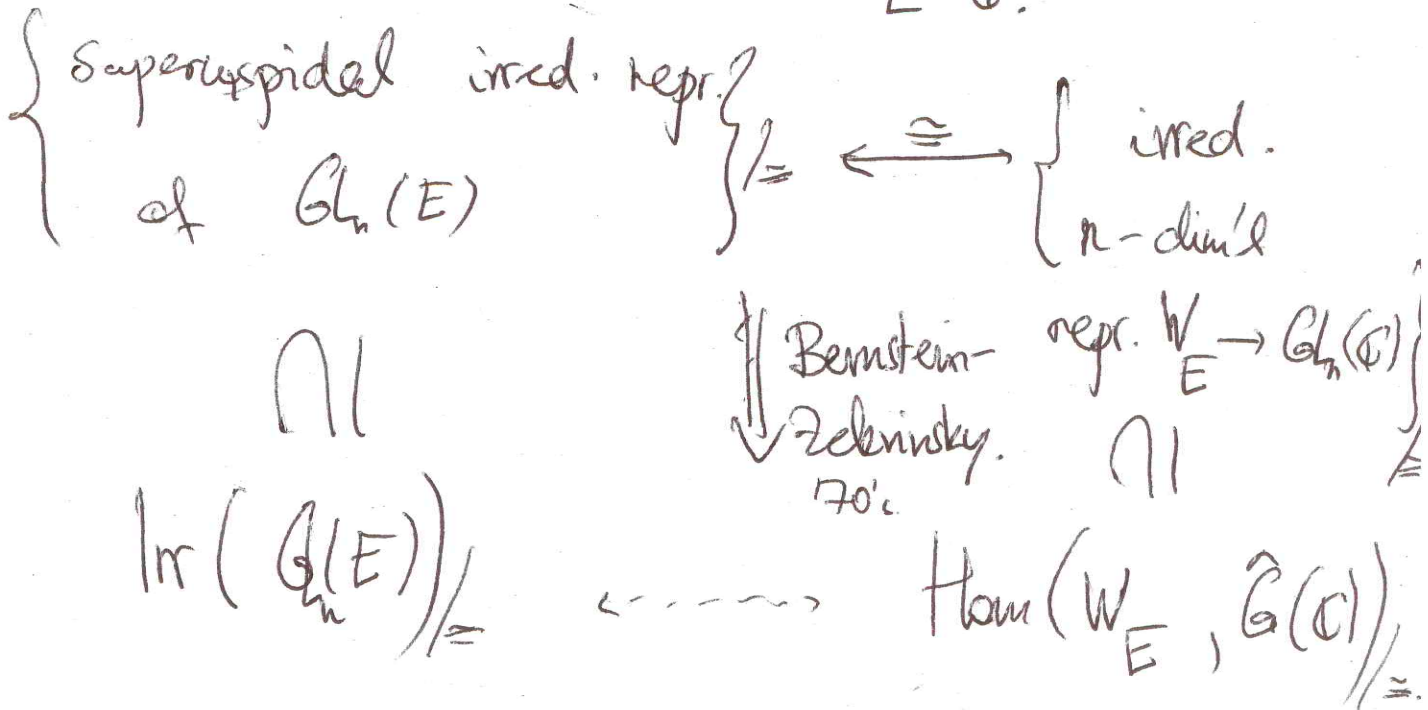
$$W_E^{\text{ab}} \cong E^\times \cong \mathcal{O}_E^\times \times \mathbb{Z}$$

get desired  
- bijection.

(cf.  $\text{Gal}(E/E)^{\text{ab}} \cong \widehat{E^\times} \cong \mathcal{O}_E^\times \times \widehat{\mathbb{Z}}.$ )

2).  $G = GL_n$ . Then  $\hat{G} = GL_n$ .

Thm (Laumon-Rapoport-Stuhler '80  $E = \mathbb{F}_q((t))$   
 Harris-Taylor; Henniart '01  $E/\mathbb{Q}_p$   
 $L = \mathbb{C}$ .)



Example.  $E'/E$  degree  $n$  extension

$$\chi: W_{E'} \rightarrow W_{E'}^{ab} \cong E'^{\times} \longrightarrow \mathbb{C}^{\times} \text{ character, 'generic'}$$

$\rightsquigarrow \rho = \text{hd}_{W_{E'}}^{W_E} \chi'$  irred. n-dim'l repr. of  $W_E$ .  
 $\} \text{ "automorphic induction"}$   
 supercuspid. repr.  $\pi_{\rho} \text{ of } GL_n(E)$ .  
 $\} \pi_{\rho} = \pi_{\chi'}$





# Goal of course:

- ~~Give~~ Give construction of map

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad} & \mathcal{Y}^{\pi} \\ \text{in repr.} & & \text{L-param.} \end{array}$$

that works uniformly for any reductive group  $G$ ,

and is purely local.

- In doing so, "explain" where

$W_E$  and  $\hat{G}$  "come from".

- Formulate a form of local

Langlands correspondence as equivalence of categories, and (essentially) construct a functor in one direction.

- Extend everything from char. 0  
coefficients to coeff.  $\Lambda$ ,  $p \in \Lambda^\times$ .

Idea. Develop the geometric Langlands  
program on the Fargues-Fontaine curve,  
using geometry of perfectoid spaces/  
diamonds.

Basic References: - Berkeley Lectures

on  $p$ -adic geometry

- Lecture Notes for a Montreal  
"Barbados" workshop.

# Big Picture

Contemplation of "space"  $\text{Spec } E^n$ .

$\text{Spec } E$

coherent

étale

vector bundles,  $E$ -vector spaces,

$G$ -torsors

$$\pi_1^{\text{ét}}(\text{Spec } E) = \text{Gal}(E/E)$$

$$[* / G](\text{Spec } E) = \bigsqcup_{\alpha \in H^1_{\text{ét}}(\text{Spec } E, G)} [* / G_{\alpha}(E) \supseteq [* / G(E)]$$

quotient stack,

classifying  $G$ -torsors

$\alpha \in H^1_{\text{ét}}(\text{Spec } E, G)$

$G$ -torsors up to isom.

$G_{\alpha}$  inner form of  $G$ .

$$\rightsquigarrow \text{Rep}(G(E)) = \text{Shv}([* / G(E)])$$

$$\supseteq \text{Shv}([* / G](\text{Spec } E))$$

Bernstein, Vogan.

want to change  $\text{Gal}(\bar{E}/E)$  into  $W_E$ .

For scheme  $X/\mathbb{F}_q$ , replace  $X$  by  $X_{\mathbb{F}_q} \xrightarrow{\text{Frob}}$ .

$$\pi_1(X_{\mathbb{F}_q}/\text{Frob}) = \pi_1^{\text{ét}}(X_{\mathbb{F}_q}) \times \hat{\mathbb{Z}}$$

$$\pi_1(X) = \pi_1^{\text{ét}}(X_{\mathbb{F}_q}) \times \hat{\mathbb{Z}}.$$

This suggests replacing

$\text{Spec } \mathbb{F}_q((t))$  by  $\text{Spec } \bar{\mathbb{F}}_q((t))/\text{Frob}$ .

$\text{Spec } E$  by  $\text{Spec } \bar{E}^c/\text{Frob}$ .

completion of max'l unram. extension.

Spec  $\check{E}/\text{Frob.}$

coherent

étale

$\text{Isoc}_E = \left\{ \begin{array}{l} \check{E}\text{-vector spaces} \\ V + \text{Frob-lin. isom. } \phi: V \cong V \end{array} \right\}$ 
 $\pi_1 = W_E$   
 U.I.  
 E-v.s.  
 R. Dieudonné-Mannin.

$\bigoplus_{\lambda \in \mathcal{B}} \text{Isoc}_E^\lambda$

isocrystals "pure of slope  $\lambda$ ".

$\text{Isoc}_E^\lambda \cong D_\lambda$ -vector spaces,  
 where  $D_\lambda / E$  central division algebra of

invariant  $\lambda \in \mathcal{B} \rightarrow \mathbb{Q}/\mathbb{Z}$ .

G-torsors in  $\text{Isoc}_E$ . (Kottwitz).

$$G\text{-Isoc} \cong \bigsqcup_{b \in \mathcal{B}(E; G)} [* / G_b(E)]$$

$G_b$  inner form of Levi subgroup of  $G$ .

$$H^1(E, G) \hookrightarrow \mathcal{B}(E, G)$$

$$\alpha \mapsto b, \quad G_b = G_\alpha.$$

Kottwitz, Kaeltha: profitable to consider all  $G$

need to be more geometric.  
want a geometric stack of  
 $G$ -isocrystals.

Several ways to define a stack of  
 $G$ -isocrystals:

1)  $\text{Isoc}_E$  is  $E$ -linear category, so  
for any  $E$ -algebra  $A$ , can consider  
 $G$ -torsors in  $\text{Isoc}_E \otimes_E A$ .

$\rightarrow$  Artin stack  $/E$ ,  
||

$$\bigcup_{b \in \mathcal{B}(E, G)} [* / G_b]$$

$\uparrow$  as algebraic group.

$\leadsto$  alg repr. of alg group  $G_b$ , not  
desired category.

~~Better~~

1). Replace  $\overline{\mathbb{F}_q}$  by any (perfect)  
 $\overline{\mathbb{F}_q}$ -algebra  $R$ . (discrete).

$$\text{Spec } \overline{\mathbb{F}_q}(t) / \text{Frob} \rightsquigarrow \text{Spec } R(t) / \text{Frob}.$$

$$\text{Spec } \overset{\vee}{E} / \text{Frob} \rightsquigarrow \text{Spec } \left( W(R) \otimes_{W(\overline{\mathbb{F}_q})} E \right) / \text{Frob}.$$

$$\overset{\vee}{E} = W(\overline{\mathbb{F}_q}) \otimes_{W(\overline{\mathbb{F}_q})} E$$

define stack on perfect  $\overline{\mathbb{F}_q}$ -alg.

$$\begin{array}{l} R \\ \text{G-Isoc} \end{array} \longmapsto \left\{ \begin{array}{l} \text{G-torsors on } \text{Spec } R(t) / \text{Frob} \\ \text{resp. } \text{Spec } \left( W(R) \otimes_{W(\overline{\mathbb{F}_q})} E \right) / \text{Frob} \end{array} \right\}$$

Thm (Rappoport-Kichartz, Caraiani-S., Ivanov, Anschütz)

$G$ -Isoc is stack for  $v$ - (arc-) topology.  
( $\rightarrow$  fppf...).

For any  $b \in \mathcal{B}(E, G)$ , get locally closed

$$G\text{-Isoc}_b \subseteq G\text{-Isoc}$$

substack

$\parallel$  locus where  $\parallel$  isom. to  $b$ .

$$[* / G_b(E)]$$

$$LG / \sigma\text{-conj } LG$$

Xiao-Zhu, Hemo-z.

Gaitsgory, Genestier-V. Lafforgue  
Zhu.

They define

$$D(G\text{-Isoc}, \overline{\mathcal{O}_E})$$

has semi-orthog.

decomposition into pieces

derived cat. of  
smooth repr.

$$D(G\text{-Isoc}_b, \overline{\mathcal{O}_E}) \cong D(G_b(E), \overline{\mathcal{O}_E})$$

$\overline{\mathcal{O}_E} \cong \mathbb{C}$



How to get a relation to

$$W_E \cong \hat{G} \cong ?$$

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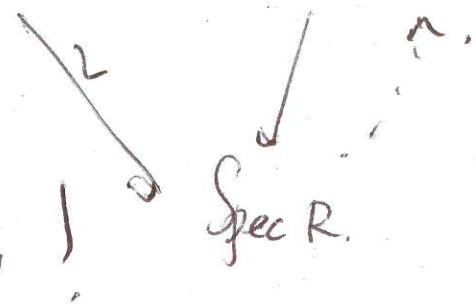
Answer: Hecke operators.

These are related to modifications of  $G$ -torsors, i.e.

$$\mathcal{L}_{G_1}, \mathcal{L}_{G_2} \quad G\text{-torsors} / \left( \text{Spec } R(A) / \text{Frob} \right)$$

+ isom  $\mathcal{L}_{G_1} \cong \mathcal{L}_{G_2}$  away from some

divisor  $D \subseteq \text{Spec } R(A)$ .

This requires sections ! 

$\Rightarrow$  need to take  $R$  a Banach algebra.

2). To any perfectoid affinoid alg.

$(R, R^+)$   $\overline{\mathbb{A}_q}$ ,

Can associate

Fargues-Fontaine curve

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$\parallel$

$\mathbb{D}^*$

$\text{Spa}(R, R^+) / \text{Frob}$

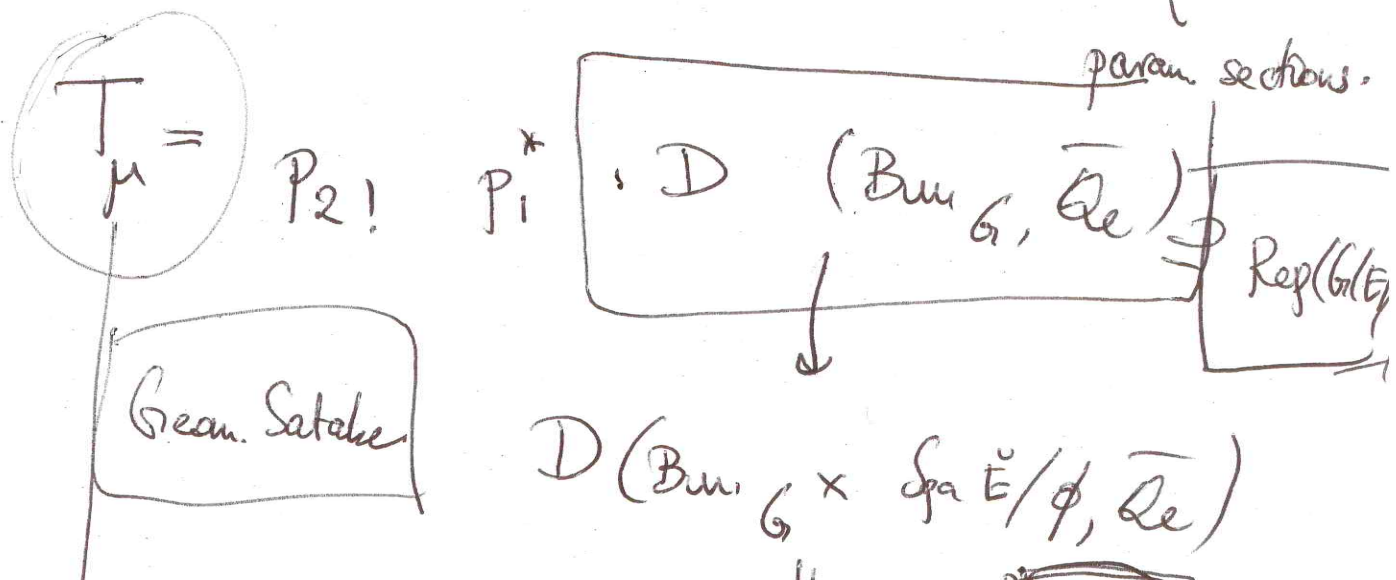
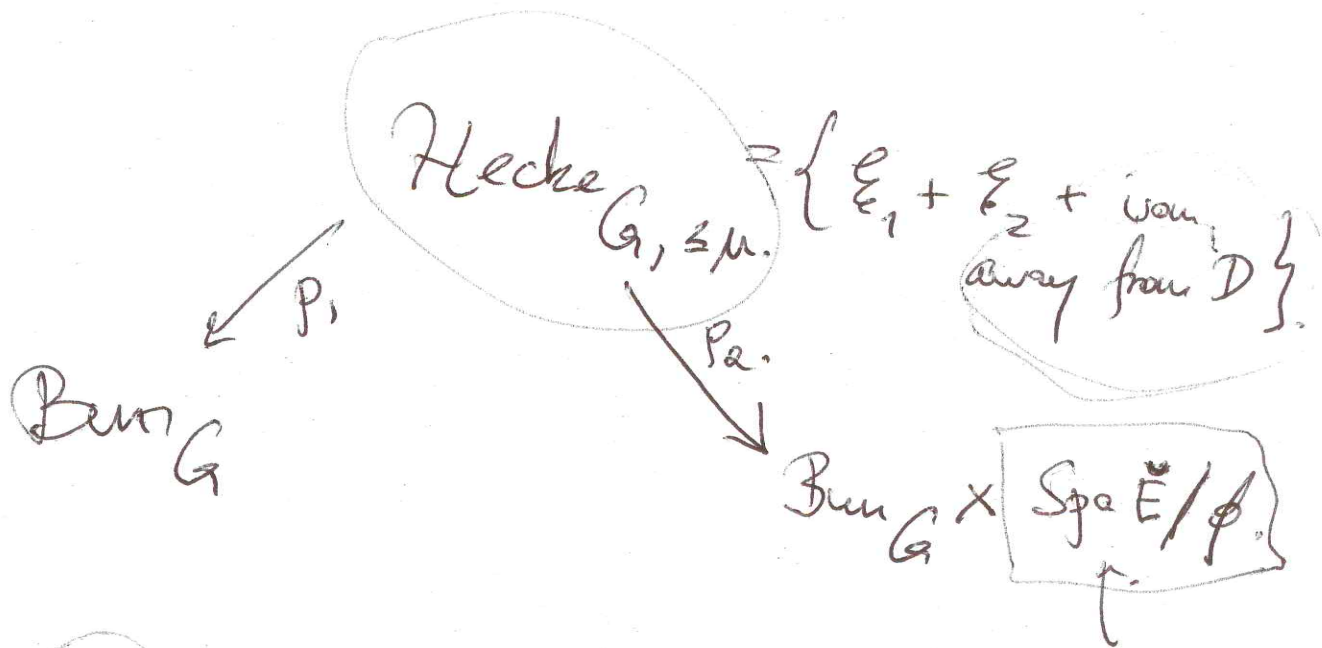
(resp. similar object in mixed char.)

$\leadsto$  moduli space of  $G$ -torsors on  
the Fargues-Fontaine curve.

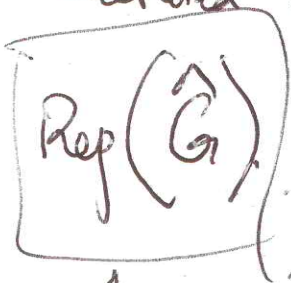
$\text{Bun}_G$

$\mathbb{D}(\text{Bun}_G, \overline{\mathbb{A}_q})$

$\mathbb{R} \text{ conj}$   
 $\mathbb{D}(G\text{-Isoc}, \overline{\mathbb{A}_q})$



enumerated by.



This categorical structure is precisely what is needed to define  $L$ -parameters.