SYMMETRIC SPACES

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Let M be a connected Riemannian manifold. We say that M is a *symmetric space* if for each point $p \in M$ there is an isometry $S_p : M \to M$ with $S_p(p) = p$ and $dS_p(p) = -\operatorname{id} : T_pM \to T_pM$. We call S_p the *geodesic symmetry* or *geodesic reflection* at p since for any unit speed geodesic c in M with $c(0) = p$ we have $S_p(c(t)) = c(-t)$. The founding father of the theory of symmetric spaces is Elie Cartan. Most of what we discuss in these lecture notes is due to him.

The model spaces of constant curvature are symmetric spaces, but there are many others. The classification of symmetric spaces is intimately connected with the classification of semisimple Lie algebras.

This is not the final version of the notes, comments and criticism are welcome.

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1. Preliminaries

Let M be a connected Riemannian manifold, D be the Levi-Civita connection of M and exp be the exponential map associated to D.

Let $p \in M$. An *orthonormal frame* of M at p is an orthogonal isomorphism $\varphi : \mathbb{R}^n \to T_pM$. If φ is an orthonormal frame of M at p then we call p the *foot point* of φ. The set OM of all orthonormal frames of M has a canonical smooth structure such that for any other manifold N, a map $f : N \to OM$ is smooth iff for any fixed vector $v \in \mathbb{R}^n$ the map $N \ni q \mapsto f(q)(v) \in TM$ is smooth. With respect to this smooth structure on OM, the projection $\pi : OM \to M$ to the foot point is a smooth submersion. We denote by O_pM the fiber of π over p, that is, the set of all orthonormal frames of M at p.

Denote by $\mathfrak{I}(M)$ the group of isometries of M. We endow $\mathfrak{I}(M)$ with the compact–open topology. In this topology, a sequence (f_n) of isometries of M converges to an isometry f of M iff it converges uniformly to f on compact subsets of M. It is immediate that the compact–open topology turns $\mathfrak{I}(M)$ into a topological group.

Let $p \in M$ and $f : M \to M$ be an isometry of M. Then

$$\exp_{f(p)} \circ f_{*p} = f \circ \exp_p \, . \tag{1.1}$$

It follows immediately that f is determined by its value and differential at p and that for any fixed frame $\varphi \in O_pM$ the map

$$\mathfrak{I}(M) \to OM \, , \quad f \mapsto f_{*p} \circ \varphi \tag{1.2}$$

is an embedding of $\mathfrak{I}(M)$ into OM. In particular, $\mathfrak{I}(M)$ is a locally compact topological group. It is not too hard to prove that the image is a smooth submanifold of OM and that the corresponding smooth structure is independent of the choice of p and φ and that $\mathfrak{I}(M)$ with this smooth structure is a Lie group. We will not need these facts in this generality. In the case of symmetric spaces, there is an easier argument which we will discuss further below.

Consider a Riemannian covering $\pi : M \to \bar{M}$ with M and $\bar{M}$ complete and connected. Let Γ be the group of covering transformations of π. Then $\Gamma \subset \mathfrak{I}(M)$. Denote by $Z(\Gamma)$ the centralizer of Γ in $\mathfrak{I}_0(M)$.

1.3. Lemma. *If $\bar{M}$ is homogeneous, then $Z(\Gamma)$ is transitive on M and hence M is homogeneous.*

Proof. Let $\bar{X}$ be a Killing field on $\bar{M}$ and X be the lift of $\bar{X}$ to M. Then X is a Killing field on M. Let f be the flow of X, $\bar{f}$ be the flow of $\bar{X}$. Then $\pi(f_t(p)) = \bar{f}_t(\pi(p))$ for all $p \in M$ and $t \in \mathbb{R}$. Let $\gamma \in \Gamma$ and $p \in M$. For all $t \in \mathbb{R}$

$$\pi(f_t(\gamma p)) = \bar{f}_t(\pi(\gamma p)) = \bar{f}_t(\pi(p)) = \pi(f_t(p)) \, .$$

It follows that there is a $\gamma_t \in \Gamma$ with

$$f_t(\gamma p) = \gamma_t f_t(p) \, .$$

The left hand side is continuous in t. Since Γ acts properly discontinuously and freely, it follows that γ_t is constant in t. Substituting $t = 0$ we get $\gamma_t = \gamma$ for all t. Hence $f_t(\gamma p) = \gamma f_t(p)$ and hence Γ centralizes the flow f_t.

Now $\bar{M}$ is homogeneous. Hence for each point $\bar{p}$ of M and each tangent vector $\bar{v}$ of $\bar{M}$ at $\bar{p}$, there is a Killing field $\bar{X}$ on $\bar{M}$ with $\bar{X}(\bar{p}) = \bar{v}$. Lifting Killing fields from M to $\bar{M}$ shows that the analogous property holds for Killing fields on M whose flow commutes with Γ. It follows easily that the centralizer $Z(\Gamma)$ of Γ in $\mathfrak{I}_0(M)$ is transitive on M as claimed. □

1.4. Lemma. *If $Z(\Gamma)$ is transitive on M and π is normal, then $\bar{M}$ is homogeneous.*

Proof. Since π is normal, Γ is transitive on the fibers of π. Hence $g \in Z(\Gamma)$ maps fibers of π to fibers of π. Therefore each $g \in Z(\Gamma)$ induces an isometry of $\bar{M}$. □

An isometry $f : M \to M$ is called a *transvection* if there is a point $pinM$ and a piecewise smooth curve c from p to $f(p)$ such that $df(p) = P_c$, where P_c denotes parallel translation along c.

1.5. Lemma. *Suppose f is a transvection and $c : [a, b] \to M$ a piecewise smooth curve from $p = c(a)$ to $f(p) = c(b)$ such that $df(p) = P_c$. Let $q \in M$ be another point and σ be a piecewise smooth curve from p to q. Then $df(q) = P_{\tilde{c}}$, where $\tilde{c} = \sigma^{-1} * c * (f \circ \sigma)$.*

Proof. Since f is an isometry, we have

$$P_{(f\circ\sigma)} \cdot df(p) \cdot P_\sigma^{-1} = df(q)\,.$$

By assumption, $df(p) = P_c$. □

It follows that the composition of transvections is a transvection. It is immediate from the definition that the inverse of a transvection is a transvection. Hence the set $\mathfrak{T}(M)$ of transvections is a subgroup of the isometry group $I(M)$ of M. Now for any isometry g and transvection f of M, $d(gfg^{-1})(gp) = P_{gc}$, where p and c are chosen as in the definition, that is, $df(p) = P_c$. Hence $\mathfrak{T}(M)$ is normal in $I(M)$.

2. Locally Symmetric and Symmetric Spaces

Let M be a connected Riemannian manifold. Denote by D the Levi–Civita connection of M. We say that M is a *locally symmetric space* if for each point $p \in M$ there is an $r > 0$ and an isometry $S_p : B_r(p) \to B_r(p)$ with $S_p(p) = p$ and $dS_p(p) = -\operatorname{id} : T_pM \to T_pM$. Since isometries map geodesics to geodesics, S_p reflects geodesics through p. For that reason we call S_p the *(local) geodesic symmetry* or *(local) geodesic reflection* at p.

2.1. Lemma. *The following conditions are equivalent:*

(1) *M is locally symmetric;*
(2) *R is parallel;*
(3) *for any geodesic c in M, $R_c = R(\,.\,,\dot{c})\dot{c}$ is parallel along c.*

Proof. Suppose first that M is locally symmetric in the sense of the definition and let $p \in M$. Then $S_p^*(DR) = DR$ since S_p is an isometry. Hence for vectors $u, x, y, z \in T_pM$,

$$\begin{aligned} -(D_u R)(x,y,z) &= dS_p((D_u R)(x,y,z)) \\ &= (D_{dS_p(u)} R)(dS_p(x), dS_p(y), dS_p(z)) \\ &= (D_{-u} R)(-x,-y,-z) = D_u R(x,y,z)\,. \end{aligned}$$

We conclude that $DR = 0$, that is, that R is parallel.

Suppose now that R is parallel and let c be a geodesic in M. Let E be a parallel vector field along c. Since $\dot c$ is parallel along c and $DR = 0$, we get

$$(D_t R_c) \cdot E = D_t(R_c \cdot E) = D_t(R(E, \dot c)\dot c) = 0\,.$$

Hence R_c is parallel along c.

Suppose now that R_c is parallel along c for each geodesic c in M. Let $p \in M$ and $r > 0$ be smaller than the injectivity radius of M at p. Consider the geodesic symmetry

$$S_p = \exp_p \circ (-\operatorname{id}) \circ \exp_p^{-1} : B_r(p) \to B_r(p)\,.$$

We prove that S_p is an isometry. To that end, we let $c : (-r, r) \to M$ be a unit speed geodesic with $c(0) = p$ and $E_1, \dots, E_n$ be a parallel frame along c. We set $R_c \cdot E_i = R_i^k E_k$. Since R_c is parallel along c, the functions R_i^k are constant. Hence if $J = j^k E_k$ is a Jacobi field along c, then

$$(j^k)'' + R_i^k j^i = 0\,, \quad 1 \le i \le n\,.$$

Since the Jacobi equation has constant coefficients, it follows that $j^k(-t) = -j^k(t)$ if $J(0) = 0$. Since $E_1, \dots, E_n$ is a parallel basis along c, we conclude that then

$$\|dS_p(J(t))\| = \|J(-t)\| = \|J(t)\|.$$

It follows that S_p is an isometry. □

The last part of this argument also gives the following result.

2.2. Theorem. *Let M and $\tilde M$ be locally symmetric spaces, let $p \in M$, $\tilde p \in \tilde M$ and $L : T_pM \to T_{\tilde p}\tilde M$ be a linear isomorphism with $L^*\tilde g_{\tilde p} = g_p$ and $L^*\tilde R_{\tilde p} = R_p$. Let $r > 0$ be smaller than the injectivity radius of M at p and of $\tilde M$ at $\tilde p$. Then $f = e\tilde{x}p_p \circ L \circ \exp_p^{-1} : B_r(p) \to B_r(\tilde p)$ is an isometry with $f(p) = \tilde p$ and $df(p) = L$.*

We recall that the torsion tensor of the Levi-Civita connection of Riemannian manifolds vanishes and hence is parallel. Therefore the results of ??? apply and have the following application.

2.3. Theorem. *Let M and $\tilde M$ be locally symmetric spaces. Assume that M is simply connected and that $\tilde M$ is complete. Let $p \in M$ and $\tilde p \in \tilde M$ be points and $L : T_pM \to T_{\tilde p}\tilde M$ be a linear isomorphism with $L^*\tilde g_{\tilde p} = g_p$ and $L^*\tilde R_{\tilde p} = R_p$. Then there is a local isometry $f : M \to \tilde M$ with $f(p) = \tilde p$ and $df(p) = L$.*

We now discuss the relation between locally symmetric and symmetric spaces. We note first that a symmetric space is locally symmetric, whereas the opposite is clearly not true. However, Theorem 2.3 has the following immediate consequence.
2.4. THEOREM. *Let M be a complete and simply connected locally symmetric space. Then M is a symmetric space.*

Vice versa, a symmetric space is automatically complete as we will see now.
2.5. PROPOSITION. *Let M be a symmetric space. Then M is complete and homogeneous.*

Proof. Let $c : (a,b) \to M$ be a maximal geodesic. We need to show that $a = -\infty$ and $b = \infty$. Suppose for example that $b < \infty$ and choose $t_0 \in (a,b)$ with $b - t_0 < t_0 - a$. Set $p = c(t_0)$. Since S_p is an isometry, the curve $\tilde{c}(t) := S_p(c(2t_0 - t))$, $2t_0 - b < t < 2t_0 - a$, is also a maximal geodesic. Moreover, $\dot{\tilde{c}}(t_0) = \dot{c}(t_0)$, hence $\tilde{c} = c$. On the other hand we have $2t_0 - a > b$, a contradiction. Hence maximal geodesics on M are defined on all of $\mathbb{R}$ and hence M is complete.

Let $p_0, p_1 \in M$. Since M is complete, there is a geodesic $c : [0,1] \to M$ from p_0 to p_1. Then the geodesic reflection S_p in the point $p = c(1/2)$ reverses c and therefore maps p_0 to p_1. Hence M is homogeneous. □

3. INFINITESIMAL DATA

Let M be a locally symmetric space and let $p_0 \in M$. Choose $r > 0$ such that for any $p \in B_r(p_0)$, the (local) geodesic symmetry S_p is defined on $B_r(p)$. This holds for example if for any $p \in B_r(p_0)$, the exponential map

$$\exp : B_r(0_p) \to B_r(p)$$

is a diffeomorphism. If M is a symmetric space, we may choose $r = \infty$. In this section, we study the Lie algebra $\mathfrak{g}^*$ of Killing fields on $B_r(p_0)$.

We first consider Killing fields whose (local) flow preserves p_0. Let $A : T_{p_0}M \to T_{p_0}M$ be an endomorphism. Then

$$L_t = \exp(tA) = e^{tA} = 1 + A + \frac{1}{2}A^2 + \cdots, \quad t \in \mathbb{R}, \tag{3.1}$$

is a smooth 1–parameter group of automorphisms of $T_{p_0}M$. Differentiating we get that these automorphisms preserve the inner product $\langle \cdot , \cdot \rangle_0$ on $T_{p_0}M$ if and only if A is skew symmetric and that they preserve the curvature tensor R_0 of M at p_0, that is, they satisfy $L_t^* R_0 = R_0$, if and only if

$$AR_0(u,v)w - R_0(Au,v)w - R_0(u,Av)w - R_0(u,v)Aw = 0 \tag{3.2}$$

for all $u, v, w \in T_{p_0}M$. Note that this can be understood as a formula for the commutator $[A, R_0(u,v)]$, namely

$$[A, R_0(u,v)] = R_0(Au,v) - R_0(u,Av) \tag{3.3}$$

We denote by $\mathfrak{k}$ the space of all skew symmetric endomorphisms of $T_{p_0}M$ satisfying the commutator rule (3.3). Note that $\mathfrak{k}$ is a Lie algebra, where we take the usual commutator of endomorphisms as Lie bracket.

3.4. LEMMA. *Let* $\mathfrak{k}^* = \{X^* \in \mathfrak{g}^* \mid X^*(p_0) = 0\}$. *Then for any* $X^* \in \mathfrak{k}^*$ *we have* $DX^*(p_0) \in \mathfrak{k}$ *and the map*

$$\mathfrak{k}^* \to \mathfrak{k}\,, \quad X^* \mapsto DX^*(p_0)\,,$$

is an injective anti–homomorphism between the Lie algebras $\mathfrak{k}^*$ *and* $\mathfrak{k}$. *The map is surjective if* M *is a simply connected symmetric space or if for any* $p \in B_r(p_0)$, *the exponential map* $\exp : B_r(0_p) \to B_r(p)$ *is a diffeomorphism.*

Proof. Let $X^* \in \mathfrak{k}^*$. Then the flow f_t of X^* fixes p_0. Now X^* is a Killing field, hence $DX^*(p_0)$ is a skew symmetric endomorphism of $T_{p_0}M$. Since f_t is a local isometry, we have $f_t^* R_0 = R_0$ and hence $DX^*(p_0) \in \mathfrak{k}$. A Killing field is determined by its value and covariant derivative at a point, hence our map is injective.

Let $X^*, Y^* \in \mathfrak{k}^*$ and set $A = DX^*(p_0)$, $B = DY^*(p_0)$. Then $[X^*, Y^*](p_0) = 0$ and for any $u \in T_{p_0}M$ we have

$$\begin{aligned} D_u[X^*, Y^*](p_0) &= D_u\{D_{X^*}Y^* - D_{Y^*}X^*\} \\ &= D^2Y^*(u, X^*(p)) + D_{D_uX^*}Y^* - D^2X^*(u, Y^*(p)) - D_{D_uY^*}X^*\,. \end{aligned}$$

Now $X^*(p_0) = Y^*(p_0) = 0$, hence the right hand side is equal to

$$D_{D_uX^*}Y^* - D_{D_uY^*}X^* = BAu - ABu = -[A, B]\,u\,.$$

We see that our map reverses the sign of Lie brackets and hence it is an anti–homomorphism of Lie algebras.

Now assume that M is a simply connected symmetric space or that for any $p \in B_r(p_0)$, the exponential map $\exp : B_r(0_p) \to B_r(p)$ is a diffeomorphism. By Theorem 2.3 or Theorem 2.2 respectively, any $A \in \mathfrak{k}$ gives rise to a 1–parameter family of isometries $f_t : B_r(p_0) \to B_r(p_0)$ with $f_t(p_0) = p_0$ and $df_t(p_0) = L_t = \exp(tA)$. Furthermore, f_t is smooth in t and $f_s \circ f_t = f_{s+t}$ for all $s, t \in \mathbb{R}$. Hence the family f_t, $t \in \mathbb{R}$, is a 1–parameter group of isometries of $B_r(p_0)$. Since $f_t(p_0) = p_0$ for all t, the corresponding Killing field

$$X^*(p) = \partial_t(f_t(p))|_{t=0}$$

on $B_r(p_0)$ satisfies $X^*(p_0) = 0$. Hence $X^* \in \mathfrak{k}^*$. Let $u \in T_{p_0}M$ and $\sigma = \sigma(s)$ be a curve with $\sigma(0) = p_0$ and $\dot\sigma(0) = u$. Then

$$\begin{aligned} D_uX^* &= D_s\partial_t(f_t(\sigma(s)))|_{s=t=0} \\ &= D_t\partial_s(f_t(\sigma(s)))|_{t=s=0} \\ &= D_tL_tu|_{t=0} \;=\; \partial_tL_tu|_{t=0} \;=\; Au \end{aligned}$$

and hence $DX^*(p_0) = A$. This proves surjectivity of our map. ∎

Next we consider a different type of isometries and Killing fields. By the choice of r, the geodesic symmetry S_p is defined on $B_r(p)$ for any $p \in B_r(p_0)$. Let $c : (-T, T) \to M$ be a geodesic through p_0 with $T = r/\|\dot{c}\|$. Then the image of c is in $B_r(p_0)$. Set $f_t = S_{c(t/2)} S_p$. The domain of definition of f_t is $B_r(p_0) \cap B_r(c(t/2))$ and $f_t(c(s)) = c(s+t)$ if $s, t, s+t \in (-T, T)$.

Let E be a parallel vector field along c. Then $S_{p_0 *} E$ is parallel along $S_{p_0} \circ c$. Since $S_{p_0}(c(s)) = c(-s)$ and $S_{p_0 *}(p_0) = -\operatorname{id}$, we get

$$S_{p_0 *} E(s) = -E(-s)\,.$$

Similarly,

$$S_{c(t/2)*} E(s) = -E(t-s)\,.$$

We conclude that

$$f_{t*} E(s) = E(s+t)$$

and hence that $df_t(c(s))$ is parallel translation along c from $c(s)$ to $c(s+t)$. Hence f_t is a transvection along c. If $s, t, s+t \in (-T, T)$, then

$$f_s \circ f_t = f_{s+t}$$

on the common domain of definition of left and right hand side. Therefore we call the family of maps f_t the (local) 1–parameter group of transvections along c.

For any $p \in B_r(p_0)$, there is an $\varepsilon > 0$ such that $p \in B_r(p_0) \cap B_r(c(t/2))$ for all $t \in (-\varepsilon, \varepsilon)$. We set

$$X^*(p) = \partial_t (f_t(p))|_{t=0}\,.$$

Then X^* is a Killing field on $B_r(p_0)$. We call X^* the *infinitesimal transvection* along c.

3.5. Lemma. *Let $u \in T_{p_0} M$ and $c : (-T, T) \to B_r(p_0)$ be the geodesic with $c(0) = p$ and $\dot{c}(0) = u$. Then $X^* \in \mathfrak{g}^*$ is the infinitesimal transvection along c if and only if*

$$X^*(p_0) = u\,, \qquad DX^*(p_0) = 0\,. \tag{3.6}$$

In particular, the space $\mathfrak{p}^* = \{X^* \in \mathfrak{g}^* \mid DX^*(p_0) = 0\}$ of infinitesimal transvections along geodesics through p_0 is a linear subspace of $\mathfrak{g}^*$.

Proof. Let X^* be the infinitesimal transvection along c. Then $X^*(p_0) = u$ since $f_t(p_0) = c(t)$. As for the covariant derivative of X^* in p_0 let $v \in T_{p_0} M$ and $\sigma = \sigma(s)$ be a smooth curve with $\sigma(0) = p_0$ and $\dot{\sigma}(0) = v$. Then

$$\begin{aligned} D_v X^* &= D_s \partial_t (f_t(\sigma(s)))|_{t=s=0} \\ &= D_t \partial_s (f_t(\sigma(s)))|_{s=t=0} \\ &= D_t (f_{t*} v)|_{t=0} \;=\; D_t (P_t v)|_{t=0} \;=\; 0\,. \end{aligned}$$

Now a Killing field (on a connected set) is determined by its value and covariant derivative at one point. Hence the stated properties determine X^* uniquely. □

3.7. PROPOSITION. *The Lie algebra* $\mathfrak{g}^*$ *of Killing fields on* $B_r(p_0)$ *splits as a direct sum,*

$$\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^* .$$

We have

$$[\mathfrak{k}^*, \mathfrak{k}^*] \subset \mathfrak{k}^* , \quad [\mathfrak{k}^*, \mathfrak{p}^*] \subset \mathfrak{p}^* , \quad [\mathfrak{p}^*, \mathfrak{p}^*] \subset \mathfrak{k}^* .$$

Furthermore, for $X^*, Y^* \in \mathfrak{p}^*$ *with* $X^*(p_0) = u$, $Y^*(p_0) = v$ *and* $Z^* \in \mathfrak{k}^*$ *with* $DZ^*(p_0) = A$ *we have*

$$D[X^*, Y^*](p) = R_0(u, v) \quad \text{and} \quad [X^*, Z^*](p_0) = Au .$$

In particular, we have $R_0(u, v) \in \mathfrak{k}$ *for all* $u, v \in T_{p_0}M$.

Proof. The first assertion $[\mathfrak{k}^*, \mathfrak{k}^*] \subset \mathfrak{k}^*$ is obvious. Let $X^* \in \mathfrak{p}^*$, $Z^* \in \mathfrak{k}^*$ with $X^*(p_0) = u$ and $DZ^*(p_0) = A$. Then $DZ^*(p_0) = 0$ and hence

$$[X^*, Z^*](p_0) = D_{X^*} Z^*(p_0) = Au .$$

Moreover, for $w \in T_{p_0}M$

$$\begin{aligned} D_w[X^*, Z^*] &= D_w D_{X^*} Z^* - D_w D_{Z^*} X^* \\ &= D^2 Z^*(w, X^*(p)) - D^2 X^*(w, Z^*(p)) \\ &\quad + D_{D_w X^*} Z^* - D_{D_w Z^*} X^* . \end{aligned}$$

The last two terms on the right hand side vanish since $DX^*(p_0) = 0$. The second term vanishes since $Z^*(p_0) = 0$. Since Z^* is a Killing field the first term is equal to $R_0(Z^*(p_0), w)X^*(p_0) = 0$. Hence $D[X^*, Z^*](p_0) = 0$ and therefore $[X^*, Y^*] \in \mathfrak{p}$.

If $X^*, Y^* \in \mathfrak{p}$, then $[X^*, Y^*](p_0) = 0$ and hence $[X^*, Y^*] \in \mathfrak{k}^*$. As for the covariant derivative at p_0, we have

$$\begin{aligned} D_w[X^*, Y^*] &= D^2 Y^*(w, X^*(p_0)) - D^2 X^*(w, Y^*(p_0)) \\ &= -R_0(Y^*(p_0), w)X^*(p_0) + R_0(X^*(p_0), w)Y^*(p_0) \\ &= R_0(u, v)w , \end{aligned}$$

where $X^*(p_0) = u$, $Y^*(p_0) = v$. This completes the proof. □

As an application of our discussion we get the remarkable fact that the Euclidean space $T_{p_0}M$ together with the curvature tensor R_0 on T_pM determines the Lie algebra $\mathfrak{g}^*$ completely. Namely let $\mathfrak{k}$ as above be the space of skew symmetric endomorphisms satisfying the commutator rule (3.3) and set $\mathfrak{p} := T_{p_0}M$, $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. For $(A, u), (B, v) \in \mathfrak{g}$ set

(3.8) $$[(A, u), (B, v)] := ([A, B] - R(u, v), Av - Bu) .$$

It is imediate that $[\cdot, \cdot]$ is a skew symmetric bilinear form on $\mathfrak{g}$. Using (3.2), a straightforward computation shows that $[\cdot, \cdot]$ also satisfies the Jacobi equation and hence turns $\mathfrak{g}$ into a Lie algebra. Our discussion implies the following result.

3.9. THEOREM. *The evaluation map* $\mathfrak{g}^* \to \mathfrak{g}$, $X^* \mapsto (DX^*(p), X^*(p))$ *is an anti-isomorphism of Lie algebras.*

Clearly the map $s : \mathfrak{g} \to \mathfrak{g}$, $(A, u) \mapsto (A, -u)$, is an involutive isomorphism of $\mathfrak{g}$ with eigenspace $\mathfrak{k}$ for the eigenvalue $+1$ and eigenspace $\mathfrak{p}$ for the eigenvalue -1. Note that $\mathfrak{p}$ is invariant under all ad_X, $X \in \mathfrak{k}$. Furthermore, the inner product $\langle \cdot, \cdot \rangle$ is invariant under s and all ad_X, $X \in \mathfrak{k}$, are skew symmetric with respect to $\langle \cdot, \cdot \rangle$. A triple $(\mathfrak{g}, s, \langle \cdot, \cdot \rangle)$ with such properties is called a *Riemannian symmetric Lie algebra*. Thus starting from a locally symmetric space we arrive naturally at a Riemannian symmetric Lie algebra.

4. Orthogonal Symmetric Lie Algebras

In our previous discussion we saw that a locally symmetric space gives rise to a Lie algebra $\mathfrak{g}$ and an involutive automorphism $s : \mathfrak{g} \to \mathfrak{g}$. Such a pair $(\mathfrak{g}, s)$ will be called a *symmetric Lie algebra.* Let $(\mathfrak{g}, s)$ be a symmetric Lie algebra. Then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{4.1}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of s for the eigenvalues $+1$ and -1 respectively. Since s is an automorphism,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\,, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,. \tag{4.2}$$

It follows that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ and that $\mathfrak{k}$ and $\mathfrak{p}$ are perpendicular with respect to the Killing form B of $\mathfrak{g}$,

$$B(\mathfrak{k}, \mathfrak{p}) = 0\,. \tag{4.3}$$

We say that a symmetric Lie algebra $(\mathfrak{g}, s)$ is *orthogonal* if there is an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ which is invariant under s and such that all ad_X, $X \in \mathfrak{k}$, are skew symmetric with respect to $\langle \cdot\,, \cdot \rangle$.

We say that a symmetric Lie algebra $(\mathfrak{g}, s)$ is *effective* if $\mathfrak{k}$ does not contain a non–trivial ideal of $\mathfrak{g}$. In what follows, $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra with corresponding inner product $\langle \cdot\,, \cdot \rangle$.

4.4. Lemma. *The Killing form B of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$.*

Proof. Let $X \in \mathfrak{k}$. Since ad_X is skew symmetric, $B(X, X) \le 0$ and equality implies $\mathrm{ad}_X = 0$. In the latter case X belongs to the center of $\mathfrak{g}$ and then $\mathbb{R} \cdot X$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$. But then $X = 0$ since $(\mathfrak{g}, s)$ is effective. □

We say that a subspace $U \subset \mathfrak{g}$ is $\mathrm{ad}_{\mathfrak{k}}$*–invariant* if $\mathrm{ad}_X U \subset U$ for all $X \in \mathfrak{k}$.

4.5. Lemma. *Let $U, V \subset \mathfrak{p}$ be subspaces. If U is $\mathrm{ad}_{\mathfrak{k}}$–invariant, then $B(U, V) = 0$ implies $[U, V] = 0$.*

Proof. Let $X \in U$, $Y \in V$. Then $[X, Y] \in \mathfrak{k}$, hence $[[X, Y], X] \in U$. But then

$$B([X, Y], [X, Y]) = B([[X, Y], Y], Y) = 0\,,$$

hence $[X, Y] = 0$ by Lemma 4.4. □

4.6. Proposition. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$.*

Proof. By (4.2) and the Jacobi identity, $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$ where $\mathfrak{h}^\perp = \{X \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{h}\}$, see ???. Now $\mathfrak{p} \subset \mathfrak{h}$, hence $\mathfrak{h}^\perp \subset \mathfrak{k}$ by (4.3). Since $\mathfrak{h}^\perp$ is an ideal and $(\mathfrak{g}, s)$ is effective we get $\mathfrak{h}^\perp = \{0\}$. □

We say that $(\mathfrak{g}, s)$ is of *Euclidean type* if $[\mathfrak{p}, \mathfrak{p}] = 0$. We say that $(\mathfrak{g}, s)$ is of *compact type* respectively *noncompact type* if $\mathfrak{g}$ is semisimple of compact type respectively noncompact type. We say that $(\mathfrak{g}, s)$ is *irreducible* if the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible.

4.7. Proposition. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra. Then there is a decomposition of $\mathfrak{g}$ into s–invariant ideals,*

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \ldots + \mathfrak{g}_m\,,$$

such that $(\mathfrak{g}_0, s)$ is of Euclidean type and $(\mathfrak{g}_i, s)$ is irreducibel with $\mathfrak{g}_i$ semisimple for $i > 0$. For each $i > 0$ there is a constant $\lambda_i \neq 0$ such that $\langle\cdot\,,\cdot\rangle = \lambda_i B$ on $\mathfrak{p}_i$. Moreover, $(\mathfrak{g}_i, s)$ is of compact type or noncompact type depending on whether $\lambda_i < 0$ respectively $\lambda_i > 0$.

Proof. Let $U \subset \mathfrak{p}$ be a subspace invariant under $\mathrm{ad}_{\mathfrak{k}}$. Since all ad_X, $X \in \mathfrak{k}$, are skew symmetric, the orthogonal complement V of U with respect to $\langle\cdot\,,\cdot\rangle$ is also invariant under $\mathrm{ad}_{\mathfrak{k}}$. Hence there is a decomposition

$$\mathfrak{p} = \mathfrak{p}'_1 + \cdots + \mathfrak{p}'_l,$$

where the $\mathfrak{p}'_i$ are irreducibel under $\mathrm{ad}_{\mathfrak{k}}$. Now let $\mathfrak{p}_0$ be the sum of the $\mathfrak{p}'_i$ with $[\mathfrak{p}'_i, \mathfrak{p}'_i] = 0$ and enumerate the other ones as $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. By Lemma 4.5, $[\mathfrak{p}_0, \mathfrak{p}_0] = \{0\}$ and $[\mathfrak{p}_i, \mathfrak{p}_j] = \{0\}$ for $i \neq j$. For $i > 0$, set $\mathfrak{k}_i = [\mathfrak{p}_i, \mathfrak{p}_i]$ and $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i$. Then $\mathfrak{g}_i$ is s–invariant. By the Jacobi identity $[\mathfrak{k}_i, \mathfrak{p}_j] = \{0\}$ for $i \neq j$. Again by the Jacobi identity we $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$. It also follows that $\mathfrak{k}_i$ is irreducibel on $\mathfrak{p}_i$. Therefore there is a constant β_i such that

$$B(X, Y) = \beta_i \cdot \langle X, Y\rangle$$

for all $X, Y \in \mathfrak{p}_i$. Now by Lemma 4.5, $\beta_i = 0$ implies $\mathfrak{p}_i, \mathfrak{p}_i] = 0$ in contradiction to the choice of $\mathfrak{p}_0$ and $i > 0$. Since B is negative definite on $\mathfrak{k}$ and $B(\mathfrak{k}, \mathfrak{p}) = 0$ we conclude that $\mathfrak{g}_i$ is semisimple.

Set $\mathfrak{h} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_m$ and let $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{h}\}$. Then $\mathfrak{g}_0$ and $\mathfrak{h}$ are ideals in $\mathfrak{g}$ and $\mathfrak{g}_0$ contains $\mathfrak{p}_0$. Since $\mathfrak{h}$ is semisimple we have $\mathfrak{g}_0 \cap \mathfrak{h} = \{0\}$. Since $\mathfrak{h}$ is s–invariant and s is an automorphism of $\mathfrak{g}$, $\mathfrak{g}_0$ is also s–invariant. Hence $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ with $\mathfrak{k}_0 \subset \mathfrak{k}$. Hence $(\mathfrak{g}_0, s)$ is of Euclidean type. □

For a given orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ with inner product $\langle\cdot\,,\cdot\rangle$, let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and extend s to a complex linear automorphism $s_{\mathbb{C}}$ of $\mathfrak{g}_C$. Furthermore extend $\langle\cdot\,,\cdot\rangle$ to a Hermitian inner product $\langle\cdot\,,\cdot\rangle_H$ on $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ is a real subalgebra of $\mathfrak{g}_{\mathbb{C}}$, invariant under $s_{\mathbb{C}}$. Denote by s^* the restriction of s_C to $\mathfrak{g}^*$. Then $(\mathfrak{g}^*, s^*)$ is a symmetric Lie algebra, orthogonal with respect to the real part $\langle\cdot\,,\cdot\rangle^*$ of $\langle\cdot\,,\cdot\rangle_H$. We call $(\mathfrak{g}^*, s^*)$ together with $\langle\cdot\,,\cdot\rangle^*$ the *dual orthogonal symmetric Lie algebra.* We can also think of $\mathfrak{g}^* = \mathfrak{g}$ as a vector space, where we change the Lie bracket by

$$[X, Y] = \begin{cases} +[X, Y] & \text{if } X \text{ or } Y \in \mathfrak{k} \\ -[X, Y] & \text{if } X, Y \in \mathfrak{p} \end{cases} \tag{4.8}$$

and where $s^* = s$ and $\langle\cdot\,,\cdot\rangle^* = \langle\cdot\,,\cdot\rangle$. Note that $B^* = B$ on $\mathfrak{k}^* = \mathfrak{k}$ and $B^* = -B$ on $\mathfrak{p}^* = \mathfrak{p}$. The following two propositions are immediate from this description.

4.9. PROPOSITION. *Let* $(\mathfrak{g}, s)$ *be an orthogonal symmetric Lie algebra with dual* $(\mathfrak{g}^*, s^*)$. *Then*

(1) $(\mathfrak{g}, s)$ *is of Euclidean type iff* $(\mathfrak{g}, s) \cong (\mathfrak{g}^*, s^*)$.
(2) $(\mathfrak{g}, s)$ *is of compact type iff* $(\mathfrak{g}^*, s^*)$ *is of noncompact type.*

4.10. PROPOSITION. *Let* $(\mathfrak{g}, s)$ *be an orthogonal symmetric Lie algebra with dual* $(\mathfrak{g}^*, s^*)$ *and let* $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \ldots + \mathfrak{g}_m$ *be the decomposition of* $\mathfrak{g}$ *into* s*–invariant ideals as in Proposition 4.7. Then* $\mathfrak{g}^* = \mathfrak{g}_0 + \mathfrak{g}_1^* + \ldots + \mathfrak{g}_m^*$ *is the corresponding decomposition of* $\mathfrak{g}^*$.

Together with duality, the following result reduces the classification of simply connected symmetric spaces to the classification of simple real Lie algebras of compact type and their involutive automorphisms.

4.11. THEOREM. *Up to isomorphism, the irreducible effective orthogonal symmetric Lie algebras* $(\mathfrak{g}, s)$ *of compact type fall into two classes:*

(1) $\mathfrak{g}$ *is a simple real Lie algebra of compact type and* s *is an involutive automorphism of* $\mathfrak{g}$.
(2) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ *where* $\mathfrak{h}$ *is a simple real Lie algebra* $\mathfrak{h}$ *f compact type and* $s(X, Y) = (Y, X)$.

Proof. Since $(\mathfrak{g}, s)$ is irreducibel and of compact type, $B < 0$ on $\mathfrak{p}$ and hence on $\mathfrak{g}$. Hence $\mathfrak{g}$ is simisimple. Let

$$\mathfrak{g} = \mathfrak{g}_1 + \ldots + \mathfrak{g}_m$$

be the decomposition of $\mathfrak{g}$ into simple ideals. Since s is an automorphism of $\mathfrak{g}$ there is an $i \in \{1, \ldots, m\}$ such that $s(\mathfrak{g}_1) = \mathfrak{g}_i$. Since s is involutive, $s(\mathfrak{g}_i) = \mathfrak{g}_1$ and therefore $\mathfrak{g}_1 + \mathfrak{g}_i$ is s–invariant. By irreducibility $\mathfrak{g}_1 + \mathfrak{g}_i = \mathfrak{g}$. There are two possibilities:

1) $i = 1$: Then $(\mathfrak{g}, s)$ is of the first type.

2) $i \neq 1$: Since s is involutive, s is an isomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_i$. Set $\mathfrak{h} = \mathfrak{g}_1 \cong \mathfrak{g}_i$, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}$ with s as predicted for the second type. ☐

5. RIEMANNIAN SYMMETRIC PAIRS

Let $(\mathfrak{G}, \mathfrak{K})$ be a pair consisting of a Lie group $\mathfrak{G}$ and a closed subgroup $\mathfrak{K}$. We say that $(\mathfrak{G}, \mathfrak{K})$ is a *symmetric pair* if $M = \mathfrak{G}/\mathfrak{K}$ is connected and if there is an involutive automorphism $\sigma : \mathfrak{G} \to \mathfrak{G}$ with

$$\mathfrak{F}_0 \subset \mathfrak{K} \subset \mathfrak{F}, \tag{5.1}$$

where $\mathfrak{F} = \{g \in \mathfrak{G} \mid \sigma(g) = g\}$ and $\mathfrak{F}_0$ denotes the component of the identity of $\mathfrak{F}$. In what follows, $(\mathfrak{G}, \mathfrak{K})$ is a symmetric pair. Denote by σ_* the differential of σ at the neutral element $e \in \mathfrak{G}$ and set

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma_* X = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \sigma_* X = -X\}, \tag{5.2}$$

the eigenspaces of σ_* for the eigenvalues 1 and -1 respectively. Since σ_* is involutive,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}. \tag{5.3}$$

Since $\mathfrak{F}_0 \subset \mathfrak{K} \subset \mathfrak{F}$, $\mathfrak{k}$ is the Lie algebra of $\mathfrak{K}$. Since $\mathfrak{K} \subset \mathfrak{F}$, $\sigma(kgk^{-1}) = k\sigma(g)k^{-1}$ for all $k \in \mathfrak{K}$ and all $g \in \mathfrak{G}$, hence σ_* commutes with all Ad_k, $k \in \mathfrak{K}$. Therefore

$$\mathrm{Ad}_k(\mathfrak{k}) \subset \mathfrak{k}\,, \quad \mathrm{Ad}_k(\mathfrak{p}) \subset \mathfrak{p} \quad \text{for all} \quad k \in \mathfrak{K}\,. \tag{5.4}$$

Furthermore,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{p}\,, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,. \tag{5.5}$$

The first and second inclusion follow from (5.4) above. As for the third inclusion, let $X, Y \in \mathfrak{p}$. Since σ_* is an automorphism of $\mathfrak{g}$, we have

$$\sigma_*[X, Y] = [\sigma_* X, \sigma_* Y] = [-X, -Y] = [X, Y]$$

and hence $[X, Y] \in k$. Note that the first two inclusions also follow by a similar argument.

$\mathfrak{G}$ acts from the left on M. We denote by $gp \in M$ the image of $p \in M$ under left multiplication by $g \in \mathfrak{G}$. Sometimes it will be advantageous to distinguish between the element $g \in \mathfrak{G}$ and the diffeomorphism of M given by left multiplication with g. Then we will use λ_g to denote the latter, $\lambda_g(p) := gp$.

We denote by $p_0 = [\mathfrak{K}]$ the distinguished point of M and by $\pi : \mathfrak{G} \to M$ the canonical map, $\pi(g) = [g\mathfrak{K}] = \lambda_g(p_0)$. We use $\pi_* = d\pi(e) : \mathfrak{p} \to T_pM$ to identify $\mathfrak{p}$ and T_pM. An easy computation shows that for all $k \in \mathfrak{K}$

$$\pi_* \circ \mathrm{Ad}_k = d\lambda_k(p_0) \circ \pi_*\,. \tag{5.6}$$

Thus with respect to the identification $\mathfrak{p} \equiv T_pM$ via π_*, the isotropy representation of $\mathfrak{K}$ on T_pM corresponds to the restriction of the adjoint representation of $\mathfrak{K}$ to $\mathfrak{p}$.

We say that a symmetric pair is *Riemannian* if there is an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{p}$ which is invariant under all Ad_k, $k \in \mathfrak{K}$. Note that such inner products are in one–to–one correspondence with $\mathfrak{G}$-invariant Riemannian metrics on M.

5.7. Theorem. *Let $(\mathfrak{G}, \mathfrak{K})$ be a Riemannian symmetric pair with corresponding involution σ of $\mathfrak{G}$ and scalar product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{p}$. Let $M = \mathfrak{G}/\mathfrak{K}$ and endow M with the $\mathfrak{G}$–invariant Riemannian metric corresponding to $\langle \cdot\,, \cdot \rangle$. Then we have:*

1. *M is a symmetric space. The geodesic symmetry S at p_0 is $S([gK]) = [\sigma(g) \cdot K]$. In particular, $S \circ \lambda_g = \lambda_{\sigma(g)} \circ S$.*
2. *For $X \in \mathfrak{p}$, $e^{tX} p_0$, $t \in \mathbb{R}$, is the geodesic through p_0 with initial velocity $\pi_* X$ and left multiplication by e^{tX}, $t \in \mathbb{R}$, is the 1–parameter group of transvections along this geodesic.*
3. *With respect to the identification $\mathfrak{p} \equiv T_{p_0}M$, the curvature tensor R and Ricci tensor* Ric *of M are given by*

$$R(X, Y)Z = -[[X, Y], Z]\,, \quad \mathrm{Ric}(X, Y) = \frac{1}{2}B(X, Y)\,,$$

where B denotes the Killing form of $\mathfrak{g}$.

(4) *With respect to the identification $\mathfrak{p} \equiv T_{p_0}M$, a subspace $\mathfrak{q} \subset \mathfrak{p}$ is tangent to a totally geodesic submanifold of M through p_0 if and only if $[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset \mathfrak{q}$. If the latter inclusion holds, then $N = \exp(\mathfrak{q})$ is such a submanifold and a symmetric space in its own right.*

Proof. Since $\langle \cdot\,,\cdot\rangle$ is invariant under $\mathrm{Ad}_{\mathfrak{K}}$, there is a unique $\mathfrak{G}$–invariant Riemannian metric on M, also denoted $\langle \cdot\,,\cdot\rangle$, such that $\langle \cdot\,,\cdot\rangle_{p_0}$ corresponds to the given inner product on $\mathfrak{p}$ under the identification $T_{p_0}M \equiv \mathfrak{p}$. By $\mathfrak{G}$–invariance, the differentials λ_{g*} preserve norms. Now let $p = gp_0 \in M$ and $u \in T_pM$. Choose $v \in T_{p_0}M$ with $\lambda_{g*}v = u$.

$$\|S_*u\| = \|S_*\lambda_{g*}v\| = \|\lambda_{\sigma(g)*}S_*v\| = \|-\lambda_{\sigma(g)*}v\| = \|v\| = \|\lambda_{g*}v\| = \|u\|\,,$$

and hence S is an isometry. This proves Assertion 1.

Let $X \in \mathfrak{p}$ and X^* be the corresponding Killing field of M,

$$X^*(p) = \partial_t(e^{tX}p)|_{t=0}\,.$$

Let $u \in T_{p_0}M$. Then

$$e^{tX}S(p) = S(e^{-tX}p)$$

and hence $S_*X^* = -X^*$. Since S is an isometry we get

$$-D_uX^* = S_*D_uX^* = D_{S_*u}S_*X^* = D_uX^*$$

and hence $DX^*(p_0) = 0$. Hence X^* is the infinitesimal transvection with $X^*(p_0) = \pi_*X$, hence Assertion 2.

Assertion 3 ...

It remains to prove Assertion 4 about totally geodesic submanifolds. Note that for a totally geodesic submanifold N through p_0, $R(u,v)w \in T_{p_0}N$ for all $u,v,w \in T_{p_0}N$. Hence the necessity of the condition on $\mathfrak{q}$ is immediate from the formula for R.

Suppose now that $[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset \mathfrak{q}$. Let $\mathfrak{l} = [\mathfrak{q},\mathfrak{q}]$. Then $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{l} + \mathfrak{q}$ is a Lie subalgebra of $\mathfrak{q}$. Let $\mathfrak{H}$ be the corresponding connected Lie subgroup of $\mathfrak{G}$. Then the orbit N of p_0 under $\mathfrak{H}$, $N = \mathfrak{H}p_0$, is a submanifold of M. Now N is totally geodesic at p_0 since $N = \exp(\mathfrak{q})p_0$ locally about p_0. Sincs $\mathfrak{H}$ is transitive on N and $\mathfrak{H}$ acts isometrically on M, N is totally geodesic everywhere. Since M is a complete Riemannian manifold and N is connected we have $N = \exp(\mathfrak{q})p_0$. In particular, N is complete. Since N is invariant under S, $S|N$ is the geodesic symmetry of N in p_0. Since N is homogeneous, N is symmetric. □

6. Transvections and the Symmetry Subgroup

We continue to assume that M is a symmetric space. Then the group $\mathfrak{T}(M)$ of transvections of M is transitive on M. More precisely, we have the following statement.

6.1. Lemma. *Let* $(\mathfrak{G}, \mathfrak{K})$ *be a symmetric pair and* $M = \mathfrak{G}/\mathfrak{K}$. *Let* $c : [a, b] \to M$ *be a piecewise smooth curve. Then there is* $g_0 \in \mathfrak{G}$ *with* $\lambda_{g_0}(c(a)) = c(b)$ *and* $d\lambda_{g_0}(c(a)) = P_c$.

6.2. *Remark.* If M is simply connected, and $\mathfrak{G} = \mathfrak{I}(M)$, then Lemma 6.1 is immediate from Theorem 2.3

Proof of Lemma 6.1. Consider the submersion $\pi : \mathfrak{G} \to M$, $\pi(g) = gp_0 = \lambda_g(p_0)$, where $p_0 = [K]$ is the distinguished point of M. For any $g \in \mathfrak{G}$, the *vertical space* in g is

$$V_g = \ker d\pi(g) = \Lambda_{g*}\mathfrak{k} \subset T_g\mathfrak{G}\,,$$

where Λ_g denotes left multiplication by g in $\mathfrak{G}$. Furthermore, the *horizontal space*

$$H_g = \Lambda_{g*}\mathfrak{p} \subset T_g\mathfrak{G}$$

is a complement of V_g in $T_g\mathfrak{G}$. The distributions V and H are left invariant and hence smooth.

Without loss of generality we may and do assume that c is smooth. Let $g : [a, b] \to \mathfrak{G}$ be a horizontal lift of c, that is, g is a smooth curve in $\mathfrak{G}$ with $\pi \circ g = c$ and $\dot g(t) \in H_{g(t)}$ for all $t \in [a, b]$. We show that $g_0 = g(b) \cdot g(a)^{-1}$ satifies the required properties. Since g is a lift of c, we have $g_0 c(a) = c(b)$, the first property.

Fix $t \in [a, b]$ and set $f_\tau = g(t)^{-1} g(t + \tau)$. Since the curve g is horizontal

$$\partial_\tau f_\tau|_{\tau=0} = \Lambda^{-1}_{g(t)*}\dot g(t) \in \mathfrak{p}\,.$$

Hence the Killing field X^*_t on M defined by

$$X^*_t(p) = \partial_\tau(f_\tau(p))|_{\tau=0}$$

is the infinitesimal transvection along the geodesic through p_0 with initial velocity $X^*(p_0)$. Therefore $DX^*(p_0) = 0$.

Now let $u \in T_{c(a)}M$, then $v = \lambda^{-1}_{g(a)*}u \in T_{p_0}M$ and

$$\begin{aligned} D_\tau\{\lambda_{g(t+\tau)*}\lambda^{-1}_{g(a)*}u\}|_{\tau=0} &= D_\tau\{\lambda_{g(t)*}f_{\tau*}v\}|_{\tau=0} \\ &= \lambda_{g(t)*}D_\tau\{f_{\tau*}v\}|_{\tau=0} \\ &= \lambda_{g(t)*}D_v X^* = 0\,. \end{aligned}$$

Therefore $\lambda_{g(t+\tau)*}\lambda^{-1}_{g(a)}u$ is parallel along c and hence $\lambda_{g(b)*}\lambda^{-1}_{g(a)} = P_c$. □

Now we consider the group $\mathfrak{T}_\mathfrak{G}(M)$ of $g \in \mathfrak{G}$ such that λ_g is a transvection of M. Lemma 6.1 and its proof have the following application.

6.3. Theorem. *Let* $(\mathfrak{G}, \mathfrak{K})$ *be a symmetric pair. Then* $\mathfrak{T}_\mathfrak{G}(M)$ *is a normal Lie subgroup of* $\mathfrak{G}$ *with Lie algebra* $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ *and* $\mathfrak{T}_\mathfrak{G}(M)$ *acts transitively on* M.

6.4. *Remark.* The group $\mathfrak{T}(M)$ of transvections of M is a connected Lie group. For a symmetric pair $(\mathfrak{G}, \mathfrak{K})$, each component of the subgroup of $\mathfrak{G}$ which acts trivially on M contains exactly one connected component of $\mathfrak{T}_\mathfrak{G}(M)$.

Proof of Theorem 6.3. For any $X \in \mathfrak{p}$ and $t \in \mathbb{R}$ we have $\exp(tX) \in \mathfrak{T}_{\mathfrak{G}}(M)$. Hence $\mathfrak{T}_{\mathfrak{G}}(M)$ contains the connected Lie subgroup $\mathfrak{H}$ of $\mathfrak{G}$ generated by $\exp(\mathfrak{p})$. By Lemma 6.5, the Lie algebra of $\mathfrak{H}$ is $[\mathfrak{p},\mathfrak{p}]+\mathfrak{p}$. Since $\exp(\mathfrak{p})p_0$ contains a neighborhood of p_0 in M, $\mathfrak{H}$ is transitive on M.

On the other hand, for any $g \in \mathfrak{T}_{\mathfrak{G}}(M)$ there is a curve $c : [0,1] \to M$ from p_0 to $\lambda_g(p_0)$ with $d\lambda_g(p_0) = P_c$. Let $g(t)$, $0 \le t \le 1$, be the horizontal lift of c to $\mathfrak{G}$ with $g(0) = e$. Now the horizontal distribution H is tangential to $\mathfrak{H}$ and $\dot g(t) \in H_{g(t)}$ for all $t \in [a,b]$, hence $g(t) \in \mathfrak{H}$ for all t. Hence $g(1) \in \mathfrak{H}$. By the proof of Lemma 6.1 we have $d\lambda_{g(1)} = P_c$, hence $\lambda_{g(1)} = \lambda_g$. We conclude that $\mathfrak{H}$ is the connected component of the identity of $\mathfrak{T}_{\mathfrak{G}}(M)$. □

6.5. Lemma. *Let $\mathfrak{G}$ be a connected Lie group. Let W be a subset of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ with*

(1) *if $X \in W$ and $t \in \mathbb{R}$ then $tX \in W$;*
(2) *W generates $\mathfrak{g}$ as a Lie algebra.*

Then $\exp(W)$ *generates $\mathfrak{G}$ as a group.*

Proof. Let $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ generated by $\exp(W)$ and

$$V = \{X \in \mathfrak{g} \mid e^{tX} \in \mathfrak{H} \text{ for all } t \in \mathbb{R}\}\,.$$

Then $W \subset V$ by Assumption 1. Let U be the linear subspace of $\mathfrak{g}$ generated by V. Now for all $h \in \mathfrak{H}$ and $X \in V$

$$\exp(t \operatorname{Ad}_h X) = h e^{tX} h^{-1} \in \mathfrak{H}\,,$$

hence $\operatorname{Ad}_h X \in V$. Therefore U is invariant under $\operatorname{Ad}_{\mathfrak{H}}$. Hence for any $X \in U$, $Y \in V$ and $t \in \mathbb{R}$

$$e^{\operatorname{ad} tY} X = \operatorname{Ad}_{\exp(tY)} X \in U\,.$$

Therefore $[U,U] \subset U$. Now $W \subset U$ and hence $U = \mathfrak{g}$. Therefore V contains a basis $X_1, \dots, X_n$ of $\mathfrak{g}$. But then the map

$$\mathbb{R}^n \ni a = (a_1, \dots, a_n) \mapsto \exp(a_1 X_1) \dots \exp(a_n X_n)$$

has maximal rank in $a = 0$. Hence the image contains a neighborhood of the neutral element of G. □

By Lemma 6.1 the isotropy group of $\mathfrak{T}(M)$ at p_0 is the holonomy group $\mathfrak{H}_{p_0}(M)$. Since the inclusion $\mathfrak{T}(M) \subset \mathfrak{I}(M)$ is continuous, $\mathfrak{H}_{p_0}(M)$ is closed in $\mathfrak{T}(M)$ and hence

$$M = \mathfrak{T}(M)/\mathfrak{H}_{p_0}(M)\,. \tag{6.6}$$

Since the Lie algebra of $\mathfrak{T}(M)$ is $[\mathfrak{p},\mathfrak{p}]+\mathfrak{p}$, $\mathfrak{T}(M)$ is the smallest subgroup of $\mathfrak{I}(M)$ containing $\mathfrak{p}$.

Let $\mathfrak{S}(M)$ be the normal subgroup of $\mathfrak{I}(M)$ consisting of products of geodesic reflections and $\mathfrak{S}^{ev}(M) \subset \mathfrak{S}(M)$ be the normal subgroup of elements which are products of an even number of reflections.

Let S be the geodesic reflection in p_0. For any $p \in M$ we have $SS_p = (S_pS)^{-1}$ since S and S_p have order 2. Furthermore, $SS_pS^{-1} = S_q$ with $q = S(p)$, hence $(S_pS)^{-1} = S_qS$.

For all $p, q \in M$ we have $S_pS_q = (S_pS)(S_qS)^{-1}$ and hence $\mathfrak{S}_{ev}(M)$ can also be characterized as the subgroup of $\mathfrak{I}(M)$ consisting of products S_pS, $p \in M$. Hence $\mathfrak{S}^{ev}(M)$ is connected.

We now represent M by a symmetric pair $(\mathfrak{G}, \mathfrak{K})$ and let $\mathfrak{S}^{ev}_{\mathfrak{G}}(M)$ be the subgroup of $g \in \mathfrak{G}$ with $\lambda_g \in \mathfrak{S}^{ev}(M)$.

6.7. Theorem. $\mathfrak{S}^{ev}_{\mathfrak{G}}(M) = \mathfrak{T}_{\mathfrak{G}}(M)$

Proof. Let $X \in \mathfrak{p} \subset \mathfrak{t}(M)$. Then $\exp(X) = S_pS$ with $p = \exp(X/2)p_0$. Hence $\exp(\mathfrak{p}) \subset \mathfrak{S}_{ev}(M)$ and therefore $\mathfrak{T}(M) \subset \mathfrak{S}^{ev}(M)$ by Theorem 6.3 and Lemma 6.5.

Vice versa, let $p, q \in M$ and $c : [a, b] \to M$ be a geodesic polygon from p to q. Choose a subdivision $a = t_0 < t_1 \ldots < t_k = b$ of $[a, b]$ such that $c|[t_{i-1}, t_i]$ is a geodesic, $1 \leq i \leq k$. Let S_i be the geodesic reflection in $c(t_i)$, $0 \leq i \leq k$. Then

$$S_pS_q = (S_0S_1)(S_1S_2)\cdots(S_{k-1}S_k)\,.$$

Now $S_{i-1}S_i$ is a transvection along the complete geodesic c_i which is defined by $c_i|[t_{i-1}, t_i] = c|[t_{i-1}, t_i]$, $1 \leq i \leq k$. Hence $\mathfrak{S}^{ev}(M) \subset \mathfrak{T}(M)$. ☐

6.8. Theorem. *Let* $(\mathfrak{G}, \mathfrak{K})$ *be a symmetric pair and assume that* $\mathfrak{G}$ *is semisimple. Then*

$$\mathfrak{G} = \mathfrak{T}_{\mathfrak{G}}(M) \quad \text{and} \quad \mathfrak{I}_0(M) = \mathfrak{G}/\mathfrak{H}\,,$$

where $\mathfrak{H} = \{g \in \mathfrak{G} \mid \lambda_g = \mathrm{id}\}$.

Proof. Without loss of generality we may assume that $\mathfrak{G}$ acts effectively on M. We let $\mathfrak{G}' = \mathfrak{I}_0(M)$ and $\mathfrak{g}'$ be the Lie algebra of $\mathfrak{g}'$. We show that $\mathfrak{g}' = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$.

We have $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$, where $\mathfrak{k}'$ is the Lie algebra of the isotropy group $\mathfrak{K}'$ of p_0 in $\mathfrak{I}_0(M)$ and $\mathfrak{p}'$ corresponds to the infinitesimal transvections through p_0. Now $\mathfrak{p} = \mathfrak{p}'$ since $\mathfrak{G}$ contains all the transvections along geodesics through $\mathfrak{p}_0 = [K]$.

Since $\mathfrak{g}$ is semisimple, the Killing form B of $\mathfrak{g}$ is non-degenerate. Hence the Rici tensor of M is non-degenerate on $T_{p_0}M$. Therefore the Killing form B' of $\mathfrak{g}'$ is non-degenerate.

Suppose $\mathfrak{g}' \neq [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$. Then there exists a $Z \in \mathfrak{k}' \backslash \{0\}$ with $B'([X, Y], Z) = 0$ for all $X, Y \in \mathfrak{p}$. On the other hand, since the Killing field Z^* corresponding to Z is nonzero and $Z^*(p_0) = 0$ we have $DZ^*(p_0) \neq 0$. Since $DZ^*(p_0)$ corresponds to ad_Z under the usual identification $T_{p_0}M \cong \mathfrak{p}$ there is a $Y \in \mathfrak{p}$ with $[Y, Z] \neq 0$. Now B' is non–degenerate on $\mathfrak{p}$, hence there is an $X \in \mathfrak{p}$ with

$$0 \neq B'(X, [Y, Z]) = B'([X, Y], Z)$$

in contradititction to the choice of Z. Hence $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p} = \mathfrak{g}$, hence $\mathfrak{T}(M) = \mathfrak{G} = I_0(M)$ as claimed. ☐

7. Riemannian Coverings and Symmetric Spaces

Clearly any connected covering space of a symmetric space is itself a symmetric space with respect to the induced metric. We reverse the point of view and ask when a space covered by a symmetric spaces is itself symmetric. The answer we find is pretty much complete, see Theorem 7.2 and Lemma 7.5.

Let $(\mathfrak{G}, \mathfrak{K})$ be a Riemannian symmetric pair with involutive automorphism σ of $\mathfrak{G}$ and inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{p}$. Assume that $\mathfrak{G}$ is connected and denote by $\mathfrak{Z}(\mathfrak{G})$ the center of $\mathfrak{G}$.

First we assume in addition that $\mathfrak{G}$ is effective on M, that is, $\mathfrak{G} \subset \mathfrak{I}(M)$, the group of isometries of M. Then the automorphism σ of $\mathfrak{G}$ is conjugation by the geodesic symmetry S of M at $p_0 = [\mathfrak{K}]$. In other words, conjugation by S is an involutive automorphism of $\mathfrak{I}(M)$ leaving $\mathfrak{G}$ invariant. We are mainly interested in the case where $\mathfrak{G}$ is the group $\mathfrak{T}(M)$ of transvections of M.

7.1. Lemma. *The centralizer $\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ of $\mathfrak{G}$ in $\mathfrak{I}(M)$ is invariant under conjugation by S and acts freely on M.*

Let $h \in \mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ and $g \in \mathfrak{G}$ be isometries with $gp_0 = hp_0$. Then $g \in \{g \in \mathfrak{G} \mid g\sigma(g)^{-1} \in \mathfrak{Z}(\mathfrak{G})\}$ and h is equal to right translation by g: $h(fp_0) = f(gp_0)$ for all $f \in \mathfrak{G}$. Moreover, conjugation by S is given by

$$ShS = ShS^{-1} = h^{-1}.$$

In particular, $\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ is an abelian group and any subgroup of $\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ is invariant under conjugation by S.

Proof. Since $\mathfrak{G}$ is invariant under conjugation by S, the same holds for its centralizer $\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ in $\mathfrak{I}(M)$.

Now let $g \in \mathfrak{G}$ and $h \in \mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$. Since g and h commute we have $h(gp_0) = g(hp_0)$. Now $\mathfrak{G}$ is transitive on M and hence $h = \mathrm{id}$ if and only if $hp_0 = p_0$. Therefore $\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ acts freely on M.

Let $h \in \mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$ and $g \in \mathfrak{G}$ with $gp_0 = hp_0$. Then there is an isometry f of M with $fp_0 = p_0$ such that $g = hf$. Then $SfS = f$ and therefore

$$g\sigma(g)^{-1} = gSg^{-1}S = hSh^{-1}S \in \mathfrak{G} \cap \mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G}).$$

Since $\mathfrak{G} \cap \mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G}) = \mathfrak{Z}(\mathfrak{G})$ we conclude that $g \in \{g \in \mathfrak{G} \mid g\sigma(g)^{-1} \in \mathfrak{Z}(\mathfrak{G})\}$. This proves the first assertion.

The proof of the second assertion is easy: Let $p \in M$ and choose $f \in \mathfrak{G}$ with $fp_0 = p$. Then

$$hp = h(fp_0) = f(hp_0) = f(gp_0).$$

Hence h is equal to right translation by g as claimed.

Now $ShS \in \mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G})$. By Lemma 7.3 we also have $SgS = g^{-1}k$ for some $k \in \mathfrak{K}$. Therefore

$$ShS(p_0) = S(hp_0) = S(gp_0) = SgS(p_0) = g^{-1}k(p_0) = g^{-1}(p_0),$$

and hence ShS is right translation by g^{-1}. Therefore $ShS = h^{-1}$. □

We consider a Riemannian covering $\pi : M \to \bar{M}$, where M and $\bar{M}$ are symmetric spaces. We let $\mathfrak{G} = \mathfrak{T}(M)$ be the group of transvections of M and denote by $\mathfrak{Z}(M)$ the centralizer of $\mathfrak{T}(M)$ in $\mathfrak{I}(M)$, $\mathfrak{Z}(M) := \mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{T}(M))$. Recall that $\mathfrak{T}(M)$ is connected.

7.2. Theorem. *The group Γ of covering transformations of π is a discrete subgroup of $\mathfrak{Z}(M)$. Vice versa, if $\Gamma \subset \mathfrak{Z}(M)$ is a discrete subgroup, then $\bar{M} = M/\Gamma$ is a symmetric space and $\pi : M \to \bar{M}$ is a Riemannian covering.*

Proof. Let $\mathfrak{S}(M)$ be the subgroup of $\mathfrak{I}(M)$ generated by the geodesic symmetries of M. Now if $p \in M$ and S denotes the geodesic symmetry of M at p, then $\pi \circ S = \bar{S} \circ \pi$, where $\bar{S}$ is the geodesic symmetry of $\bar{M}$ at $\bar{p} = \pi(p)$.

Fix $\gamma \in \Gamma$. Then for any point $q \in M$

$$\pi(S(\gamma q)) = \bar{S}(\pi(\gamma q)) = \bar{S}(\pi(q)) = \pi(S(q))\,.$$

Hence there is $\gamma_q \in \Gamma$ with $S(\gamma q) = \gamma_q S(q)$. Since Γ acts freely and properly discontinuously on M we get that γ_q does not depend on q. Hence S normalizes Γ. It follows that $\mathfrak{S}(M)$ and $\mathfrak{T}(M) \subset \mathfrak{S}(M)$ normalize Γ. Since Γ is a discrete subgroup of $\mathfrak{I}(M)$ and $\mathfrak{T}(M)$ is a connected Lie group, $\mathfrak{T}(M)$ centralizes Γ. Hence $\Gamma \subset \mathfrak{Z}(M)$. Now Γ is discrete since the action of Γ on M is properly discontinuous.

Vice versa, let Γ be a discrete subgroup of $\mathfrak{Z}(M)$. Since $\mathfrak{Z}(M)$ acts freely on M, Γ acts freely and properly discontinuously on M. Hence $\bar{M} = M/\Gamma$ is a smooth Riemannian manifold and $\pi : M \to \bar{M}$ is a Riemannian covering. By Lemma 7.1, Γ is invariant under conjugation by reflections. Hence geodesic reflections are well defined isometries on $\bar{M}$ and hence $\bar{M}$ is a symmetric space. □

We now return to the more general situation of a symmetric pair $(\mathfrak{G}, \mathfrak{K})$, where $\mathfrak{G}$ is not assumed to be effective on $M = \mathfrak{G}/\mathfrak{K}$. We consider $\mathfrak{N} = \{g \in G \mid g\sigma(g)^{-1} \in \mathfrak{Z}(\mathfrak{G})\}$.

7.3. Lemma. *The set $\mathfrak{N}$ is a closed, σ–invariant subgroup of $\mathfrak{G}$ and $\mathfrak{N}$ normalizes $\mathfrak{K}$. For $g \in \mathfrak{N}$ we have $\sigma(g) = g^{-1}$ modulo $\mathfrak{K}$. In particular, $\mathfrak{N}/\mathfrak{K}$ is an abelian group.*

Proof. Let $\mathfrak{F}$ be the subgroup of $\mathfrak{G}$ fixed by σ and $\mathfrak{F}_0$ be the identity component of $\mathfrak{F}$. Then $\mathfrak{F}_0 \subset \mathfrak{K} \subset \mathfrak{F}$. Let $M_0 = \mathfrak{G}/\mathfrak{F}_0$. Then M_0 is connected, hence a symmetric space. Since M_0 is complete, the exponential map $T_{p_0}M_0 \to M_0$ is surjective, where $p_0 = [\mathfrak{F}]$. Hence for each $g \in \mathfrak{G}$ there are $k \in \mathfrak{F}_0$ and $X \in \mathfrak{p}$ with $g = e^X k$. Since

$$\sigma(k) = k \quad \text{and} \quad \sigma(e^X) = e^{-X}$$

we get

$$g\sigma(g)^{-1} = (e^X k) \cdot \sigma(e^X k)^{-1} = e^{2X}\,.$$

Hence $g = e^X k \in \mathfrak{N}$ iff $e^{2X} \in \mathfrak{Z}(\mathfrak{G})$. Now $\mathfrak{Z}(\mathfrak{G})$, the center of $\mathfrak{G}$, is a normal subgroup of $\mathfrak{G}$. Hence $e^{2X} \in \mathfrak{Z}(\mathfrak{G})$ implies $e^{-2X} \in \mathfrak{Z}(\mathfrak{G})$ and also

$$g^{-1}\sigma(g^{-1}) = (k^{-1}e^{-X}) \cdot \sigma(k^{-1}e^{-X})^{-1} = k^{-1}e^{-2X}k \in \mathfrak{Z}(\mathfrak{G})\,.$$

It follows that $\mathfrak{N}^{-1} = \mathfrak{N}$. Now let $g, h \in \mathfrak{N}$. Since $h\sigma(h)^{-1}$ is in the center $\mathfrak{Z}(\mathfrak{G})$ of $\mathfrak{G}$ we get

$$(gh)\sigma(gh)^{-1} = gh\sigma(h)^{-1}\sigma(g)^{-1} = g^{-1}\sigma(g)^{-1}h\sigma(h)^{-1} \in \mathfrak{Z}(\mathfrak{G}) .$$

It follows that $\mathfrak{N} \cdot \mathfrak{N} = \mathfrak{N}$, hence $\mathfrak{N}$ is a subgroup of G. It also follows that

$$\mathfrak{N} \to \mathfrak{Z}(\mathfrak{G}), \quad g \mapsto g\sigma(g)^{-1} ,$$

is a homomorphism. The kernel of this homomorphism is $\mathfrak{F}$. Hence $\mathfrak{F}$, and therefore also $\mathfrak{F}_0$, is normal in $\mathfrak{N}$. Now

$$\sigma(e^X \cdot \mathfrak{F}_0) = e^{-X} \cdot \mathfrak{F}_0 = (e^X)^{-1} \cdot \mathfrak{F}_0 ,$$

hence for any $g \in \mathfrak{N}$, $\sigma(g) = g^{-1}$ modulo $\mathfrak{F}_0$. Now a group for which the map which sends an element to its inverse is a homomorphism is abelian. Hence $\mathfrak{N}/\mathfrak{F}_0$ is abelian.

Since $\mathfrak{F}_0 \subset \mathfrak{K} \subset \mathfrak{N}$ we conclude that $\mathfrak{N}/\mathfrak{K}$ is abelian and that $\sigma(g) = g^{-1}$ modulo $\mathfrak{K}$. □

For any g in the normalizer $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$ of $\mathfrak{K}$ in $\mathfrak{G}$ we have $g \cdot \mathfrak{K} = \mathfrak{K} \cdot g$. Hence for any $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$, right translation ρ_g by g is a well defined diffeomorphism of $M = \mathfrak{G}/\mathfrak{K}$,

$$\rho_g(hp_0) := hgp_0 , \quad p_0 = [\mathfrak{K}] .$$

These right translations will play an important role in our discussion below. Note that the group $\mathfrak{N}$ from Lemma 7.3 is contained in $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$, hence right translations by elements of $\mathfrak{N}$ are defined.

Since right translations by elements from $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$ as above commute with left translations by elements from $\mathfrak{G}$, any such right translation — if it happens to be an isometry of M — is contained in the centralizer

$$\mathfrak{Z}_{\mathfrak{I}(M)}(\mathfrak{G}) = \{f \in \mathfrak{I}(M) \mid f \circ \lambda_g = \lambda_g \circ f \text{ for all } g \in \mathfrak{G}\} \tag{7.4}$$

of $\mathfrak{G}$ in $\mathfrak{I}(M)$.

Now we assume in addition that G is semisimple. As in Theorem 6.8, let $\mathfrak{H} = \{g \in \mathfrak{G} \mid \lambda_g = \mathrm{id}\}$. Let $\mathfrak{h}$ be the Lie algebra of $\mathfrak{H}$. Note that $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$, hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. By Theorem 6.8 we have $\mathfrak{T}_{\mathfrak{G}}(M) = \mathfrak{G}$ and $\mathfrak{I}_0(M) = \mathfrak{G}/\mathfrak{H}$. By Proposition 4.7 we also have

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_1 + \cdots + \mathfrak{g}_m ,$$

where the $\mathfrak{g}_i$ are semisimple σ–invariant ideals of $\mathfrak{g}$ and $(\mathfrak{g}_i, s)$ is irreducible with $\langle \cdot , \cdot \rangle = \lambda_i B$ on $\mathfrak{p}_i$, where $\lambda_i \neq 0$, $1 \leq i \leq m$. Define a non–degenerate bilinear form A on $\mathfrak{g}$ by $A(\mathfrak{h}, \mathfrak{g}_i) = A(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for all i and $j \neq i$, $A(X, Y) = \lambda_i B$ on $\mathfrak{g}_i$ and $A = B$ on $\mathfrak{h}$. Recall that on $\mathfrak{h}$ and $\mathfrak{g}_i$, B is the Killing form of $\mathfrak{h}$ and $\mathfrak{g}_i$ respectively. It follows easily that for any $X \in \mathfrak{g}$, ad_X is skew symmetric with respect to A. Since $\mathfrak{G}$ is connected, we get that A is invariant under Ad_g for all $g \in \mathfrak{G}$. Note that $A|\mathfrak{p} = \langle \cdot , \cdot \rangle$.

7.5. Lemma. *Assume that $\mathfrak{G}$ is semisimple and let $\mathfrak{N} = \{g \in G \mid g\sigma(g)^{-1} \in Z_G\}$ as above. Then $\mathfrak{N} = \mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$, the normalizer of $\mathfrak{K}$ in $\mathfrak{G}$ and $\mathfrak{N}/\mathfrak{K} \cong \mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G}) = \mathfrak{Z}(M)$.*

Proof. By Lemma 7.3 we have $\mathfrak{N} \subset \mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$. Vice versa, let $g \in \mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$. We need to show that ρ_g is an isometry of M. To that end, we define a diffeomorphism κ_g of M by

$$\kappa_g(hp_0) = g^{-1}hgp_0\,,$$

that is, $\kappa_g = \lambda_h^{-1} \circ \rho_g$. For all $h \in \mathfrak{G}$ we have $\rho_g \circ \lambda_h = \lambda_{hg} \circ \kappa_g$. Now let $v \in T_{hp_0}M$, $v = d\lambda_h(p_0) \cdot u$, where $u \in T_{p_0}M$. Then

$$\begin{aligned} d\rho_g(hp_0) \cdot v &= d\rho_g(hp_0) \cdot d\lambda_h(p_0) \cdot u \\ &= d\lambda_{hg}(p_0) \cdot d\kappa_g(p_0) \cdot u\,. \end{aligned}$$

Now $d\kappa_g(p_0)$ preserves the inner product in $T_{p_0}M$ because A is invariant under all Ad_g, $g \in \mathfrak{G}$. It follows that ρ_g is an isometry of M. Clearly $\rho_g \in \mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G})$. We already know that the map $\mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G}) \to \mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$, sending $h \in \mathfrak{Z}_{\mathfrak{I}}(\mathfrak{G})$ to the right translation ρ_g, where $hp_0 = gp_0$, is an injective homomorphism. We just showed that it is also surjective. □

8. The Cartan Immersion of Symmetric Spaces

In this section we assume that $\mathfrak{G}$ is semisimple and that $\mathfrak{K}$ is compact. As usual, let $M = \mathfrak{G}/\mathfrak{K}$ and $\pi : \mathfrak{G} \to M$ be the map with $\pi(g) = gp_0$, where $p_0 = [\mathfrak{K}]$. The group $\mathfrak{G}$ acts on itself by $g * h := gh\sigma(g)^{-1}$. We let $\mathfrak{P}$ be the orbit of the neutral element,

$$\mathfrak{P} = \{g\sigma(g)^{-1} \mid g \in \mathfrak{G}\}\,. \tag{8.1}$$

Then $\mathfrak{P}$ is an embedded submanifold of $\mathfrak{G}$ with $T_e\mathfrak{P} = \mathfrak{p}$. To see this, we choose open neighborhoods U of 0 in $\mathfrak{k}$ and V of 0 in $\mathfrak{p}$ such that

$$\exp : U + V \to \exp(U + V) =: W \subset \mathfrak{G}$$

is a diffeomorphism onto its image. Then

$$\exp(V) = \{g \in W \mid \sigma(g) = g^{-1}\}\,.$$

On the other hand, if $\sigma(h) = h^{-1}$ for some $h \in \mathfrak{G}$, then this property holds for any element in the $\mathfrak{G}$–orbit of h. Hence $\mathfrak{P} \cap W = \exp(V)$. Hence $\mathfrak{P} \cap W$ is an embedded submanifold in W. Since $\mathfrak{P}$ is an orbit, it follows that $\mathfrak{P}$ is an embedded submanifold as claimed.

We endow $\mathfrak{P}$ with the $\mathfrak{G}$–invariant Riemannian metric which is equal to the given inner product on $\mathfrak{p}$ (when identified with $T_{p_0}M$).

Since $\sigma(k) = k$ for all $k \in \mathfrak{K}$, the map $\varphi : \mathfrak{G} \to \mathfrak{G}$, $\varphi(g) = g\sigma(g)^{-1}$ factorizes over M: there is a (smooth) map

$$\Phi : M \to \mathfrak{G} \quad \text{with} \quad \Phi \circ \pi = \varphi\,. \tag{8.2}$$

The map Φ is called the *Cartan immersion* of M.

8.3. Theorem. *The submanifold $\mathfrak{P}$ is closed and $\mathfrak{P} = \exp(\mathfrak{p})$. Furthermore, $\mathfrak{P}$ is totally geodesic with respect to the bi–invariant connection of $\mathfrak{G}$ and a symmetric space with respect to the $\mathfrak{G}$–invariant metric above.*

The image of Φ is equal to $\mathfrak{P}$ and $\Phi : M \to \mathfrak{P}$ is a covering with $\|\Phi_* v\| = 2\|v\|$ for all $v \in TM$. The covering Φ is normal with $|\mathfrak{F}/\mathfrak{K}|$ leaves and its group of covering transformations consists of the right translation by elements of $\mathfrak{F}/\mathfrak{K}$.

Proof. Since the exponential map $T_{p_0}M \to M$ is surjective, each element of $\mathfrak{G}$ is of the form $g = \exp(X)k$ with $k \in \mathfrak{K}$, $X \in \mathfrak{p}$ and such that $\|X\| = d(p_0, gp_0)$. Now if $g = \exp(X)k$, then $g\sigma(g)^{-1} = \exp(2X)$, hence the image of φ and of Φ respectively is equal to $\exp(\mathfrak{p})$.

If (h_n) is a sequence in $\exp(\mathfrak{p})$ with $h_n \to h \in \mathfrak{G}$, then $h_n = \exp(X_n)$ with $X_n \in \mathfrak{p}$ such that $\|X_n\| = d(p_0, h_n p_0)$. But then $\|X_n\|$ is uniformly bounded and hence $X_n \to X \in \mathfrak{p}$ after passing to a subsequence if necessary. But then $h = \exp(X)$ and therefore $\exp(\mathfrak{p})$ is closed.

Now $\mathfrak{G}$ is a Lie group and the Ad_G–invariant non–degenerate bilinear form A as in Lemma 7.5 gives rise to a bi–invariant pseudo–Riemannian metric on G, also denote A. The left (or right) translates of 1–parameter subgroups of G are the geodesics of A. Hence $\mathfrak{P}$ is totally geodesic with respect to A and therefore, since $\mathfrak{G}$ is connected, $\mathfrak{P} = \exp(\mathfrak{p})$. It follows immediately from the definiton that $\|\phi_* X\| = 2\|X\|$. Via the identification $\pi_* : \mathfrak{p} \to T_{p_0}M$, A corresponds to the Riemannian metric on M. Therefore also $\|\Phi_* X\| = 2\|X\|$. Hence Φ is a covering.

Now $\mathfrak{K}$ is normal in $\mathfrak{F}$ by Lemma 7.3, hence the group of covering transformation $M \to P$ can be identified with right translation by element of $\mathfrak{F}/\mathfrak{K}$. It follows that $\mathfrak{P}$ is symmetric. (This can also be seen directly.) □

9. Rank and Maximal Flats

Throughout, we represent a given symmetric space M by a Riemannian symmetric pair (G, K), where G is connected and K is compact. We let σ be the corresponding automorphism of G and denote by $s = \sigma_*$ the differential of σ at the neutral element $e \in G$. As usual, we let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G into $(+1)$ and (-1)–eigenspace of s.

A subspace $\mathfrak{a} \subset \mathfrak{p}$ is a subalgebra if and only if $\mathfrak{a}$ is abelian. This is an immediate consequence of $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Clearly, any 1–dimensional subspace of $\mathfrak{p}$ is abelian. It follows that any $X \in \mathfrak{p}$ is contained in a maximal abelian subspace of $\mathfrak{p}$. This section is devoted to the discussion of maximal abelian subspaces of $\mathfrak{p}$ and the corresponding totally geodesic submanifolds of M.

9.1. Lemma. *Let $\mathfrak{g} = \mathfrak{g}_0 + \cdots + \mathfrak{g}_m$ be a decomposition into s–invariant ideals and $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i$ be the decomposition of $\mathfrak{g}_i$ into the eigenspaces of s as usual. Then $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace if and only if $\mathfrak{a} = \mathfrak{a}_0 + \cdots + \mathfrak{a}_m$, where $\mathfrak{a}_i \subset \mathfrak{p}_i$ is a maximal abelian subspace.*

In particular, if $\mathfrak{g} = \mathfrak{g}_0 + \cdots + \mathfrak{g}_m$ is the decomposition as in ???, then a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ contains $\mathfrak{p}_0$ since $[\mathfrak{p}_0, \mathfrak{p}_0] = \{0\}$. Together with Lemma 9.1 this reduces the proof of (most of) the statements below to the case where $\mathfrak{g}$ is semisimple, that is, where $\mathfrak{g}_0 = \{0\}$.

In a similar vein, if $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is of the noncompact type and $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ is the dual involutive Riemannian Lie algebra, then $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian if and only if $i\mathfrak{a} \subset i\mathfrak{p}$ is maximal abelian. Again by Lemma 9.1 we can restrict in the proof of most of the statements below to the case where $\mathfrak{g}$ is of compact type.

9.2. Theorem. *Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Then the connected Lie subgroup A of G with Lie algebra $\mathfrak{a}$ is closed and $F = Ap_0$ is a closed, totally geodesic, flat submanifold of M.*

The submanifolds $F = Ap_0$ as above are called *maximal flats.*

Proof of Theorem 9.2. We note first that A is abelian and connected, hence the closure $\bar{A}$ is an abelian and connected subgroup of G. Now $P = \exp(\mathfrak{p})$ is a closed submanifold of G, see ???, therefore $\bar{A} \subset P$. It follows that the tangent space $\bar{\mathfrak{a}}$ of $\bar{A}$ at the neutral element of G is an abelian subspace of $\mathfrak{g}$ with $\mathfrak{a} \subset \bar{\mathfrak{a}} \subset \mathfrak{p}$. By maximality of $\mathfrak{a}$ we conclude $\mathfrak{a} = \bar{\mathfrak{a}}$. Hence $A = \bar{A}$, the first assertion.

By ???, $F = Ap_0$ is a totally geodesic, flat submanifold of M. It remains to show that F is closed. To that end, let (p_n) be a sequence in F converging to $p \in M$. Then we have $p_n = g_n p_0$ for appropriately chosen $g_n \in A$. Now since K is compact and the sequence (p_n) converges, the sequence (g_n) is contained in a compact part of G. Passing to a subsequence if necessary we may therefore assume that (g_n) converges to some $g \in G$. By what we proved above we have $g \in A$ and hence $p \in F$. □

9.3. LEMMA. *Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Then there is $X \in \mathfrak{a}$ whose centralizer in $\mathfrak{p}$ is equal to $\mathfrak{a}$,*

$$\mathfrak{z}_{\mathfrak{p}}(X) \cap \mathfrak{p} = \mathfrak{a}\,.$$

Vectors $X \in \mathfrak{a}$ with centralizer in $\mathfrak{p}$ equal to $\mathfrak{a}$ are called *regular.*

Proof of Lemma 9.3. By what we said above we can assume that $\mathfrak{g}$ and hence G is of compact type. Then $A = \exp(\mathfrak{a})$ is a torus. Hence there existis $X \in \mathfrak{a}$ with $\exp(tX)$, $t \in \mathbb{R}$, dense in A.

Let $Z \in \mathfrak{p}$ commute with X, $[Z, X] = 0$. Then $\exp(sZ)$ and $\exp(tX)$ commute for all $s, t \in \mathbb{R}$. By the density of $\exp(tX)$, $t \in \mathbb{R}$, in A it follows that $\exp(sZ)$ commutes with A for all $s \in \mathbb{Z}$. Differentiating at $s = 0$, we conclude that $[Z, \mathfrak{a}] = \{0\}$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ we get $Z \in \mathfrak{a}$. □

9.4. LEMMA. *Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Let $X \in \mathfrak{a}$ be regular and $Y \in \mathfrak{p}$. On K, consider the function*

$$f(k) = B(Ad_k Y, X)\,.$$

Then $\operatorname{Ad}_k Y \in \mathfrak{a}$ for any critical point k of f.

Proof. Let k be a critical point of f. Then for any $Z \in \mathfrak{k}$,

$$\begin{aligned} 0 &= \partial_t(f(e^{tZ}k))|_{t=0} \\ &= B([Z, Ad_k Y], X) = B(Z, [Ad_k Y, X]). \end{aligned}$$

Since B is negative definite on $\mathfrak{k}$, we conclude, $[\operatorname{Ad}_k Y, X] = 0$. Since X is regular, $\operatorname{Ad}_k Y \in \mathfrak{a}$. □

9.5. THEOREM. *If $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ are maximal abelian, then there is $k \in K_0$ with $\operatorname{Ad}_k \mathfrak{a} = \mathfrak{a}'$. Furthermore,*

$$\cup_{k \in K_0} \operatorname{Ad}_k \mathfrak{a} = \mathfrak{p}\,.$$

Proof. Recall that for any $k \in K$, Ad_k is an automorphism of $\mathfrak{g}$ and that $\operatorname{Ad}_k \mathfrak{p} = \mathfrak{p}$. Now if $Y \in \mathfrak{a}'$ is regular, then $\mathfrak{a}'$ is the centralizer of Y in $\mathfrak{p}$. Therefore for any $k \in K$, $\operatorname{Ad}_k \mathfrak{a}'$ is the centralizer of $\operatorname{Ad}_k Y$ in $\mathfrak{p}$. Now by Lemma 9.4 there is $k \in K_0$ with $\operatorname{Ad}_k Y \in \mathfrak{a}$. But then $\mathfrak{a} \subset Ad_k \mathfrak{a}'$ by what we just said. Hence $\dim \mathfrak{a} \le \dim \mathfrak{a}'$. Reversing the roles of $\mathfrak{a}$ and $\mathfrak{a}'$, we conclude $\dim \mathfrak{a} = \dim \mathfrak{a}'$ and $\mathfrak{a} = Ad_k \mathfrak{a}'$. □

By Theorem 9.5, the dimension of maximal abelian subspaces of $\mathfrak{p}$ coincides; it is called the *rank* of M. By definition, the rank of M is equal to the dimension of maximal flats in M.

9.6. COROLLARY. *Let F, F' be maximal subspaces, $p \in F$ and $p' \in F'$. Then there is $g \in G$ with $gF = F'$ and $gp = p'$.*

We discuss an important application to compact Lie groups. As a preparation we need the following result.

9.7. Lemma. *Let G be a compact connected Lie group and $T \subset G$ a closed, connected, abelian subgroup — a torus. Let $g \in Z_G(T)$, the centralizer of T in G. Then there is a torus $T' \subset G$ containing g and T. In particular, $Z_G(T)$ is connected.*

Proof. Let A be the closure of the subgroup of G generated by g and T. Then A is a compact abelian Lie subgroup of G and the component A_0 of the identity of A is a torus. Now A/A_0 is generated by $[g] = gA_0$ since A_0 contains T. Since A/A_0 is finite, there exists an $n \geq 1$ with $[g^n] = 1$, that is, with $g^n \in A_0$. Recall that A_0 is a torus, hence there is $a \in A_0$ such that $\{a^m \mid m \in \mathbb{Z}\}$ is dense in A_0. Furthermore, there is $b \in A_0$ with $(bg)^n = a$. Then the closure of $\{(bg)^k \mid k \in \mathbb{Z}\}$ contains A_0 and meets all the components of A, hence contains A. Now there is $X \in \mathfrak{g}$ with $\exp X = bg$. Hence the closure T' of $\{\exp(tX) \mid t \in \mathbb{R}\}$ is a closed, connected, abelian subgroup of G containing A. Hence T' is a torus containing g and T. □

9.8. Theorem. *Let G be a compact connected Lie group. Then*

(1) *maximal abelian subgroups of G are tori;*
(2) *any tow maximal tori in G are conjugate.*

In particular, any element of G is contained in a maximal torus.

Proof. This is immediate from our discussion of compact connected Lie group as symmetric spaces and the discussion of maximal flat subspaces above. □

10. Roots and Weyl Chambers

We assume that $(\mathfrak{g}, s, \langle \cdot\,, \cdot \rangle)$ is an orthogonal symmetric Lie algebra of compact type. As usual, we decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ into the eigenspaces of s. Without loss of generality we can assume in our discussion below that the inner product on $\mathfrak{p}$ is equal to $-B$, where B is the Killing form of $\mathfrak{g}$.

We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Typical elements from $\mathfrak{a}$ will be denoted H, whereas typical elements from $\mathfrak{g}$ will be denoted X, Y, Z.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexefication of $\mathfrak{g}$. For any $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$, we let

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}_H X = i\alpha(H)X \text{ for all } H \in \mathfrak{a}\}\,. \tag{10.1}$$

It is immediate from the Jacobi equation that

$$[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}\,. \tag{10.2}$$

Each ad_H, $H \in \mathfrak{a}$, is skewsymmetric, hence for any ad_H–invariant subspace $V \subset \mathfrak{g}_{\mathbb{C}}$, the orthogonal complement is ad_H–invariant as well. Since all ad_H, $H \in \mathfrak{a}$, commute with each other, we get a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \sum_{\alpha} \mathfrak{g}_{\mathbb{C}}^{\alpha}\,. \tag{10.3}$$

In the direct sum decomposition (10.3), we only list those α with $\mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}$. These α are called the *roots* and the corresponding spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}$ are called *root spaces*. The set of roots is denoted Δ, the subset of nonzero roots is denoted Δ^*. Note that $\Delta \leq \dim \mathfrak{g}$.

For $X \in \mathfrak{g}_{\mathbb{C}}$, denote by $\bar{X}$ the conjugate element. Let $\alpha \in \Delta$, $H \in \mathfrak{a}$ and $X \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Then $H = \bar{H}$ and hence

$$[H, \bar{X}] = \overline{[H, X]} = -i\alpha(H)\bar{X}\,.$$

Hence

$$\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}\,. \tag{10.4}$$

Since s is an involutive automorphism of $\mathfrak{g}$ and $sH = -H$, we also have

$$[H, sX] = s[sH, X] = -i\alpha(H)sX$$

and therefore

$$s(\mathfrak{g}_{\mathbb{C}}^{\alpha}) = \mathfrak{g}_{\mathbb{C}}^{-\alpha}\,. \tag{10.5}$$

Thus the set of root spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is invariant under s and under conjugation and, furthermore, $\Delta = -\Delta$.

Each $\alpha \in \Delta^*$ defines a hyperplane in $\mathfrak{g}$, namely $\{\alpha = 0\}$. These hyperplanes are called *walls*. We say that $H \in \mathfrak{a}$ is *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \Delta^*$, that is, if H is not contained in a wall. Otherwise we call H *singular*. Below we will show that this notion of regular and singular coincides with the one introduced before. The connected components of the set of regular elements of $\mathfrak{a}$ are called

Weyl chambers. The choice of a Weyl chamber $\mathcal{C}$ determines a partition of Δ^* into *positive* and *negative* roots,

$$\Delta^+ = \{\alpha \in \Delta^* \mid \alpha > 0 \text{ on } \mathcal{C}\}\,, \qquad \Delta^- = \{\alpha \in \Delta^* \mid \alpha < 0 \text{ on } \mathcal{C}\}\,. \tag{10.6}$$

We have

$$\Delta^* = \Delta^+ \dot\cup \Delta^- \quad \text{and} \quad \Delta^- = -\Delta^+\,. \tag{10.7}$$

Once and for all we choose a Weyl chamber $\mathcal{C}$ and the corresponding partition of the roots into positive and negative roots. In examples such a choice is more or less natural. For $\alpha \in \Delta^+$ we set

$$\mathfrak{g}^\alpha = \mathfrak{g} \cap (\mathfrak{g}^\alpha_{\mathbb{C}} + \mathfrak{g}^{-\alpha}_{\mathbb{C}})\,. \tag{10.8}$$

Then $\mathfrak{g}^\alpha$ is invariant under s and hence

$$\mathfrak{g}^\alpha = \mathfrak{k}^\alpha + \mathfrak{p}^\alpha \tag{10.9}$$

with

$$\mathfrak{k}^\alpha = \mathfrak{k} \cap \mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{p}^\alpha = \mathfrak{p} \cap \mathfrak{g}^\alpha\,. \tag{10.10}$$

We also set

$$\mathfrak{g}^0 = \mathfrak{g} \cap \mathfrak{g}^0_C = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})\,. \tag{10.11}$$

Note that

$$\mathfrak{k}^0 = \mathfrak{k} \cap \mathfrak{g}^0 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \quad \text{and} \quad \mathfrak{p}^0 = \mathfrak{p} \cap \mathfrak{g}^0 = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}\,. \tag{10.12}$$

Let $\alpha \in \Delta$ and $X, Y \in \mathfrak{g}$ with $X - iY \in \mathfrak{g}^\alpha_{\mathbb{C}}$. Then

$$\begin{aligned}[][H, X] - i[H, Y] = [H, X - iY] &= i\alpha(H)(X - iY) \\ &= -i\alpha(H)X + \alpha(H)Y\end{aligned}$$

and hence

$$[H, X] = \alpha(H)Y, \quad [H, Y] = -\alpha(H)X\,. \tag{10.13}$$

Motivated by this computation we say that $X \in \mathfrak{k}^\alpha$ and $Y \in \mathfrak{p}^\alpha$ are *related* if $X - iY \in \mathfrak{g}^\alpha_C$.

10.14. Lemma. *Let $\alpha \in \Delta^+$ and fix $H \in \mathfrak{a}$ with $\alpha(H) \neq 0$. Then $\frac{1}{\alpha(H)} \operatorname{ad}_H$ is an orthogonal isomorphism between $\mathfrak{k}^\alpha$ and $\mathfrak{p}^\alpha$ independent of the choice of H and $Y \in \mathfrak{p}^\alpha$ is related to $X \in \mathfrak{k}^\alpha$ if and only if $Y = \frac{1}{\alpha(H)} \operatorname{ad}_H X$.*

Proof. Let $X \in \mathfrak{k}^\alpha$. By the definition of $\mathfrak{k}^\alpha$, there is $Y \in \mathfrak{g}$ with

$$X - iY \in \mathfrak{g}^\alpha_{\mathbb{C}}\,, \quad X + iY \in \mathfrak{g}^{-\alpha}_{\mathbb{C}}\,.$$

Hence X and Y are related and $[H, X] = \alpha(H)Y$, $[H, Y] = -\alpha(H)X$. Vice versa, for $Y \in \mathfrak{p}^\alpha$ there is $x \in \mathfrak{k}^\alpha$ such that X and Y are related. Hence for H regular, $\frac{1}{\alpha(H)} \operatorname{ad}_H$ is an isomorphism between $\mathfrak{k}^\alpha$ and $\mathfrak{p}^\alpha$.

It remains to show that $\|X\| = \|Y\|$. Now s is orthogonal and $\mathfrak{g}_{\mathbb{C}}^{\alpha} \perp \mathfrak{g}_{\mathbb{C}}^{-\alpha}$. Hence

$$\begin{aligned} 4\|X\|^2 &= \|(X - iY) + s(X - iY)\|^2 \\ &= \|(X - iY)\|^2 + \|(X - iY)\|^2 \\ &= \|(X - iY) - s(X - iY)\|^2 = 4\|Y\|^2 . \end{aligned}$$

Hence $\frac{1}{\alpha(H)} \operatorname{ad}_H$ is an orthogonal transformation as claimed. □

Let $Z = X - iY \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ as above. Then for all $H \in \mathfrak{a}$,

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = 0 .$$

Now $[X, Y] \in \mathfrak{p}$, hence $[X, Y]$ is in $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$.

10.15. Lemma. *Let $\alpha \in \Delta^+$ and $X \in \mathfrak{k}^{\alpha}$, $Y \in \mathfrak{p}^{\alpha}$ be related unit vectors. Then $H_\alpha = [X, Y]$ does not depend on the choice of X, Y. In fact*

$$\langle H_\alpha, H\rangle = \alpha(H) \quad \text{for all } H \in \mathfrak{a} .$$

Proof. We have

$$B(H_\alpha, H) = B([X, Y], H) = B(X, [Y, H]) = \alpha(H)B(X, Y) .$$

Now $B = -\langle\cdot\,,\cdot\rangle$, hence the claim. □

10.16. Lemma. *Let $H \in \mathfrak{a}$. Then the centralizer of H in $\mathfrak{g}$ is*

$$\mathfrak{z}_{\mathfrak{g}}(H) = \mathfrak{g}^0 + \sum_{\substack{\alpha>0 \\ \alpha(H)=0}} \mathfrak{g}^{\alpha} .$$

In particular, H is regular if and only if $\mathfrak{z}_{\mathfrak{g}}(H)$ is equal to $\mathfrak{g}^0$ or respectively if and only if $\mathfrak{z}_{\mathfrak{p}}(H) = \mathfrak{a}$.

Proof. We compute centralizers in $\mathfrak{g}_{\mathbb{C}}$. Let $X \in \mathfrak{g}_{\mathbb{C}}$ and decompose $X = \sum_\alpha X_\alpha$ with $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Then

$$[H, X] = \sum_{\alpha(H)\neq 0} i\alpha(H)X_\alpha .$$

Hence $[H, X] = 0$ if and only if $X_\alpha = 0$ for all $\alpha \in \Delta$ with $\alpha(H) \neq 0$. Hence X is in the centralizer of H in $\mathfrak{g}_{\mathbb{C}}$ if and only if $X \in \sum_{\alpha(H)=0} \mathfrak{g}_{C}^{\alpha}$. The assertion about $\mathfrak{z}_{\mathfrak{g}}(H)$ is an immediate consequence. □

We now assume that $(\mathfrak{G}, \mathfrak{K})$ is a Riemannian symmetric pair of compact type with associated orthogonal symmetric Lie algebra $(\mathfrak{g}, s, -B)$ as above. We let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian and $\mathfrak{A}$ be the connected subgroup of $\mathfrak{G}$ with Lie algebra $\mathfrak{a}$. Then

$$\mathfrak{N}_{\mathfrak{K}}(\mathfrak{a}) = \{k \in \mathfrak{K} \mid \operatorname{Ad}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{a}\} = \{k \in \mathfrak{K} \mid k\mathfrak{A}k^{-1} = \mathfrak{A}\} \tag{10.17}$$

and

$$\mathfrak{Z}_{\mathfrak{K}}(\mathfrak{a}) = \{k \in \mathfrak{K} \mid \operatorname{Ad}_k |\mathfrak{a} = \operatorname{id}_{\mathfrak{a}}\} = \{k \in \mathfrak{K} \mid kgk^{-1} = g \text{ for all } g \in \mathfrak{A}\} \tag{10.18}$$

are called the *normalizer* and *centralizer* of $\mathfrak{a}$ in $\mathfrak{K}$ respectively. The Lie algebras of $\mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$ and $\mathfrak{Z}_{\mathfrak{K}}(\mathfrak{a})$ are denoted $\mathfrak{n}_{\mathfrak{K}}(\mathfrak{a})$ and $\mathfrak{z}_{\mathfrak{K}}(\mathfrak{a})$ respectively.

10.19. LEMMA. $\mathfrak{n}_{\mathfrak{K}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{K}}(\mathfrak{a})$

Proof. Let $X \in \mathfrak{n}_{\mathfrak{K}}(\mathfrak{a})$. Then $[H, X] \in \mathfrak{a}$ for all $H \in \mathfrak{a}$ and hence

$$B([H,X],[H,X]) = B([[H,X],H],X) = 0\,.$$

Now B is negative definite on $\mathfrak{g}$ since $\mathfrak{g}$ is of compact type. Hence $[H, X] = 0$. □

Now $\mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$ and $\mathfrak{Z}_{\mathfrak{K}}(\mathfrak{a})$ are closed subgroups of the compact group $\mathfrak{K}$. Hence both groups are compact. By the lemma, $\mathfrak{W} = \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})/\mathfrak{Z}_{\mathfrak{K}}(\mathfrak{a})$ is finite. We call $\mathfrak{W}$ the *Weyl group.*

10.20. LEMMA. *Let $\alpha \in \Delta^*$ be a root and s_α be the reflection of $\mathfrak{a}$ about the wall $\{\alpha = 0\}$. Then there is $k \in \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$ with $\mathrm{Ad}_k = s_\alpha$ on $\mathfrak{a}$.*

Proof. Let $X \in \mathfrak{k}^\alpha$, $Y \in \mathfrak{p}^\alpha$ be related unit vectors and $H_\alpha = [X, Y]$. Then

$$\mathrm{ad}_X^{2n-1}(H_\alpha) = (-\alpha(H_\alpha))^n Y\,, \quad \mathrm{ad}_X^{2n}(H_\alpha) = (-\alpha(H_\alpha))^n H_\alpha\,.$$

Now $\alpha(H_\alpha) = \|H_\alpha\|^2$, hence

$$\begin{aligned}\mathrm{Ad}_{\exp(tX)}(H_\alpha) &= \exp(t\,\mathrm{ad}_X)(H_\alpha)\\ &= \sum_{n\geq 0} \frac{(-1)^n}{(2n)!}(t\|H_\alpha\|)^{2n} H_\alpha + \sum_{n\geq 1} \frac{(-1)^n}{(2n-1)!}(t\|H_\alpha\|)^{2n-1} Y\\ &= \cos(t\|H_\alpha\|)H_\alpha - \|H_\alpha\|\sin(t\|H_\alpha\|)Y\,.\end{aligned}$$

Therefore for $t = \pi/\|H_\alpha\|$,

$$\mathrm{Ad}_{\exp(tX)}(H_\alpha) = -H_\alpha\,.$$

For $H \perp H_\alpha$, we have $[X, H] = -\alpha(H)Y = 0$ and hence

$$\mathrm{Ad}_{\exp(tX)}(H) = H$$

for such H. Hence $\mathfrak{k} = \exp(tX)$ with $t = \pi/\|H_\alpha\|$ is an element of $N_K(\mathfrak{a})$ realizing the reflection s_α. □

10.21. THEOREM. *The Weyl group $\mathfrak{W} = \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})/\mathfrak{Z}_{\mathfrak{K}}(\mathfrak{a})$ acts simply transitively on the Weyl chambers of $\mathfrak{a}$. Furthermore, $\mathfrak{W}$ is generated by the reflections about the hyperplanes $\{\alpha = 0\}$, $\alpha \in \Delta^*$.*

Proof. By Lemma 10.20, the reflections s_α about the hyperplanes $\{\alpha = 0\}$ belong to $\mathfrak{W}$. Now let $\mathcal{C}$, $\mathcal{C}'$ be Weyl chambers. Since pairwise intersections $\{\alpha = 0\} \cap \{\beta = 0\}$ between linearily independent roots are subspaces of $\mathfrak{a}$ of codimension two, there is a continuous curve $c : [0, 1] \to \mathfrak{a}$ with $c(0) \in \mathcal{C}$ and $c(1) \in \mathcal{C}'$ such that c does not meet such intersections and such that $c\|[t_i, t_{i+1}]$ is contained in a closed Weyl chamber $\bar{\mathcal{C}}_i$ with $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_k = \mathcal{C}'$ for some appropriate subdivision $t_0 = 0 < t_1 < \ldots < t_{k+1} = 1$ of $[0, 1]$. Then

$$c(t_i) \in \bar{\mathcal{C}}_{i-1} \cap \bar{\mathcal{C}}_i \subset \{\alpha_i = 0\}$$

for some root $\alpha_i \in \Delta^*$, $1 \le k$, hence

$$s_{\alpha_i}(\bar{\mathcal{C}}_{i-1}) = \bar{\mathcal{C}}_i$$

and therefore

$$s_{\alpha_k} \cdots s_{\alpha_1}(\mathcal{C}) = \mathcal{C}' \, .$$

This shows that the subgroup of $\mathfrak{W}$ generated by the reflections s_α, $\alpha \in \Delta^*$, is transitive on the set of Weyl chambers in $\mathfrak{a}$.

It remains to show that $\mathfrak{W}$ acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$. To that end it suffices to show that the stabilizer in $\mathfrak{W}$ of a prefered Weyl chamber $\mathcal{C}$ is trivial. Suppose $w(\mathcal{C}) = \mathcal{C}$ for some $w \in \mathfrak{W}$. Since $\mathfrak{W}$ is finite, there is $n \ge 1$ such that $w^n = 1$. Choose $H_0 \in \mathcal{C}$ and set

$$H = H_0 + w(H_0) + \cdots + w^{n-1}(H_0) \, .$$

Since $\mathcal{C}$ is a convex cone and $w(\mathcal{C}) = \mathcal{C}$ we have $H \in \mathcal{C}$. Since $w^n = 1$ we have $w(H) = H$. Let $\mathfrak{T}$ be the closure of $\{\exp(tH) \mid t \in \mathbb{R}\}$. Then $\mathfrak{T}$ is a closed connected abelian subgroup of $\mathfrak{G}$, hence a torus. Furthermore, $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{T}) = \mathfrak{Z}_{\mathfrak{G}}(\{\exp(tH) \mid t \in \mathbb{R}\})$, hence $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{z}_{\mathfrak{g}}(H)$, where $\mathfrak{t}$ is the Lie algebra of $\mathfrak{T}$. By ???, $\mathfrak{Z}_{\mathfrak{G}}(H) = \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{T})$ is connected.

Now choose $k \in \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$ with $\mathrm{Ad}_k = w$ on $\mathfrak{a}$. Then $\mathrm{Ad}_k(H) = H$, hence $k \in \mathfrak{Z}_{\mathfrak{K}}(\mathfrak{T})$. Therefore $k = \exp(X)$ for some $X \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})) = \mathfrak{z}_{\mathfrak{g}}(H)$. Now $H \in \mathcal{C}$, hence $\alpha(H) \neq 0$ for all $\alpha \in \Delta^*$ and hence $\mathfrak{z}_{\mathfrak{g}}(H) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$. But then $X \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, hence $k \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{a})$. This implies $w = 1$. □

10.22. Lemma. *Let $\mathfrak{b} \subset \mathfrak{a}$ be a subspace and suppose $\mathrm{Ad}_k(\mathfrak{b}) \subset \mathfrak{a}$ for some $\mathfrak{k} \in K$. Then there is $w \in W$ with $\mathrm{Ad}_k(H) = w(H)$ for all $H \in \mathfrak{b}$.*

Proof. Let $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{b})$ be the centralizer of $\mathfrak{b}$ in $\mathfrak{G}$. Now $\mathfrak{b}$ is invariant under the involutions $\sigma_* = s$ of $\mathfrak{g}$, $s(\mathfrak{b}) = \mathfrak{b}$. Hence

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{b}) + \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) \, .$$

Let $H \in \mathfrak{a}$, $X \in \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$, $k \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{b})$. Then

$$Ad_k(X) \in \mathfrak{p} \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) \, .$$

Hence for all $H' \in \mathfrak{b}$,

$$[H', [H, \mathrm{Ad}_k(X)]] = [[H', H], \mathrm{Ad}_k(X)] + [H, [H', \mathrm{Ad}_k(X)]] = 0 \, ,$$

and therefore $[H, \mathrm{Ad}_k(X)] \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{b})$.

Now consider the connected component $\mathfrak{Z}$ of the identity of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{b})$. On $\mathfrak{Z} \cap \mathfrak{K}$, consider the smooth function

$$f(k) = B(H, Ad_k(X)) \, ,$$

where $H \in \mathfrak{a}$, $X \in \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ are as above and where we assume that H is regular. If k is a critical point of f, then

$$0 = B(H, [\mathrm{Ad}_k(X), Y]) = B([H, \mathrm{Ad}_k(X)], Y)$$

for all $Y \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{b})$. Now $[H, \mathrm{Ad}_k(X)] \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{b})$ by what we said above, hence $[H, \mathrm{Ad}_k(X)] = 0$. Since X is regular, we conclude that $\mathrm{Ad}_k(X) \in \mathfrak{a}$.

Now choose $k = k_0$ as in the assertion and X regular in $\mathrm{Ad}_{k_0}^{-1}(\mathfrak{a}) \supset \mathfrak{b}$. Then

$$Ad_{k_0}^{-1}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{p}}(X) \subset \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) .$$

Let $k = k_1 \in \mathfrak{Z} \cap \mathfrak{K}$ be a critical point of the function f as above. Then $\mathrm{Ad}_{k_1}(X) \in \mathfrak{a}$, hence

$$\mathrm{Ad}_{k_1}^{-1}(\mathfrak{a}) \subset \mathfrak{z}_{\mathfrak{p}}(X) = \mathrm{Ad}_{k_0}^{-1}(\mathfrak{a}) ;$$

therefore

$$\mathrm{Ad}_{k_1}^{-1}(\mathfrak{a}) = \mathrm{Ad}_{k_0}^{-1}(\mathfrak{a}) .$$

Hence $k = k_0 k_1^{-1} \in \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$. Since $k_1 \in \mathfrak{Z} \cap \mathfrak{K}$, $\mathrm{Ad}_k = Ad_{k_0}$ on $\mathfrak{b}$ so that $w = Ad_k|\mathfrak{a}$ satisfies the requirement. □

10.23. Corollary. *For any* $X \in \mathfrak{p}$, $\mathrm{Ad}_{\mathfrak{K}}(X) = Ad_{\mathfrak{K}_0}(X)$. *For any* $p \in M$, $\mathfrak{K}p = \mathfrak{K}_0 p$.

Proof. It suffices to consider the case where $X, Y \in \mathfrak{a}$ and $Y = Ad_k(X)$ for some $k \in \mathfrak{K}$. Then by Lemma 10.22 we can assume $k \in \mathfrak{N}_{\mathfrak{K}}(\mathfrak{a})$. But $\mathfrak{W}$ is generated by reflections and these are by elements in $\mathfrak{K}_0$, hence the assertion. □

10.24. Corollary. *Let* $H \in \mathfrak{a}$, $Y \in \mathfrak{p}$ *and suppose that* $\exp(H)p_0 = \exp(Y)p_0 =: p$. *Suppose furthermore that* p *is not conjugate to* p_0 *along the geodesic* $c_H(t) = \exp(tH)p_0$, $t \in \mathbb{R}$. *Then* $[H, Y] = 0$.

In particular, there is $k \in \mathfrak{K}$ *with* $\mathrm{Ad}_k(Y) =: H' \in \mathfrak{a}$, $\mathrm{Ad}_k(H) = H$ *and* $\exp(H')p_0 = p$.

Proof. By assumption, there is $k \in \mathfrak{K}$ with $\exp(H) = \exp(Y)k$; hence for all $t \in \mathbb{R}$,

$$e^{-H} e^{tY} e^{H} = k^{-1} e^{tY} k = \exp(t\, \mathrm{Ad}_k^{-1} Y) .$$

Therefore

$$\exp(-ad_H)(Y) = \mathrm{Ad}_{\exp(H)}^{-1}(Y) = Ad_k^{-1}(Y) \in \mathfrak{p} .$$

Now

$$Y = Y_0 + \sum \eta_\alpha Y_\alpha$$

with $Y_0 \in \mathfrak{a}$ and unit vectors $Y_\alpha \in \mathfrak{p}^\alpha$, $\alpha > 0$. Let $X_\alpha \in \mathfrak{k}^\alpha$ be related with Y_α. Then for all $H \in \mathfrak{a}$,

$$\begin{aligned} \exp(-ad_H)(Y_\alpha) &= Y_\alpha - \alpha(H)X_\alpha - \frac{1}{2}(\alpha(H))^2 Y_\alpha + \frac{1}{6}(\alpha(H))^3 X_\alpha + \dots \\ &= \cos(\alpha(H))Y_\alpha - \sin(\alpha(H))X_\alpha . \end{aligned}$$

Since the resulting vector is in $\mathfrak{p}$ we conclude $\alpha(H) \in \pi\mathbb{Z}$ if $\eta_\alpha \neq 0$. But then p is conjugate to p_0 along c_H, a contradiction. □

11. Examples

Grassmannians
Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $G_{\mathbb{K}}(p,q)$ be the Grassmann manifold of p-planes in $\mathbb{K}^{p+q}$. We show that $G_{\mathbb{K}}(p,q)$ together with a natural metric is a Riemannian symmetric space.

The natural action of $\mathfrak{G} = S0(p+q)$ on $G_{\mathbb{R}}(p,q)$ is transitive. The stabilizer of the p-plane $\mathbb{R}^p \times \{0\} \subset \mathbb{R}^{p+q}$ is $\mathfrak{K} = S(0(p) \times 0(q))$, hence

$$G_{\mathbb{R}}(p,q) = S0(p+q)/S(0(p) \times 0(q)) .$$

Similarly, the natural action of $\mathfrak{G} = U(p+q)$ on $G_{\mathbb{C}}(p,q)$ is transitive. The stabilizer of the p-plane $\mathbb{C}^p \times \{0\} \subset \mathbb{C}^{p+q}$ is $\mathfrak{K} = U(p) \times U(q)$, hence

$$G_{\mathbb{C}}(p,q) = U(p+q)/U(p) \times U(q) .$$

In either case, let S be the reflection of $\mathbb{K}^{p+q}$ in $\mathbb{K}^p \times \{0\}$. Then conjugation with S is an involutive automorphism of $\mathfrak{G}$ with $\mathfrak{K}$ as its set of fixed points. Hence $(\mathfrak{G}, \mathfrak{K})$ is a symmetric pair.

We write square $(p+q)$-matrices as blocks of four matrices corresponding to the prefered decomposition $\mathbb{K}^{p+q} = \mathbb{K}^p \times \mathbb{K}^q$. In the case of a matrix M with real entries, M^* denotes the transposed matrix of M. In the case where the entries of M are complex, M^* denotes the transposed of the conjugate matrix $\bar{M}$ of M. With this notation

$$\mathfrak{k} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \middle| \, U^* = -U, \, V^* = -V \right\}, \tag{11.1}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix} \middle| \, X \in \mathrm{Mat}_{\mathbb{K}}(q,p) \right\} \cong \mathrm{Mat}_{\mathbb{K}}(q,p) . \tag{11.2}$$

Furthermore,

$$\left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & X^*V - UX^* \\ VX - XU & 0 \end{pmatrix} .$$

Since $U^* = -U$ and $V^* = -V$ the RHS belongs to $\mathfrak{p}$. We also have

$$\left[\begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Y^* \\ Y & 0 \end{pmatrix} \right] = \begin{pmatrix} Y^*X - X^*Y & 0 \\ 0 & YX^* - XY^* \end{pmatrix} .$$

There is an Ad_G–invariant scalar product on $\mathfrak{g}$,

$$\langle A, B \rangle = -\frac{1}{2} \operatorname{tr}(AB) .$$

Its restriction to $\mathfrak{p}$ turns $(\mathfrak{G}, \mathfrak{K})$ into a Riemannian symmetric pair. With respect to the identification $\mathfrak{p} \equiv Mat_{\mathbb{R}}(q,p)$ indicated by the notation above, the curvature tensor is given by

$$R(X,Y)Z = XY^*Z + ZY^*X - YX^*Z - ZX^*Y .$$

Inside $G_{\mathbb{K}}(p,q)$, we consider the open subset $G^-_{\mathbb{K}}(p,q)$ of p-planes on which the non-degenerate symmetric bilinear form with fundamental matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{11.3}$$

is negative definite. The group $\mathfrak{G} = 0(p,q)$ respectively $\mathfrak{G} = U(p,q)$ of linear transformation preserving this form is transitive on $G^-_{\mathbb{K}}(p,q)$ and

$$G^-_{\mathbb{R}}(p,q) = 0(p,q)/0(p) \times 0(q)\,, \quad G^-_{\mathbb{C}}(p,q) = U(p,q)/U(p) \times U(q)$$

The reflection of $\mathbb{K}^{p+q}$ in $\mathbb{K}^p \times \{0\}$ with respect to A coincides with the above reflection S. Conjugation with S is an involutive automorphism of $\mathfrak{G}$ which has $\mathfrak{K} = 0(p) \times 0(q)$ respectively $\mathfrak{K} = U(p) \times U(q)$ as its set of fixed points. Hence $(\mathfrak{G}, fK)$ is a symmetric pair. We have

$$\mathfrak{k} = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \middle| \, U^* = -U,\, V^* = -V \right\}, \tag{11.4}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \middle| \, X \in \mathrm{Mat}_{\mathbb{K}}(q,p) \right\} \cong \mathrm{Mat}_{\mathbb{K}}(q,p)\,. \tag{11.5}$$

The Lie bracket in $\mathfrak{k}$ is as above, but there is a change in the Lie brackets $[\mathfrak{k}, \mathfrak{p}]$ and $[\mathfrak{p}, \mathfrak{p}]$,

$$\left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & UX^* - X^*V \\ VX - XU & 0 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} \right] = \begin{pmatrix} Y^*X - X^*Y & 0 \\ 0 & YX^* - XY^* \end{pmatrix}.$$

We see that with respect to the identification $\mathfrak{p} \equiv \mathrm{Mat}_{\mathbb{R}}(q,p)$, the Lie bracket $[\mathfrak{k}, \mathfrak{p}]$ remains the same, but the Lie bracket $[\mathfrak{p}, \mathfrak{p}]$ changes sign. There is a corresponding change in sign for the curvature tensor.

Lie groups with bi–invariant metrics

Let $\mathfrak{L}$ be a connected Lie group with a bi-invariant metric and set $\mathfrak{G} = \mathfrak{L} \times \mathfrak{L}$.

Then the diagonal

$$\mathfrak{K} = \{(g,g) | g \in \mathfrak{L}\} \subset \mathfrak{G}$$

is the set of fixed points of the involution

$$\sigma(g,h) = (h,g)$$

of $\mathfrak{G}$. We conclude that $(\mathfrak{G}, \mathfrak{K})$ is a Riemannian symmetric pair. The map

$$f : \mathfrak{G}/\mathfrak{K} \to \mathfrak{L}, \quad (g,h) \cdot \mathfrak{K} \mapsto gh^{-1}$$

is a $\mathfrak{G}$–invariant diffeomorphism. Here $\mathfrak{G}$ acts on $\mathfrak{L}$ by $(g,h)\cdot l = glh^{-1}$. With respect to this diffeomorphism, the reflection in the neutral element $e \in \mathfrak{L}$ is the inversion of L, $l \mapsto l^{-1}$. We have

$$f_*\pi_*(X,-X) = 2X,$$

which results in the factor 1/4 in the usual formula for the curvature tensor of bi–invariant metrics on Lie groups.

12. Appendix on Lie Algebras

We discuss some elementary facts from the theory of Lie algebras. Let $\mathbb{K}$ be a field and $\mathfrak{g}$ be a Lie algebra over K. Denote by B the Killing form of $\mathfrak{g}$.

12.1. Lemma. *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be an ideal. Then*

(1) $B|\mathfrak{h}$ *is the Killing form of* $\mathfrak{h}$.
(2) $\mathfrak{h}^\perp = \{X \in \mathfrak{g} \mid B(X, \mathfrak{h}) = 0\}$ *is an ideal.*

Proof. The first assertion is clear since $\mathrm{ad}_X(\mathfrak{g}) \subset \mathfrak{h}$ for all $X \in \mathfrak{h}$. As for the second, let $X \in \mathfrak{h}^\perp$, $Y \in \mathfrak{g}$, $Z \in \mathfrak{h}$. Then

$$B([X,Y],Z) = B(X,[Y,Z]) = 0\,,$$

hence $[X,Y] \in \mathfrak{h}^\perp$. Hence $\mathfrak{h}^\perp$ is an ideal. □

12.2. Lemma. *Suppose* $\mathfrak{g}$ *is semisimple and* $\mathfrak{h} \subset \mathfrak{g}$ *is an abelian ideal. Then* $\mathfrak{h} = \{0\}$.

Proof. Let $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. Then $\mathrm{ad}_X \circ \mathrm{ad}_Y = 0$ on $\mathfrak{h}$ and the image of $\mathrm{ad}_X \circ \mathrm{ad}_Y$ is contained in $\mathfrak{h}$. Hence $B(X,Y) = 0$. Since $\mathfrak{g}$ is semisimple, $X = 0$. □

12.3. Corollary. *If* $\mathfrak{g}$ *is semisimple, then the center of* $\mathfrak{g}$ *is trivial.*

12.4. Lemma. *Suppose* $\mathfrak{g}$ *is semisimple. Let* $\mathfrak{h} \subset \mathfrak{g}$ *be an ideal. Then* $\mathfrak{h}$ *is semisimple,* $\mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$ *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$.

Proof. Let $X \in \mathfrak{h}$, $Y \in \mathfrak{h}^\perp$, $Z \in \mathfrak{g}$. Then

$$B([X,Y],Z) = B(X,[Y,Z]) = 0$$

since $X \in \mathfrak{h}$ and $[Y,Z] \in \mathfrak{h}^\perp$. Therefore $\mathfrak{h} \cap \mathfrak{h}^\perp$ is an abelian ideal in $\mathfrak{g}$. Hence $\mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$. □

12.5. Proposition. *Suppose* $\mathfrak{g}$ *is semisimple. Then there is a unique decomposition*

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_m$$

of $\mathfrak{g}$ *into simple ideals. If* $\mathfrak{h} \subset \mathfrak{g}$ *is an ideal, then there are* $i_1, \dots, i_k \in \{1, \dots, m\}$ *such that* $\mathfrak{h} = \mathfrak{g}_{i_1} + \cdots + \mathfrak{g}_{i_k}$. *If* s *is an automorphism of* $\mathfrak{g}$, *then there is a permutation* σ *of* $\{1, \dots, m\}$ *such that* $s(\mathfrak{g}_i) = \mathfrak{g}_{\sigma(i)}$.

Proof. The existence of a decomposition $\mathfrak{g} = \mathfrak{g}_1 + \ldots + \mathfrak{g}_m$ of $\mathfrak{g}$ into simple ideals is immediate from what we said above. As for the other assertions, we only need to show that any simple ideal $\mathfrak{h}$ of $\mathfrak{g}$ is equal to some $\mathfrak{g}_i$. Now $[\mathfrak{g}_i, \mathfrak{h}] = 0$ for all i implies that $\mathfrak{h}$ belongs to the center of $\mathfrak{g}$. By the above this implies $\mathfrak{h} = \{0\}$, a contradiction. On the other hand, suppose $[\mathfrak{g}_i, \mathfrak{h}] \neq \{0\}$ for some i. Then since $\mathfrak{g}_i$ and $\mathfrak{h}$ are ideals, $\mathfrak{g}_i \cap \mathfrak{h} \supset [\mathfrak{g}_i, \mathfrak{h}] \neq \{0\}$. Now $\mathfrak{g}_i \supset \mathfrak{g}_i \cap \mathfrak{h} \subset \mathfrak{h}$ and $\mathfrak{g}_i$ and $\mathfrak{h}$ are simple. Since $\mathfrak{g}_i \cap \mathfrak{h}$ is a nontrivial ideal, we get $\mathfrak{g}_i = \mathfrak{g}_i \cap \mathfrak{h} = \mathfrak{h}$. □