ON THE GEOMETRY OF METRIC SPACES

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Preface

I discuss some aspects of the inner geometry of metric spaces. My main aim is the introduction of spaces with upper curvature bounds. I also prove some elementary and basic results about such spaces. The relevance of this kind of generalized differential geometry has been emphasized by Misha Gromov. I first learned about it in the Séminaire Suisse in Bern in the summer of 1988, where we discussed Gromov's article [Gr2] on hyperbolic groups and where I was responsible for the presentation of the generalized Hadamard-Cartan Theorem.

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1. Length Spaces

Let (X, d) be a metric space[1]. A *curve* or *path* in X is a continuous map $\sigma : I \to X$, where I is some interval. The *length* $L(\sigma)$ of a curve $\sigma : [a, b] \to X$ is defined as

$$L(\sigma) = \sup \sum_{i=1}^{k} d(\sigma(t_{i-1}), \sigma(t_i)), \tag{1.1}$$

where the supremum is taken over all subdivisions

$$a = t_0 < t_1 < \cdots < t_k = b$$

of $[a, b]$. We say that σ is *rectifiable* if $L(\sigma) < \infty$. If $\sigma : [a, b] \to X$ is rectifiable, then $\sigma : [a', b'] \to X$ is rectifiable for any $[a', b'] \subset [a, b]$. Vice versa, if σ is defined on a general interval I, then we say that σ is rectifiable if $\sigma \mid [a, b]$ is rectifiable for any compact interval $[a, b] \subset I$.

Let $\sigma : I \to X$ be rectifiable. Then there is a non-negative integrable function $v = v_\sigma : I \to \mathbb{R}$, the *speed* of σ, such that

$$L(\sigma \mid [a, b]) = \int_a^b v(t)\, dt \tag{1.2}$$

for all compact $[a, b] \subset I$. We say that σ is parameterized by arc length if it has unit speed, $v \equiv 1$.

If $\varphi : [a', b'] \to [a, b]$ is a monotonic and continuous surjection, then $L(\sigma \circ \varphi) = L(\sigma)$. Vice versa, the *arc length*

$$s : [a, b] \to [0, L(\sigma)], \quad s(t) = L(\sigma| [a, t]) \tag{1.3}$$

is a non-decreasing and continuous surjection and

$$\tilde{\sigma} : [0, L(\sigma)] \to X, \quad \tilde{\sigma}(s(t)) := \sigma(t),$$

is well defined, continuous and parameterized by arc length.

The *length metric*[2] associated to d is

$$d_L(x, y) := \inf\{L(\sigma) \mid \sigma \text{ is a curve from } x \text{ to } y\}. \tag{1.4}$$

It is easy to see that d_L is a metric and that

$$(d_L)_L = d_L. \tag{1.5}$$

We say that X is a *length space* if $d = d_L$. By definition, the usual distance function on a Riemannian manifold is a length metric.

[1] We allow that d assumes ∞ as a value.

[2] The terms *inner metric* and *intrinsic metric* are also customary.

1.6. Proposition. *If X is complete and for any pair x, y of points in X and any constant $\varepsilon > 0$ there is a $z \in X$ such that*

$$d(x,z), d(y,z) \leq \frac{1}{2}\, d(x,y) + \varepsilon,$$

then X is a length space. □

Now let X be a topological space. A *length structure* on X is a functional L on the space of continuous curves $\sigma : [a,b] \to X$ which satisfies the following four rules:

1) Positivity: $0 \leq L(\sigma) \leq \infty$.
2) $L(\sigma_1 * \sigma_2) = L(\sigma_1) + L(\sigma_2)$.
3) Invariance under reparameterization: $L(\sigma \circ \phi) = L(\sigma)$ if $\phi : [a', b'] \to [a,b]$ is a monotonic and continuous surjection.
4) Semicontinuity: If $\sigma_n \to \sigma$ uniformly, then $L(\sigma) \leq \liminf L(\sigma_n)$.

The length functional (1.1) on a metric space is an example of a length structure. Vice versa, if L is a length functional on a topological space X, then the length metric d_L as in (1.4) is a pseudometric on X. The topology of d_L is, in general, different from the original topology of X.

1.1. **Metric Simplicial Complexes.** Let X be a simplicial complex of finite dimension. Let $V = V_X$ be the set of vertices of X. A point $x \in X$ is given by its barycentric coordinates $x : V \to [0,1]$. Consider the metric d_2 on X defined by

$$d_2(x,y) := \Big\{ \sum_{v \in V} |\, x(v) - y(v)|^2 \Big\}^{1/2}. \tag{1.7}$$

Restricted to any simplex S of X, d_2 coincides with the Euclidean metric on S, considered as the convex hull of the unit coordinate vectors $e_0, \dots, e_k$ in $\mathbb{R}^{k+1}$, where $k = \dim S$. Note that the usual weak topology on X is finer than the topology on X induced by d_2 with equality if and only if X is locally finite.

1.8. Proposition. *The metric d_2 is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. Then $\lim x_n(v)$ exists for any vertex v of X. Assume recursively that there are pairwise distinct vertices $v_0, \dots v_k$ such that $x(v_j) := \lim x_n(v_j) > 0$ for $0 \leq j \leq k$, and note that this implies $k \leq \dim X$. If $x(v_0) + \ldots + x(v_k) = 1$, then x is a point of X and $\lim x_n = x$. If $x(v_0) + \ldots + x(v_k) < 1$, then the rest $1 - x_n(v_0) - \ldots - x_n(v_k)$ is bounded away from 0 for all sufficiently large n and is distributed among other vertices. For such n, at most $\dim X - k$ further vertices can profit. Since (x_n) is a Cauchy sequence, it follows that there is another vertex v_{k+1} such that $\lim x_n(v_{k+1}) > 0$. After at most $\dim X + 1$ steps we obtain a limit point x in X. □

1.9. PROPOSITION. *The length metric d_E associated to d_2 is complete and $d_E \geq d_2$. Furthermore, $d_E(x,y) = d_2(x,y)$ if x, y are contained in a common simplex of X.*

Proof. The inequality $d_E \geq d_2$ is immediate from the definition of d_E and the triangle inequality for d_2. If x, y are in a common simplex S, then the line segment in S between x and y is a curve of length $d_2(x,y)$. Hence $d_E(x,y) = d_2(x,y)$ for such pairs x, y.

Let (x_n) be a Cauchy sequence in X with respect to d_E. Since $d_E \geq d_2$, (x_n) is also a Cauchy sequence with respect to d_2, hence converges to a point $x \in X$ with respect to d_2, by Proposition 1.8. For any vertex v of X with $x(v) > 0$, we have $x_n(v) > 0$ for all n sufficiently large. For all such n, x_n and x are in a common simplex of X, and then $d_E(x_n, x) = d_2(x_n, x)$ by the first part of the proof. Hence $d_E(x_n, x)$ tends to 0 as n tends to ∞. □

1.10. LEMMA. *For any curve $\sigma : [a,b] \to X$ there is a subdivision*

$$a = t_0 < t_1 < \cdots < t_{2k} = b$$

of $[a,b]$ such that $\sigma(t_{2i-1})$ and $\sigma(t)$, $t \in [2i-2, 2i]$, are contained in a common simplex of X, $1 \leq i \leq k$.

Proof. Let $t \in [a,b]$. There is $\varepsilon > 0$ such that $\sigma(s)(v) > 0$ for all vertices v with $\sigma(t)(v) > 0$ and all $s \in [a,b]$ with $|\, s - t| < \varepsilon$. Hence for any such s, $\sigma(s)$ and $\sigma(t)$ are contained in a common simplex. □

Let $\sigma : [a,b] \to X$ be a curve. We say that σ is *piecewise linear* if there is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $\sigma|[t_{i-1}, t_i]$ is a line segment in a simplex of X, $1 \leq i \leq k$. If σ is piecewise linear, then σ is rectifiable of length $\sum d_2(\sigma(t_{i-1}), \sigma(t_i))$.

1.11. COROLLARY. *In the notation of Lemma 1.10, replacing the legs $\sigma|[t_{j-1}, t_j]$ of σ by the corresponding line segments from $\sigma(t_{j-1})$ to $\sigma(t_j)$, $1 \leq j \leq 2k$, we obtain a piecewise linear curve of length at most the length of σ, connecting the same endpoints, and homotopic to σ modulo endpoints.*

In the literature, the modification of curves as in Corollary 1.11 is refered to as the *Birkhoff curve shortening process.*

Let $F = (L_S)$ be a family of length structures on the simplices S of X. We say that F is *coherent* if

$$L_S(\sigma) = L_T(\sigma) \tag{1.12}$$

for all pairs of adjacent simplices $S \subset T$ and curves σ with image in S. We say that F is *uniform* if there is a constant $\lambda \geq 1$ such that

$$\lambda^{-1} L_E \leq L_S \leq \lambda L_E \tag{1.13}$$

for all simplices S, where L_E denotes the length structure on S determined by the metric d_2. Let d_F be the pseudometric on X associated to the family F,

$$d_F(x,y) = \inf \sum L(\sigma_i), \tag{1.14}$$

where the infimum is taken over all concatenations of finitely many consecutive paths $\sigma = \sigma_1 * \cdots * \sigma_m$ joining x to y, where each leg σ_j is contained in a simplex S_j and $L(\sigma_j)$ is the length of σ_j in S_j with respect to the given length structure on S_j.

Lemma 1.10 implies that d_E is the metric associated to the length structures L_E on the simplices.

1.15. Theorem. *If F is coherent and uniform, then (X, d_F) is a complete length space.*

Proof. Fix λ as in (1.13). By Lemma 1.10, $\lambda^{-1} d_E \le d_F \le \lambda d_E$. Hence d_F is complete. It is clear from the definition of d_F and Proposition 1.6 that d_F is a length metric. □

1.2. **Notes.** Rinow's book [Ri] contains a rather detailed discussion of rectifiable curves and length metrics. The first chapter of the lecture notes [Gr1] of Gromov contains some interesting examples of length metrics. Our notion of length structure is somewhat different from the notion there.

2. Geodesic Spaces

A curve $\sigma : I \to X$ is called a *geodesic* if it has constant speed $v \ge 0$ and any $t \in I$ has a neighborhood U in I such that

$$d(\sigma(t'), \sigma(t'')) = v \cdot |t' - t''| \tag{2.1}$$

for all t', t'' in U. We say that a geodesic $\sigma : I \to X$ is *minimal* if (2.1) holds for all $t', t'' \in I$. We say that X is a *geodesic space* if for any pair $x, y \in X$ there is a minimal geodesic from x to y.

2.2. Proposition. *If X is complete and any pair $x, y \in X$ has a* midpoint*, that is, a point $m \in X$ such that*

$$d(x,m) = d(y,m) = \frac{1}{2}\, d(x,y),$$

then X is geodesic. More generally, if there is a constant $R > 0$ such that any pair of points $x, y \in X$ with $d(x,y) < R$ has a midpoint, then any such pair can be connected by a minimizing geodesic. □

2.1. **The Hopf-Rinow Theorem.** In the metric theory of Riemannian manifolds, the theorem of Hopf and Rinow clarifies the existence of minimal geodesics. Cohn-Vossen extended their theorem to locally compact length spaces, see [Co].

2.3. LEMMA. *Let X be a locally compact length space. Then for any $x \in X$ there is an $r > 0$ such that the following hold:*
(1) If $d(x, y) \leq r$, then there is a minimal geodesic from x to y.
(2) If $d(x, y) > r$, then there is a point $z \in X$ with $d(x, z) = r$ and

$$d(x, y) = r + d(z, y).$$

Proof. Since X is locally compact, there is an $r > 0$ such that the closed metric ball $\bar{B}(x, 2r)$ is compact. Let $y \in X$. Then since X is a length space, there is a sequence $\sigma_n : [0, 1] \to X$ of curves from x to y such that $L(\sigma_n) \to d(x, y)$. Without loss of generality we can assume that σ_n has constant speed $L(\sigma_n)$.

If $d(x, y) \leq r$, then $L(\sigma_n) \leq 2r$ for n sufficiently large. Then the image of σ_n is contained in $\bar{B}(x, 2r)$ and σ_n has Lipschitz constant $2r$. Hence the sequence (σ_n) is equicontinuous and has a convergent subsequence, by the theorem of Arzela-Ascoli. The limit is a minimizing geodesic from x to y, hence (1).

If $d(x, y) > r$, then there is a point $t_n \in (0, 1)$ such that $d(\sigma_n(t_n), x) = r$. A limit z of a subsequence of $(\sigma_n(t_n))$ satisfies (2). □

We say that a geodesic $\sigma : [0, \omega) \to X$, $0 < \omega \leq \infty$, is a *ray* if σ is minimal and $\lim_{t \to \omega} \sigma(t)$ does not exist. The most important step in Cohn-Vossen's argument is the following result.

2.4. THEOREM. *Let X be a locally compact length space. Then for x, y in X there is either a minimal geodesic from x to y or else a unit speed ray $\sigma : [0, \omega) \to X$ with $\sigma(0) = x$, $0 < \omega < d(x, y)$, such that the points in the image of σ are between x and y; that is, if z is in the image of σ, then*

$$d(x, z) + d(z, y) = d(x, y).$$

Proof. Let $x, y \in X$ and assume that there is no minimizing geodesic from x to y. Let r_1 be the supremum of all r such that there is a point z between $x = x_0$ and y with $d(x_0, z) = r$ and there is a minimal geodesic from x_0 to z. Then $r_1 > 0$, by Lemma 2.3. We let x_1 be such a point with $\delta_1 = d(x_0, x_1) \geq r_1/2$ and σ_1 a minimal unit speed geodesic from x_0 to x_1. Since there is no minimizing geodesic from x_0 to y we have $x_1 \neq y$. Since x_1 is between x_0 and y, any point z in the image of σ_1 is between x_0 and y. There is no minimal geodesic σ from x_1 to y since otherwise the concatenation $\sigma_1 * \sigma$ would be a minimizing geodesic from

$x = x_0$ to y. Hence we can apply the same procedure to x_1 and obtain $r_2 > 0$, x_2 between x_1 and y with $\delta_2 = d(x_1, x_2) \geq r_2/2$ and a minimal unit speed geodesic σ_2 from x_1 to x_2. Since x_2 is between x_1 and y, any point in the image of $\sigma_1 * \sigma_2$ is between x_0 and y. In particular, $\sigma_1 * \sigma_2$ is a minimal unit speed geodesic and $x_2 \neq y$. Proceeding inductively, we obtain a minimal unit speed geodesic

$$\sigma = \sigma_1 * \sigma_2 * \sigma_3 \dots : [0, \delta_1 + \delta_2 + \delta_3 + \dots) \to X$$

such that all points on σ are between x and y. In particular, σ is minimizing. By the definition of r_1 we have

$$(r_1 + r_2 + \dots)/2 \leq \omega := \delta_1 + \delta_2 + \dots \leq r_1$$

It remains to show that σ is a ray. If this were not the case, the limit $\lim_{t \to \omega} \sigma(t) =: \bar{x}$ would exist. Since there is no minimizing geodesic from x to y we would have $\bar{x} \neq y$. But then there would be a (short) minimizing unit speed geodesic $\bar{\sigma} : [0, \bar{r}] \to X$ with $\bar{r} > 0$ such that $\bar{\sigma}(0) = \bar{x}$ and such that all the points on $\bar{\sigma}$ would be between $\bar{x}$ and y. Then all points on $\sigma * \bar{\sigma}$ would be between x and y and, in particular, $\bar{r} \leq r_n/2$ for all n by the definition of r_n. But this would contradict $\bar{r} > 0$ and $r_n \to 0$. □

2.5. Theorem of Hopf-Rinow (local). *Let X be a locally compact length space, and let $x \in X$ and $R > 0$. Then the following are equivalent:*
(1) Any geodesic $\sigma : [0, 1) \to X$ with $\sigma(0) = x$ and $L(\sigma) < R$ can be extended to the closed interval $[0, 1]$.
(2) Any minimal geodesic $\sigma : [0, 1) \to X$ with $\sigma(0) = x$ and $L(\sigma) < R$ can be extended to the closed interval $[0, 1]$.
(3) $\bar{B}(x, r)$ is compact for $0 \leq r < R$.
Each of these implies that for any pair y, z of points in $B(x, R)$ with $d(x, y) + d(x, z) < R$, there is a minimal geodesic from y to z.

Proof. The conclusions (3) ⇒ (1) and (1) ⇒ (2) are clear. We prove (2) ⇒ (3). Let $r_0 \in (0, R]$ be the supremum over all $r \in (0, R)$ such that $\bar{B}(x, r)$ is compact. We assume $r_0 < R$. Let (x_n) be a sequence in $\bar{B}(x, r_0)$. From (2) and Theorem 2.4 we conclude that there is a minimal geodesic $\sigma_n : [0, 1] \to X$ from x to x_n. Then $d(x, \sigma_n(t)) \leq t r_0$ for $0 \leq t \leq 1$, and by a diagonal argument we conclude that $\sigma_n \mid [0, 1)$ has a subsequence converging to a minimal geodesic $\sigma : [0, 1) \to X$. By (2), σ can be extended to 1 and clearly $\sigma(1)$ is the limit of the (corresponding) subsequence of $(\sigma_n(1)) = (x_n)$. Hence $\bar{B}(x, r_0)$ is compact.

Since X is locally compact, there is an $\varepsilon > 0$ such that $\bar{B}(y, \varepsilon)$ is compact for any $y \in \bar{B}(x, r_0)$. But then $\bar{B}(x, r_0 + \delta)$ is compact for $\delta > 0$ sufficiently small, a contradiction to the definition of r_0. □

2.6. Theorem of Hopf-Rinow (global). *Let X be a locally compact length space. Then the following are equivalent:*
(1) X is complete.
(2) Any geodesic $\sigma : [0,1) \to X$ can be extended to $[0,1]$.
(3) For some point $x \in X$, any minimal geodesic $\sigma : [0,1) \to X$ with $\sigma(0) = x$ can be extended to $[0,1]$.
(4) Bounded subsets of X are relatively compact.
Each of these implies that X is a geodesic space, that is, for any pair x, y of points in X there is a minimal geodesic from x to y. □

2.2. **Isometries.** Let X be a metric space. We say that $\varphi : X \to X$ is an *isometry* if φ preserves distances. For an isometry φ of X,

$$d_\varphi : X \to \mathbb{R}, \quad d_\varphi(x) = d(x, \varphi(x)), \tag{2.7}$$

is called the *displacement function* of φ. We say that φ is *semisimple* if d_φ achieves its mimimum in X. If φ is semisimple and $\min d_\varphi = 0$, then we say that φ is *elliptic*. Obviously, φ is elliptic iff it has a fixed point. If φ is semisimple and $\min d_\varphi > 0$, then we say that φ is *hyperbolic*. We say that φ is *parabolic* if d_φ does not achieve a minimum in X.

2.8. Proposition. *If X is a geodesic space and φ is a hyperbolic isometry of X, then there is a unit speed geodesic $\sigma : \mathbb{R} \to X$ and a number $t_0 > 0$ such that*

$$\varphi(\sigma(t)) = \sigma(t + t_0)$$

for all $t \in \mathbb{R}$. We say that σ is an axis *of φ.*

Proof. Let $x \in X$ with $d_\varphi(x) = \min d_\varphi$. Consider the geodesic segment ρ from x to $\varphi(x)$, and let y be the midpoint of ρ. Then

$$d(y, x) = d(y, \varphi(x)) = d(x, \varphi(x))/2.$$

Since $d(\varphi(y), \varphi(x)) = d(y, x)$, we get

$$d(y, \varphi(y)) \leq d(y, \varphi(x)) + d(\varphi(x), \varphi(y)) = d(x, \varphi(x)).$$

It follows that $d(y, \varphi(y)) = \min d_\varphi$ and that the concatenation of ρ with $\varphi(\rho)$ is a geodesic segment. Hence the concatenation σ of all the segments $\varphi^n(\rho)$, $n \in \mathbb{Z}$, is an axis of φ. □

2.3. **Notes.** Let X be the space consisting of two vertices x, y and a sequence of edges σ_n of length $1 + 1/n$, $n \geq 1$. Then X with the induced length metric is a complete length space. However, X is not a geodesic space since $d(x, y) = 1$. One way of avoiding this phenomenon is to assume the finitely many shapes condition in [Br1]. In our discussion below this problem does not appear, see Lemma 6.1 and its consequences.

3. Curvature Bounds

Let X be a metric space. A *triangle* in X consists of three unparameterized geodesic segments a, b, c in X, called the *edges* or *sides* of the triangle, whose endpoints match (in the usual way). The length of an edge is denoted by the same letter as the edge, what is what will be clear from the context. As usual, the vertex opposite to an edge will also be denoted by the same letter, but capitalized. The angle at the vertex will be denoted by the corresponding Greek letter.

For $\kappa \in \mathbb{R}$, we denote by M^n_κ the model space of dimension n and constant curvature κ. That is, M^n_κ is the round sphere of radius $1/\sqrt{\kappa}$ in Euclidean space $\mathbb{R}^{n+1}$ if $\kappa > 0$, Euclidean space $\mathbb{R}^n$ if $\kappa = 0$, and hyperbolic space H^n of curvature κ if $\kappa < 0$. We let D_κ be the diameter of M^n_κ. We assume that the reader is somewhat familiar with the trigonometry of M^2_κ. One of the main tools is the cosine formula for triangles (a, b, c) in M^2_κ of perimeter $a + b + c < 2D_\kappa$,

$$c = c_\kappa(a, b, \gamma), \tag{3.1}$$

where c_κ is an explicit function of a, b, and γ, which is monotonically increasing in a, b, and γ.

3.1. **Triangle Comparison.** If $\Delta = (a, b, c)$ is a triangle in X, a triangle $\bar{\Delta} = (\bar{a}, \bar{b}, \bar{c})$ in M^2_κ is called an *Alexandrov triangle* or *comparison triangle* for (a, b, c) if $\bar{a} = a$, $\bar{b} = b$, and $\bar{c} = c$. A comparison triangle for Δ exists and is unique (up to congruence) if its sides satisfy the triangle inequalities

$$a \le b + c, \quad b \le c + a, \quad c \le a + b \tag{3.2}$$

and if its perimeter satisfies

$$a + b + c < 2D_\kappa. \tag{3.3}$$

If the sides of Δ are minimal, then the triangle inequalities (3.2) hold.

3.4. Definition. We say that a triangle Δ is CAT(κ) if its sides satisfy the inequalities (3.2) and (3.3) and if

$$d(x, y) \le d(\bar{x}, \bar{y})$$

for all points x, y on the edges of Δ and corresponding points $\bar{x}, \bar{y}$ on the edges of the comparison triangle $\bar{\Delta}$ in M^2_κ.

The CAT condition requires that a triangle Δ is not fatter than $\bar{\Delta}$.

3.5. Lemma. *Let $\Delta = (a, b, c)$ and $\Delta' = (a', b', c')$ be triangles in X and assume that $a = b'$ and that $c \cup c'$ is a geodesic. If Δ and Δ' are CAT(κ) and $\Delta'' = (a', b, c \cup c')$ has perimeter $< 2\pi D_\kappa$, then Δ'' is also CAT(κ).*

Moreover, if the length of a is strictly smaller than the distance of the pair of points on the comparison triangle $\bar{\Delta}''$ corresponding to the points B and C on Δ'', then the distance of any pair x, y of points on the edges of Δ'' which are not on the same edge of Δ'' is strictly smaller than the distance of the corresponding pair of points on $\bar{\Delta}''$.

Proof. Match the comparison triangles $\bar{\Delta}$ and $\bar{\Delta}'$ in M^2_κ along the sides $\bar{a}$ and $\bar{b}'$. Let $x = B = A'$. Since $c \cup c'$ is a geodesic, we have

$$d(y, y') = d(y, x) + d(x, y')$$

for $y \in c$ and $y' \in c'$ sufficiently close to x. We claim that the inner angle of $\bar{\Delta} \cup \bar{\Delta}'$ at $\bar{x}$ is at least π. If not, there is a point z on $a = b'$ different from x such that the minimal geodesic in M^2_κ from $\bar{y}$ to $\bar{y}'$ passes through $\bar{z}$. Since Δ and Δ' are CAT(κ),

$$\begin{aligned} d(y, y') &= d(y, x) + d(x, y') \\ &= d(\bar{y}, \bar{x}) + d(\bar{x}, \bar{y}') \\ &> d(\bar{y}, \bar{z}) + d(\bar{z}, \bar{y}') \\ &\geq d(y, z) + d(z, y') \geq d(y, y'). \end{aligned}$$

This is a contradiction, hence the inner angle at $\bar{x}$ is at least π. In particular, the sides of Δ'' satisfy the triangle inequality (3.2).

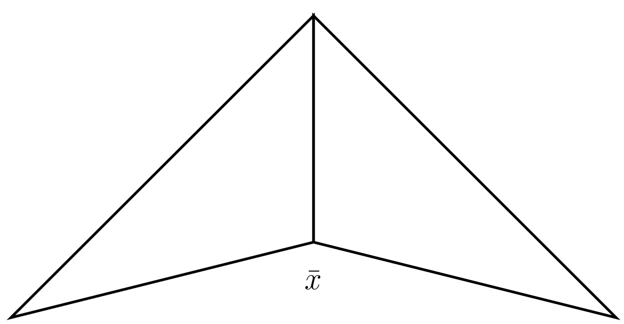

Figure 1

If the angle is equal to π, then $\bar{\Delta} \cup \bar{\Delta}'$ is the comparison triangle of Δ''. Then it follows easily that Δ'' is CAT(κ). If the angle is strictly bigger than π, the comparison triangle is obtained by straightening the broken geodesic $\bar{c} * \bar{c}'$ of $\bar{\Delta} \cup \bar{\Delta}'$, keeping the length of $\bar{b}, \bar{c}, \bar{a}'$ and $\bar{c}'$ fixed. Now let Q be the union of the two triangular surfaces bounded by $\bar{\Delta}$ and $\bar{\Delta}'$, and denote by Q_τ the corresponding surface obtained during the process of straightening at time τ. The inner distance d_τ of two points on the boundary of Q_τ, that is, the distance determined by taking the infimum of the lengths of curves in Q_τ connecting the given points, is strictly smaller than the corresponding distance at a later time. The proof of this assertion is an exercise in the trigonometry of M^2_κ. At the final time ω of the deformation, Q_ω is a triangle in M^2_κ of perimeter $< 2\pi D_\kappa$ and, therefore, the inner distance d_ω is equal to the distance in M^2_κ. The lemma follows. □

3.6. Lemma. *Let $x \in X$ and $r \in (0, D_\kappa/2)$. Assume that for any two points $y, z \in B_r(x)$ there is a minimal geodesic in X from y to z and that all triangles in $B_{2r}(x)$ with minimal sides and of perimeter $< 4r$ are CAT(κ).*

Then all geodesics in $B_r(x)$ are minimal in X, for all $y, z \in B_r(x)$ a minimal geodesic from y to z in X is contained in $B_r(x)$ and is the unique geodesic connection in $B_r(x)$ between them. The geodesic $\sigma_{yz} : [0,1] \to B_r(x)$ from y to z depends continuously on y and z.

Proof. Let $\sigma : [0,1] \to B_r(x)$ be a geodesic. Suppose that σ is not minimal. Then there is a maximal $t_\sigma \in (0,1)$ such that $\sigma_1 = \sigma|\,[0, t_\sigma]$ is minimal. Note that $L(\sigma_1) < 2r$. There is a $\delta > 0$ with $\delta < t_\sigma, 1 - t_\sigma$ such that $\sigma|[t_\sigma - \delta, t_\sigma + \delta]$ is minimal and such that $L(\sigma_1) + L(\sigma_2) < 2r$, where $\sigma_2 = \sigma|\,[t_\sigma, t_\sigma + \delta]$. Let $y = \sigma(t_\sigma)$, $z_1 = \sigma(t_\sigma - \delta)$ and $z_2 = \sigma(t_\sigma + \delta)$. We have

$$(*) \qquad d(z_1, z_2) = d(z_1, y) + d(y, z_2).$$

On the other hand, $\sigma|[0, t_\sigma + \delta]$ is not minimal, by the definition of t_σ. Let σ_3 be a minimal geodesic from $\sigma(0)$ to $z_2 = \sigma(t_\sigma + \delta)$. Then

$$(**) \qquad L(\sigma_3) < L(\sigma_1) + L(\sigma_2),$$

and σ_3 is contained in $B_{2r}(x)$ since $d(x, \sigma(0)) + d(x, z_2) < 2r$. Now the triangle $\Delta = (\sigma_1, \sigma_2, \sigma_3)$ has minimal sides and perimeter $< 4r$, and is contained in $B_{2r}(x)$. In the comparison triangle $\bar\Delta$ of Δ we get from $(**)$ and $(*)$,

$$\begin{aligned} d(\bar z_1, \bar z_2) &< d(\bar z_1, \bar y) + d(\bar y, \bar z_2) \\ &= d(z_1, y) + d(y, z_2) = d(z_1, z_2), \end{aligned}$$

a contradiction to Δ being CAT(κ). Hence σ is minimal.

Now let $y, z \in B_r(x)$ and $\sigma : [0,1] \to X$ be a minimal geodesic from y to z. Then σ is contained in $B_{2r}(x)$ and σ together with minimal geodesics from x to y and x to z forms a triangle Δ in $B_{2r}(x)$ with minimal sides and perimeter $< 4r$. Since Δ is CAT(κ) and $r < D_\kappa/2$, it follows that any point on σ has distance $< r$ to x.

Let y_1, z_1 and y_2, z_2 be two pairs of points in $B_r(x)$ and $\sigma_1, \sigma_2 : [0,1] \to X$ be minimal geodesics connecting them. Let $\alpha, \omega, \sigma : [0,1] \to X$ be minimal geodesics from y_1 to y_2, z_1 to z_2 and y_1 to z_2, respectively. If $d(y_1, y_2), d(z_1, z_2)$ are sufficiently small, then the triangles $(\sigma_1, \omega, \sigma)$ and $(\sigma_2, \alpha, \sigma)$ have perimeter $< 4r$ and are therefore CAT(κ). Then the distance between σ_1 and σ_2 can be estimated, via comparison triangles, in terms of the corresponding distances in M^2_κ. This implies the asserted uniqueness of σ_{yz} and the continuous dependence on the end points y and z. □

3.7. Remark. The assumptions and assertions in Lemma 3.6 are not optimal, compare Lemmas 6.8, 6.10 below.

3.2. **Upper Curvature Bounds.** We now come to the definition of upper curvature bounds. We start with the global version.

3.8. Definition. Let $\kappa \in \mathbb{R}$. We say that a metric space X is CAT(κ) if it satisfies the following two properties:
(1) For all $x, y \in X$ with $d(x,y) < D_\kappa$, there is a geodesic in X of length $d(x,y)$ connecting x and y.
(2) All triangles in X of perimeter $< 2D_\kappa$ are CAT(κ).

The model spaces M^n_κ are the standard examples. More generally, a complete Riemannian manifold with sectional curvature $\leq \kappa$ is CAT(κ) iff its injectivity radius is $\geq D_\kappa$. In particular, Definition 3.8 catches more than upper bounds on the sectional curvature. To remedy this we need to localize the CAT property.

3.9. Definition. We say that X has *(Alexandrov) curvature* at most κ, in symbols : $K_X \leq \kappa$, if each point $x \in X$ has a neighborhood Y such that Y together with the restricted metric is CAT(κ).

If X is CAT(κ), then the geodesic σ_{xy} as in Definition 3.8.1 is the unique geodesic in X from x to y of length $< D_\kappa$ and depends continuously on x and y. Moreover, for $x \in X$ and $r < D_\kappa/2$, the ball $B_r(x)$ is CAT(κ).

3.10. Exercises. 1) If X is CAT(κ) with respect to a metric d, then also with respect to the metric d_κ, where $d_\kappa(x,y) := \min\{d(x,y), D_\kappa\}$.
2) A metric space X is CAT(κ) iff
(1) For all $x, y \in X$ with $d(x,y) < D_\kappa$, there is a geodesic in X of length $d(x,y)$ connecting x and y.
(2) For each triangle (a,b,c) in X of perimeter $< 2D_\kappa$, we have

$$d(C, M) \leq d(\bar{C}, \bar{M}),$$

where M is the midpoint of c and $\bar{M}$ the midpoint of the side $\bar{c}$ of the comparison triangle in M^2_κ.

3.3. **Angles.** Let X be a space with Alexandrov curvature $K_X \le \kappa$. Let $Y \subset X$ be CAT(κ), and suppose that $d(x,y) < D_\kappa$ for all $x, y \in Y$. Let $\varepsilon > 0$ and $\sigma_1, \sigma_2 : [0,\varepsilon] \to Y$ be two unit speed geodesics with $\sigma_1(0) = \sigma_2(0) =: x$. For $s, t \in (0,\varepsilon)$, let Δ_{st} be the triangle spanned by $\sigma_1|[0,s]$ and $\sigma_2|[0,t]$. Let $\gamma(s,t)$ be the angle at $\bar{x}$ of the comparison triangle $\bar{\Delta}_{st}$ in M^2_κ. Then $\gamma(s,t)$ is monotonically decreasing as s, t decrease and hence

$$\angle(\sigma_1, \sigma_2) := \lim_{s,t\to 0} \gamma(s,t) \tag{3.11}$$

exists and is called the *angle* subtended by σ_1 and σ_2. The angle function satisfies the triangle inequality

$$\angle(\sigma_1, \sigma_3) \le \angle(\sigma_1, \sigma_2) + \angle(\sigma_2, \sigma_3). \tag{3.12}$$

The triangle inequality is very useful in combination with the fact that

$$\angle(\sigma_1, \sigma_2) = \pi \quad \text{if } \sigma_1^{-1} * \sigma_2 \text{ is a geodesic.} \tag{3.13}$$

Here σ_1^{-1} is defined by $\sigma_1^{-1}(t) = \sigma_1(-t)$, $-\varepsilon \le t \le 0$.

The trigonometric formulas for spaces of constant curvature show that we can use comparison triangles in M^2_λ as well, where $\lambda \in \mathbb{R}$ is arbitrary (but fixed), and obtain the same angle measure. In particular, we have the following formula:

$$\cos(\angle(\sigma_1, \sigma_2)) = \lim_{s,t\to 0} \frac{s^2 + t^2 - d^2(\sigma_1(s), \sigma_2(t))}{2st}. \tag{3.14}$$

If $y, z \in Y \setminus \{x\}$ and σ_1, σ_2 are the unit speed geodesics from x to y and z, respectively, we set $\angle_x(y,z) := \angle(\sigma_1, \sigma_2)$.

In a CAT(κ) subspace $Y \subset X$, let $x_n \to x$, $y_n \to y$, and $z_n \to z$ with $x \in Y$ and $y, z \in Y \setminus \{x\}$. Then $\angle_{x_n}(y_n, z_n)$ and $\angle_x(y_n, z_n)$ are defined for all sufficiently large n and we have

$$\angle_x(y,z) = \lim \angle_x(y_n, z_n) \ge \limsup \angle_{x_n}(y_n, z_n). \tag{3.15}$$

The proof is a good exercise.

In a CAT(κ) subspace Y and points A, B, C with pairwise distances $< D_\kappa$, we can speak of the triangle $\Delta(A,B,C)$ spanned by these points since there are unique geodesic connections between them.

3.16. Proposition. *Let X be CAT(κ), $\Delta = \Delta(A,B,C)$ be a triangle in X of perimeter $< 2D_\kappa$, and $\bar{\Delta}$ be the comparison triangle in M^2_κ.*
(1) Then $\alpha \le \bar{\alpha}$.
(2) If $d(x,y) = d(\bar{x}, \bar{y})$ for one pair of points on the boundary of Δ such that x, y do not lie on the same edge, or if $\alpha = \bar{\alpha}$, then Δ bounds a convex region in X isometric to the triangular region in M^2_κ bounded by $\bar{\Delta}$.

Proof. The first assertion follows immediately from the definition of angles since triangles in X are CAT(κ). The equality in (ii) is an easy consequence of the last assertion in Lemma 3.5. □

3.4. **Barycenters.** Let X be a metric space and Z be a subset of X. For $x \in X$ let $\operatorname{rad}(x, Z)$ be the infimum over all $r \geq 0$ with $Z \subset B_r(x)$. We call

$$\operatorname{rad} Z := \inf\{\operatorname{rad}(x, Z) \mid x \in X\} \tag{3.17}$$

the *circumradius* of Z and a point $x \in X$ with $Z \subset \bar{B}_{\operatorname{rad} Z}(x)$ a *circumcenter* of Z. Circumcenters need not exist nor need they be unique.

3.18. PROPOSITION. *Let X be a complete CAT(κ) space and $Z \subset X$. If* $\operatorname{rad} Z < D_\kappa/2$*, then Z has a unique circumcenter.*

Proof. We can assume that $r = \operatorname{rad} Z > 0$. Now $r < D_\kappa/2$. Hence for any $\varepsilon > 0$ with $r + \varepsilon < D_\kappa/2$ there is $\delta > 0$ such that, for any three points $x, y, z \in X$ with $d(x, y) \geq \varepsilon$ and $d(x, z), d(y, z) < r + \varepsilon$, we have

$$d(x, m) < \max\{d(x, z), d(y, z)\} - \delta,$$

where m is the midpoint between x and y. Now uniqueness of the circumcenter is immediate. It also follows that a sequence (x_n) in X with $\operatorname{rad}(x_n, Z) \to \operatorname{rad}(Z)$ is Cauchy. Now X is complete, hence we get existence of the circumcenter. □

Let X be a metric space and μ be a measure on X such that $g(x) = \int d^2(x, y)\mu(dy)$ is finite for one (and hence any) $x \in X$. We say that $x \in X$ is a *center of gravity* or a *barycenter* of μ if $g(x) = \inf g$. The same arguments as above give the following result.

3.19. PROPOSITION. *Let X be a complete CAT(κ) space and $Z \subset X$. If* $\operatorname{rad} Z < D_\kappa/2$*, then Z has a unique barycenter.* □

As an example, we discuss short loops on the unit sphere.

3.20. PROPOSITION. *Let $\sigma : I \to S^2$ be a closed curve of length $< 2\pi$. Then the image of σ has circumradius $\leq L(\sigma)/4 < \pi/2$ with equality iff it is a great circle arc of length $L(\sigma)/2$.*

Proof. We assume that σ is piecewise smooth and parameterized by arc length. We will show that σ is contained in an open hemisphere. The rest of the proof is left as an exercise.

For a unit tangent vector v of S^2, let $\nu(v)$ be the number of times the parameterized great circle $\gamma : [0, 2\pi] \to S^2$ with $\gamma'(0) = v$ intersects σ. For almost all v, $\nu(v)$ is finite and, since σ is closed, either 0 or ≥ 2.

Let S be the set of unit tangent vectors v of S^2 with $\nu(S) \geq 2$. Then, by Santalo's formula,

$$\begin{aligned} L(\sigma) &= \frac{1}{8\pi} \int_{US^2} \nu(v) \mu_L(dv) \\ &= \frac{1}{8\pi} \int_S \nu(v) \mu_L(dv) \geq \frac{1}{4\pi} \mu_L(S), \end{aligned}$$

where US^2 denotes the unit tangent bundle of S^2 and μ_L the Liouville measure on US^2. Now $\mu_L(US^2) = 8\pi^2$ and $L(\sigma) < 2\pi$. It follows that $\mu_L(S) < \mu_L(US^2)$, and hence there are unit tangent vectors v such that γ_v does not intersect σ. It follows that σ is contained in an open hemisphere. □

3.5. **Filling.** Let X be a metric space. A *polygon* or, more precisely, a *k-gon* in X is a closed curve $\sigma : [a, b] \to X$ together with a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ such that each leg $\sigma|[t_{i-1}, t_i]$ is a geodesic, where we count indices modulo k. Somewhat sloppily, we also use a sequences $(x_1, x_2, \dots, x_k)$ of points in X to denote a k-gon consisting of consecutive (chosen) minimal geodesics from x_{i-1} to x_i, $1 \leq i \leq k + 1 = 1$, where we do not specify any parameterization.

Let $\kappa \in \mathbb{R}$. Assume that for all points $x, y \in X$ with $d(x, y) < D_\kappa$ there is a minimal geodesic from x to y. Then to require that X is CAT(κ) is equivalent to asking that for any trigon (x_1, x_2, x_3) of length $< 2D_\kappa$ there are a triangular region T in M^2_κ, possibly degenerate, with boundary a trigon $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ such that $d(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ and a contraction[3] $f : T \to X$ with $f(\bar{x}_i) = x_i$.

3.21. THEOREM. *Let X be CAT(κ). Let $(x_1, \dots, x_k)$, $k \geq 3$, be a polygon in X of length $< 2D_\kappa$. Then there is a convex region P in M^2_κ, possibly degenerate, bounded by a polygon $(\bar{x}_1, \dots, \bar{x}_k)$ with $d(\bar{x}_{i-1}, \bar{x}_i) = d(x_{i-1}, x_i)$, where we count indices modulo k, and a contraction $f : P \to X$ with $f(\bar{x}_i) = x_i$.*

Proof. Let (x_1, x_2, x_3) be a trigon in X as in the assertion. Choose a trigon $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M^2_κ with $d(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$. Note that $d(x_i, x_j) < D_\kappa$ by the assumption on the length of the trigon. Let $\rho : [0, 1] \to X$ and $\sigma_s : [0, 1] \to X$, $0 \leq s \leq 1$, be the minimal geodesics from x_2 to x_3 and x_1 to $\rho(s)$, respectively. Similarly, let $\bar{\rho} : [0, 1] \to M^2_\kappa$ and $\bar{\sigma}_s : [0, 1] \to M^2_\kappa$, $0 \leq s \leq 1$, be the minimal geodesics from $\bar{x}_2$ to $\bar{x}_3$ and $\bar{x}_1$ to $\bar{\rho}(s)$, respectively. Define a map f on the triangular region T spanned by $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ by $f(\bar{\sigma}_s(t)) = \sigma_s(t)$. For all $r < s$ in $[0, 1]$,

[3] That is, f has Lipschitz constant 1.

the triangle in X spanned by $(x_1, \rho(r), \rho(s))$ has perimeter $< 2D_\kappa$ and hence is CAT(κ). It follows that f is a contraction.

Let $k \geq 4$, and suppose that the assertion holds for $(k-1)$-gons of length $< 2D_\kappa$. Let $(x_1, \ldots, x_k)$, be a k-gon of length $< 2D_\kappa$. Note that this implies that $d(x_i, x_j) < D_\kappa$ for all i, j. We apply the inductive assumption to the $(k-1)$-gon $(x_1, \ldots, x_{k-1})$. There is a convex region Q in M^2_κ, bounded by a polygon $(\bar{x}_1, \ldots, \bar{x}_{k-1})$ with $d(\bar{x}_{i-1}, \bar{x}_i) = d(x_{i-1}, x_i)$, where we count indices modulo $k-1$, and a contraction $g : Q \to X$ with $g(\bar{x}_i) = x_i$. If

$$d(x_{k-1}, x_1) = d(x_{k-1}, x_k) + d(x_k, x_1),$$

then we can choose $P = Q$ and $f = g$, except that we introduce a further vertex $\bar{x}_k$ on the leg from $\bar{x}_{k-1}$ to $\bar{x}_1$ of the boundary polygon.

By Proposition 3.20, $(x_1, \ldots, x_{k-1})$ is contained in an open hemisphere if $\kappa > 0$. We assume now that

$$d(x_{k-1}, x_1) < d(x_{k-1}, x_k) + d(x_k, x_1),$$

and let $\bar{x}_k$ be the point outside Q with $d(\bar{x}_k, \bar{x}_1) = d(x_k, x_1)$ and $d(\bar{x}_k, \bar{x}_{k-1}) = d(x_k, x_{k-1})$. We let T be the triangular region spanned by $(\bar{x}_1, \bar{x}_{k-1}, \bar{x}_k)$ and set $R = Q \cup T$.

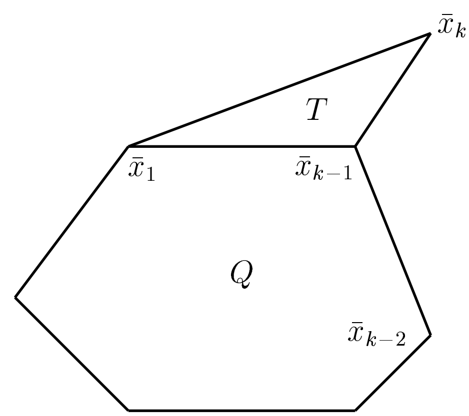

Figure 2

We choose a contraction $h : T \to X$ with $h(\bar{x}_i) = x_i$, $i = 1, k-1, k$, as in the case of trigons considered above. Since g and h are isometric along the legs of the boundaries of Q and T, respectively, it follows that $g = h$ along the leg from $\bar{x}_{k-1}$ to $\bar{x}_1$. Hence we obtain a well defined map $f : R \to X$ with $f(\bar{x}_i) = x_i$. If the interior angles of R at $\bar{x}_1$ and $\bar{x}_{k-1}$ are $\leq \pi$, then R is convex. In this case we choose $P = R$, and then $f : P \to X$ is a contraction as asserted.

We assume now that at least one of the interior angles of R at $\bar{x}_1$ or $\bar{x}_{k-1}$ is $> \pi$. Consider the length metric d_R on R induced from the metric on M^2_κ. It is easy to see that R is CAT(κ) with respect to d_R, see Subsection 4.1 below. If the interior angle at $\bar{x}_{k-1}$ is $> \pi$, then the segment from $\bar{x}_{k-2}$ to $\bar{x}_{k-1}$ concatenated with the one from $\bar{x}_{k-1}$ to $\bar{x}_k$ is a geodesic in R with respect to d_R. The analogous statement holds if the interior angle at $\bar{x}_1$ is $> \pi$. Hence with respect to d_R, $(\bar{x}_1, \dots, \bar{x}_{k-2}, \bar{x}_k)$ or $(\bar{x}_2, \dots, \bar{x}_k)$ is a geodesic $(k-1)$-gon in R to which we can then apply the inductive assumption. That is, there is a convex region P with appropriate points on the boundary and a contraction $h : P \to R$ as asserted. Now $f : R \to X$ is a contraction with respect to d_R, hence P together with $f \circ h$ satisfies the required properties. □

3.22. Theorem (Reshetnyak [Re2]). *Let X be CAT(κ) and $c : [0, L] \to X$ be a unit speed curve of length $L < 2D_\kappa$. Then there is a convex region P in M^2_κ, possibly degenerate, bounded by (the image of) a unit speed curve $\bar{c} : [0, L] \to X$, and a contraction $f : P \to X$ with $f \circ \bar{c} = c$.*

Proof. Approximate c by polygons. □

3.6. **Notes.** The acronym CAT was introduced by Gromov [Gr2]. According to [BH] the letters refer to E. Cartan, A.D. Alexandrov and Toponogov. Our formulation in Definition 3.8 was influenced by [KL]. The books [BH] and [Ri] contain more detailed introductions into spaces with upper curvature bounds. B. Kleiner [Kl] discusses regularity properties spaces with upper curvature bounds. The articles [LS1] and [LS2] contain nice and elementary results concerning such spaces.

Spaces with lower curvature bounds can be defined in a similar way. However, the world of such spaces is very different from what we encounter here, see [BBI], [BGP]. If a locally compact metric space X has upper and lower curvature bounds and each geodesic segment of X is contained in a complete geodesic, that is, a geodesic defined on the whole real line, then it is a manifold with a Riemannian metric of smoothness at least $C^{1+\alpha}$, see [ABN].

There are other notions of curvature bounds. See [Ri] for a comparison of some possibilities.

4. Constructions

In this section we discuss some elementary constructions of spaces with upper curvature bounds. The aim is to get new examples out of existing ones.

4.1. **Glueing.** Let X_1 and X_2 be CAT(κ). Let $Z_1 \subset X_1$ and $Z_2 \subset X_2$ be D_κ-convex subsets, that is, for any two points in Z_1 of distance $< D_\kappa$, the unique geodesic in X_1 connecting them is contained in Z_1, and similar for Z_2. Assume also that Z_1 and Z_2 are compact and isometric and fix an isometry $i : Z_1 \to Z_2$. Let

$$X := (X_1 \dot{\cup} X_2)/\sim, \tag{4.1}$$

where $\sim$ identifies $z \in Z_1$ with $i(z) \in Z_2$. In other words, we use i to glue X_1 to X_2. As metric we use the given metrics d_1 on X_1 and d_2 on X_2 for the corresponding subsets of X and, for $x_1 \in X_1$ and $x_2 \in X_2$,

$$d(x_1, x_2) = \min\{d_1(x_1, z) + d_2(i(z), x_2)\}, \tag{4.2}$$

where the minimum is over all $z \in Z_1$. Then X with metric d is CAT(κ). Similarly, if X_1 and X_2 have curvature $\le \kappa$ and $Z_1 \subset X_1$ and $Z_2 \subset X_2$ are compact and locally convex, then X with metric d as above also has curvature $\le \kappa$. The proof is left as an exercise.

4.2. **Cones.** Let X be a space with metric d. We define a new metric $\angle$ on X by

$$\angle(x, y) := \min\{d(x, y), \pi\}. \tag{4.3}$$

The diameter of X with metric $\angle$ is $\le \pi$. We think of $\angle$ as measuring an angle. We define the κ-*cone* over X as space

$$C_\kappa X = X \times [0, D_\kappa/2)/X \times \{0\} \tag{4.4}$$

with metric d_κ

$$d_\kappa((x, r), (y, s)) := c_\kappa(r, s, \angle(x, y)), \tag{4.5}$$

where c_κ is as in (3.1). We call the distinguished point $o := X \times \{0\}$ the *apex* or *origin* of $C_\kappa X$. For $\kappa = 1$, 0, or -1, we also speak of the *spherical, Euclidean*, or *hyperbolic cone* over X, respectively.

Note that d_κ mimics the distance in a ball B^2_κ of radius $D_\kappa/2$ in M^2_κ with respect to polar coordinates about its center, where $\angle$ stands for the corresponding distance in the unit circle S^1. In other words, B^2_κ is isometric to the κ-cone over S^1. More generally, the ball B^n_κ of radius $D_\kappa/2$ in M^n_κ is isometric to the κ-cone over S^{n-1}.

Let $\sigma = (\tau, \rho) : [a, b] \to C_\kappa X$ be a curve from (x, r) to (y, s). Then σ is rectifiable iff ρ and τ are rectifiable, and then the speeds satisfy

$$v_\sigma^2 = \operatorname{sn}_\kappa(\rho)^2 v_\tau^2 + v_\rho^2, \tag{4.6}$$

where sn_κ solves the differential equation $\operatorname{sn}''_\kappa + \kappa \operatorname{sn}_\kappa = 0$ with initial condition $\operatorname{sn}_\kappa(0) = 0$, $\operatorname{sn}'_\kappa(0) = 1$. This is the same formula as in the case of polar coordinates in B^2_κ.

Assume now that $\sigma : [a,b] \to C_\kappa X$ is rectifiable. A *comparison curve* for σ is a curve $\bar\sigma = (\bar\tau, \rho)$ in B^2_κ, where $\bar\tau : [a,b] \to S^1$ is monotone with speed v_τ. By definition, σ and $\bar\sigma$ have the same speed and, hence, the same length. In particular, if σ is a geodesic, then $\bar\sigma$ is a line segment in B^2_κ and, up to parameterization, τ is a geodesic in X. Moreover, the triangle spanned by σ and the geodesic segments from o to the endpoints of σ is κ*-flat*, that is, its convex hull is isometric to the region spanned by a comparison triangle in B^2_κ. Furthermore, if $r, s > 0$, then a geodesic from (x,r) to (y,s) passes through o iff $\angle(x,y) = \pi$.

4.7. PROPOSITION. *$C_\kappa X$ is CAT(κ) iff X is CAT(1).*

Proof. We only show that $C_\kappa X$ is CAT(κ) if X is CAT(1). The other direction will be clear from the arguments in the main Case 2a below. It suffices to consider triangles with minimal sides. Let $\Delta = (a,b,c)$ be such a triangle. We distinguish four cases.
Case 1a: The origin is a vertex of Δ, say $o = a \cap b$. Then either $o \in c$, and then $c = a \cup b$ and Δ is degenerate and hence CAT(κ). Or else we are in the case discussed above, and then Δ is κ-flat, hence CAT(κ).
Case 1b: The origin is on an edge, but is not a vertex. In this case we subdivide Δ into two triangles by introducing o as a further edge. By applying Case 1a and Lemma 3.5 we conclude that Δ is CAT(κ).
Case 2a: The origin is not on Δ. Let α, β, and γ be the geodesics in X associated to a, b, and c, respectively. That is, if we parameterize a on $[0,1]$, say, then $a(t) = (\alpha(t), \rho_a(t))$, and similarly for b and c. We first consider the main case of the proof, namely that the perimeter of the triangle $\Delta_X = (\alpha, \beta, \gamma)$ in X is $< 2\pi$. Since o is not on any edge of Δ, the lengths of the edges of Δ_X are $< \pi$. Since X is CAT(1), it follows that the edges of Δ_X are minimal. Therefore Δ_X satisfies (3.2) and (3.3), and hence Δ_X has a comparison triangle $\bar\Delta_X = (\bar\alpha, \bar\beta, \bar\gamma)$ in the unit sphere S^2. With respect to parameterizations $a = (\alpha, \rho_a)$, $b = (\beta, \rho_b)$, and $c = (\gamma, \rho_c)$ as above, we obtain a comparison triangle

$$\bar\Delta = (\bar a, \bar b, \bar c) = ((\bar\alpha, \rho_a), (\bar\beta, \rho_b), (\bar\gamma, \rho_c))$$

of Δ in B^3_κ. Now Δ_X is CAT(1). Hence by the definition of distances in $C_\kappa X$ it follows that Δ is CAT(κ).
Case 2b: The origin is not on an edge, but the associated triangle (α, β, γ) has perimeter $\geq 2\pi$. Subdividing Δ into two triangles satisfying the assumptions in Case 1 or Case 2a, we can apply Lemma 3.5 to conclude that Δ is CAT(κ). □

4.3. **Tangent Cones.** The tangent cone is a replacement of the tangent space of smooth manifolds. Tangent vectors correspond to equivalence classes of geodesic segments.

Let X be a metric space with curvature $K_X \leq \kappa$ and x be a point in X. Say that two geodesic segments $\sigma_1 : [0, b_1) \to X$ and $\sigma_2 : [0, b_2) \to X$ starting at x are equivalent if

$$\lim_{t \to 0} \frac{1}{t} d(\sigma_1(t), \sigma_2(t)) = 0. \tag{4.8}$$

The equivalence class of a geodesic segment $\sigma : [0, b) \to X$ starting at x is denoted $\sigma'(0)$ and called the *initial velocity* of σ. Any two geodesics with the same initial velocity have the same speed.

Let $C'_x X$ be the set of all initial velocities of geodesic segments starting at x. There is a natural metric d_x on $C'_x X$,

$$d_x(\sigma_1'(0), \sigma_2'(0)) := \lim_{t \to 0} \frac{1}{t} d(\sigma_1(t), \sigma_2(t)). \tag{4.9}$$

The completion of $C'_x X$ with respect to d_x is called the *tangent cone* of X at x, denoted $C_x X$.

Let $D'_x X \subset C'_x X$ be the set of equivalence classes of unit speed geodesics starting at x. It is immediate from (3.14) that two unit speed geodesics σ_1 and σ_2 starting at x are equivalent iff $\angle(\sigma_1, \sigma_2) = 0$. Hence

$$\angle_x(\sigma_1'(0), \sigma_2'(0)) := \angle(\sigma_1, \sigma_2) \tag{4.10}$$

is a metric on $D'_x X$, the *angle metric*. The completion of $D'_x X$ with respect to $\angle_x$ is called the *space of directions* at x, denoted $D_x X$.

4.11. Proposition. *Let X be a metric space with curvature $K_X \leq \kappa$ and x be a point in X. Then $C_x X$ is $CAT(0)$, $D_x X$ is $CAT(1)$, and $C_x X = C_0 D_x X$, the Euclidean cone over $D_x X$.*

Proof. The main point of the proof is to show that geodesics in $C_x X$ are limits of geodesics in X (in a sense that will become clear). For $t > 0$, let X_t be the space X endowed with metric $d_t := d/t$. Note that X_t has curvature $\leq t^2|\kappa|$ and that $t^2|\kappa| \to 0$ as $t \to 0$. Note also that angle measurement does not depend on the scale.

Let $x \in X$. Let $\sigma_0, \sigma_1 : [0, \delta) \to X$ be unit speed geodesics starting at x with $\angle(\sigma_0, \sigma_1) := \gamma \in (0, \pi)$. Let $a, b > 0$. In what follows, we only consider sufficiently small $t \in (0, 1)$ such that $(a + b)t < \min\{\delta, D_\kappa/2\}$ and such that the ball of radius $(a + b)t$ about x is CAT(κ). Consider the geodesics $\sigma_{0,t}, \sigma_{1,t} : [0, 1] \to X$,

$$\sigma_{0,t}(s) := \sigma_0(sta), \quad \sigma_{1,t}(s) := \sigma_1(stb),$$

from x to $B_t := \sigma_0(ta)$ and $A_t := \sigma_1(tb)$, respectively. Their lengths in X_t are a and b, respectively.

In the Euclidean triangle $\Delta = (A, B, C)$ determined by sides a, b which meet at an angle γ, let c be the length of the side opposite γ.

Let $\sigma_t : [0,1] \to X$ be the minimal geodesic from A_t to B_t. Then by (3.14) and the cosine formula of Euclidean geometry,

$$\begin{aligned}\cos\gamma &= \lim_{t\to 0} \frac{t^2a^2 + t^2b^2 - d^2(\sigma_0(ta), \sigma_1(tb))}{2tatb} \\ &= \frac{1}{2ab}\left(a^2 + b^2 - \frac{d^2(\sigma_0(ta), \sigma_1(tb))}{t^2}\right) \\ &= \frac{1}{2ab}\left(a^2 + b^2 - c^2\right)\end{aligned}$$

and hence

$$\lim_{t\to 0} L_t(\sigma_t) = \lim_{t\to 0} \frac{d(\sigma_0(ta), \sigma_1(tb))}{t} = c,$$

where $L_t := L/t$ is the length functional in X_t.

Let α and β be the angles of Δ at A and B and α_t, β_t, and γ_t be the angles of $\Delta_t = (A_t, B_t, x)$ at A_t, B_t and $x =: C_t$, respectively. By definition, $\gamma_t \to \gamma$ as $t \to 0$. Our next aim is to show that $\alpha_t \to \alpha$ and $\beta_t \to \beta$ as $t \to 0$. Since X has curvature $\le t^2|\kappa|$ and the side lengths of Δ_t tend to the side lengths of Δ as $t \to 0$, we have

$$\limsup_{t\to 0} \alpha_t \le \alpha \quad \text{and} \quad \limsup_{t\to 0} \beta_t \le \beta.$$

To show $\liminf_{t\to 0} \alpha_t \ge \alpha$, fix $b' > b$. Let $\Delta' = (A', B, C)$ be the Euclidean triangle determined by sides a, b' which meet at an angle γ. Let α' be the angle of Δ' at A'. Consider A on the side a' of Δ' at distance b from C and let $\alpha'' = \angle_A(A', B)$. Then $\alpha + \alpha'' = \pi$. On the other hand, Let $A'_t := \sigma_1(tb')$ and $\Delta'_t = (A'_t, B_t, x)$. Let $\alpha''_t = \angle_{A_t}(A'_t, B_t)$. By what we showed above, the side lengths of the triangle (A_t, A'_t, B_t) tend to the side lengths of (A, A', B). Hence $\limsup_{t\to 0} \alpha''_t \le \alpha''$. Now $\alpha_t + \alpha''_t \ge \pi$, hence $\liminf_{t\to 0} \alpha_t \ge \alpha$. By the analogous argument, $\liminf_{t\to 0} \beta_t \ge \beta$. Hence

$$\lim_{t\to 0} \alpha_t = \alpha \quad \text{and} \quad \lim_{t\to 0} \beta_t = \beta.$$

It then also follows that for any two points y and z on Δ and corresponding points y_t and z_t on Δ_t we have $\lim_{t\to 0} d_t(y_t, z_t) = d(y, z)$.

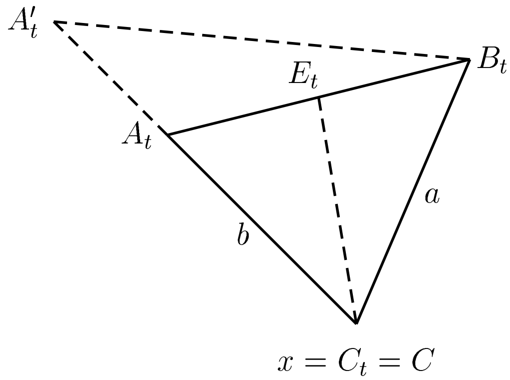

Figure 3

Now fix $r \in (0,1)$. Let $E_t = \sigma_t(r)$ and E be the corresponding point on the side c of Δ (with $d(A,E) = r\, d(A,B)$). Then

$$\lim_{t \to 0} d_t(x, E_t) = d(C,E) =: e$$

by what we proved above. Let $\rho_t : [0, et] \to X$ be the minimal geodesic from x to E_t. By what we just said, $L_t(\sigma_t)$ is approximately equal to e. Let $\varepsilon > 0$. Then

$$c \leq d_t(A_t, B_t) = L_t(\sigma_t) \leq c + \varepsilon$$

for all t sufficiently small. Let $s < t$ and $F = \rho_t(es)$. Then since the curvature of X_t is $\leq t^2|\kappa|$, hence almost ≤ 0, we get that

$$d_t(A_s, F) \leq \frac{s}{t}(d(A_t, E_t) + \varepsilon) \quad \text{and} \quad d_t(B_s, F) \leq \frac{s}{t}(d(B_t, E_t) + \varepsilon)$$

for all s and t sufficiently small. But then

$$d_s(A_s, F) \leq d(A, E) + \varepsilon \quad \text{and} \quad d_s(B_s, F) \leq d(B, E) + \varepsilon,$$

and, therefore, the triangle (A_s, B_s, F) in X_s is almost degenerate. But the curvature of X_s is almost 0, hence $d_s(F, E_s)$ tends to 0 for $s, t \to 0$. It follows that the sequence of equivalence classes of the geodesics σ_t in $T_x X$ is Cauchy with respect to the metric d_x. By construction, the limit is a point E_∞ in $T_x X$ with

$$d_s(A_\infty, E_\infty) = d(A, E) \quad \text{and} \quad d_s(B_\infty, E_\infty) = d(B, E),$$

where A_∞ and B_∞ denote the equivalence classes of σ_1 and σ_0 in $T_x X$, respectively. The rest of the proof that $T_x X$ is CAT(0) is now similar to the proof of Proposition 4.11. Note that it suffices to show that triangles with end points in $C_x' X$ are CAT(0).

For $r \geq 0$ and a direction $\sigma'(0) \in C_xX$, let $r\sigma'(0)$ be the direction determined by the geodesic segment $\sigma(rt)$, $0 \leq t < \varepsilon$. Clearly any direction $v \in C_xX$ can be represented as rv with $v \in D_xX$. By (3.14),

$$d_x^2(rv, sw) = r^2 + s^2 - 2rs\cos\angle(v,w),$$

hence C_xX is the Euclidean cone over D_xX. By Proposition 4.11, D_xX is CAT(1) with respect to the angle metric. □

4.4. **Products and Spherical Joins.** Let X_1 and X_2 be metric spaces with metric d_1 and d_2, respectively. We equip the product $X = X_1 \times X_2$ with the Euclidean product metric d given by

$$d^2((x_1,x_2),(y_1,y_2)) := d_1^2(x_1,y_1) + d_2^2(x_2,y_2). \tag{4.12}$$

Then a curve $\sigma = (\sigma_1, \sigma_2)$ in X is a geodesic iff σ_1 is a geodesic in X_1 and σ_2 is a geodesic in X_2. It follows that X is CAT(0) if X_1 and X_2 are CAT(0). More generally, if $\kappa \geq 0$, then X is CAT(κ) if X_1 and X_2 are CAT(κ).

We define the *spherical join* $X_1 * X_2$ of X_1 and X_2 to be the space of directions of $C_0X_1 \times C_0X_2$ at the distinguished point (o_1, o_2),

$$X_1 * X_2 := D_{(o_1,o_2)}(C_0X_1 \times C_0X_2). \tag{4.13}$$

If X_1 and X_2 are CAT(1), then C_0X_1 and C_0X_2 are CAT(0). Then $C_0X_1 \times C_0X_2$ is CAT(0) as well, and hence $X_1 * X_2$ is CAT(1). If Y_1 and Y_2 are spaces with upper curvature bounds and $y_1 \in Y$ and $y_2 \in Y_2$ are points with $D_{y_1}Y_1 = X_1$ and $D_{y_2}Y_2 = X_2$, then

$$D_{(y_1,y_2)}(Y_1 \times Y_2) = X_1 * X_2.$$

It is consistent with the definition that we have (or set)

$$X * \emptyset = \emptyset * X = X.$$

We view $X_1 = D_{(o_1,o_2)}(C_0X_1 \times \{o_2\})$ and $X_2 = D_{(o_1,o_2)}(\{o_1\} \times C_0X_2)$ as subspaces of $X_1 * X_2$. Under this identification, each point of X_1 is connected to each point of X_2, in $X_1 * X_2$, by a geodesic of length $\pi/2$. The nicest examples of spherical joins are unit spheres,

$$S^m * S^n = S^{m+n+1}. \tag{4.14}$$

The spherical join of S^n with a single point is a closed unit hemisphere of dimension $n+1$. More generally, the spherical join of S^n with a discrete set is a family of hemispheres, indexed by the discrete set, and glued along the common equator S^n.

4.5. **Simplicial Complexes.** An M^n_κ*-simplex* is the intersection of $n+1$ closed half spaces in M^n_κ in general position. For $\kappa = 1$, 0, or -1, we also call such a simplex a *spherical, Euclidean*, or *hyperbolic simplex*, respectively.

Let S be an M^n_κ-simplex. Then $\operatorname{diam} S < D_\kappa$ and S is convex, that is, for any two points $x, y \in S$, the minimal geodesic from x to y lies inside S. For any $x \in S$, the tangent cone $C_x S$ in the sense of Subsection 4.3 is naturally equal to the closed, convex subset of vectors v in $T_x M^n_\kappa$ such that the geodesic determined by v is contained in S for some initial time.

The boundary of S is the union of M^k_κ-simplices, $0 \le k < n$. If x is a point in the interior of a k-simplex T on the boundary of S, then $C_x S$ is naturally equal to the product $T_x T \times (C_x S \cap N_x T)$, where $N_x T$ is the normal space to T at x, that is, the orthogonal complement of $T_x T$ in $T_x M^n_\kappa$. The set $L_x S = D_x M^n_\kappa \cap N_x T$ is a spherical simplex. By what we said in Subsection 4.4,

$$D_x S = D_x T * L_x T.$$

This formula also holds for x in the interior of S since then $L_x S = \emptyset$ because of $N_x S = \{0\}$.

Let X be a finite dimensional simplicial complex. An M_κ*-structure* on X is a length structure on X with respect to which all the n-simplices of X are M^n_κ-simplices, for all $n \in \{0, \dots, \dim M\}$. For $\kappa = 1$, 0, or -1, we call such a structure also a *(piecewise) spherical, Euclidean*, or *hyperbolic structure*, respectively.

Let F be an M^n_κ-structure on X. Let x be a point in the interior of some simplex T in X. Then the *link* $L_x X$ of x in X is the simplicial complex consisting of the simplices $L_x S$, where S runs over the simplices S containing T. By what we said above, the M^n_κ-structure on X induces a spherical structure on the link $L_x X$. Moreover, $D_x X = D_x T * L_x X$.

Assume now that $\kappa \in \{1, 0, -1\}$ and that F is uniform in the sense of (1.13). In our context this means that there is a uniform lower and upper bound on the lengths of the edges of X. Let $x \in X$. Then, by uniformity, there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(x)$ is isometric to the ball of radius ε about the apex in the spherical, Euclidean, or hyperbolic cone over $D_x X$, respectively. Hence by Proposition 4.7 and its spherical and hyperbolic siblings, X has curvature ≤ 1, 0, or -1, respectively, iff $D_x X$ is CAT(1) for all x in X. Since for each $x \in X$, $D_x X = D_x T * L_x X$, and the link of a point in the interior of a simplex T occurs as a link in the space $D_v X$, where v is any vertex of T, we conclude that X has curvature ≤ 1, 0, or -1, respectively, iff $D_v X$ is CAT(1) for all vertices v of X.

4.6. **Notes.** In the Appendix of [BBu] there is a discussion of upper curvature bounds for piecewise smooth metrics on simplicial complexes of dimension two. The situation in higher dimension is unclear in general, see [Ko] for the case of two manifolds glued together along their boundaries. Note that the Euclidean plane minus the open unit disc is a space with nonpositive curvature. The corresponding assertion in higher dimension is clearly false. Compare also [ABB].

5. Buildings as Metric Spaces

A Coxeter complex is a connected simplicial complex associated to a group of reflections. The reader not familiar with Coxeter complexes and groups of reflections should think of the following simple examples.

5.1. Examples. (1) Let S be the boundary of the standard n-simplex. Then via restriction, the symmetries of S are in one-to-one correspondnece with the permutations of the vertices of S. The barycentric subdivision S' of S is invariant under the group of symmetries W of S, and S' is a Coxeter complex with Coxeter group W (of type A_n). Each orbit of W meets each chamber of S' exactly once. Moreover, W is generated by the reflections about the codimension one simplices of any of the chambers of S'.
(2) The Euclidean plane subdivided into triangles with angles

$$(\pi/p, \pi/q, \pi/r)$$

with $(p, q, r) = (3, 3, 3)$ or $(2, 4, 4)$ or $(2, 3, 6)$, respectively, is a Coxeter complex. The group W generated by reflections about (the lines spanned by) the edges is a Coxeter group (of type $\tilde{A}_2$, $\tilde{B}_2$, and $\tilde{G}_2$, respectively), and W is actually generated by the reflections about the edges of one triangle of the subdivision.

5.2. Definition. A *Tits building* is a simplicial complex X together with a family of subcomplexes $\mathcal{A}$, called *apartments*, such that each apartment is a Coxeter complex and such that
(1) for any two simplices S, T of X, there is an apartment containing both of them;
(2) for any two simplices S, T of X and apartments A, B containing them, there is an isomorphism $i : A \to B$ fixing S and T pointwise.

5.3. Remarks. (1) The isomorphism i in (2) is only assumed to be an isomorphism of apartments, not an isomorphism of X.

(2) Letting $S = T$ be the empty simplex in X, we get from (2) that any two apartments of X are isomorphic.

A Tits building is a connected chamber complex. A Tits building is called *thick* if each codimension one simplex is adjacent to at least three chambers.

Let X be a chamber complex of dimension n and with set $V = V_X$ of vertices. A *labeling* of X is a map $l : V \to L$, where L is a set of $n+1$ elements, called *labels*, such that for each chamber S of X the restriction of l to the vertices of S is a bijection to L. For example, the barycentric subdivision of a chamber complex is labeled by the dimension of the original simplices containing the vertices of the subdivision. More important for us: Coxeter complexes admit labelings, and their labelings are unique up to a bijection between the sets of labels.

5.4. Lemma. *Buildings admit labelings.*

Proof. Let X be a Tits building of dimension n, and let $S \subset X$ be a chamber. Choose a labeling $l_S : V_S \to L$, where L is a set with $n+1$ elements. Let v be a vertex in X. Let A be an apartment containing S and v. Since A is a Coxeter complex, there is exactly one labeling $l_A : V_A \to L$ extending the labeling L_S of S. Set $l(v) := l_A(v)$. By Property 5.2.2, $l(v)$ is well defined. By Property 5.2.1, l is a labeling of X. □

The following result presents one of the main tools in the theory of Tits buildings.

5.5. Lemma. *Let X be a Tits building, $A \subset X$ be an apartment and $S \subset A$ be a chamber. Then there is a unique map $r : X \to A$ with:*
(1) $r(x) = x$ for all $x \in A$.
(2) If B is an apartment containing S, then $r : B \to A$ is the unique isomorphism as in Axiom 5.2.2 fixing S.
Moreover, r is a label preserving chamber map and $r^{-1}(S) = S$.

Proof. Property (2) tells us how to define r. Definition 5.2 tells us that this works. □

We say that a Tits building is a *spherical*, *Euclidean*, or *hyperbolic building*, if its associated Coxeter complex is a sphere, a Euclidean space, or a hyperbolic space, respectively. Note that in these cases, the Coxeter groups act by isometries on their Coxeter complexes.

Let X be a Tits building of one of the above types. Let A be an apartment in X. Endow A with the corresponding standard Riemannian metric of constant sectional curvature. By restriction, this also endows the simplices of A with smooth Riemannian metrics of constant sectional curvature. Fix a chamber S in A and let T be any other simplex in X. Let B be an apartment containing S and T and $i : B \to A$ be the unique isomorphism fixing S. Define a smooth Riemannian metric on T by pulling back the Riemannian metric on its image in A under i. By Property 5.2.2, the induced metric on T does not depend on the choice of B. We call this piecewise smooth Riemannian metric the *standard metric* on X. Except for scaling in the Euclidean case, the standard metric is unique.

The standard metric induces a length metric on X. We will get to this metric in a more direct way. To that end choose an apartment A on X and endow A with the standard metric d_A. Let B be another apartment of X. Define a metric d_B on B by setting $d_B(x,y) := d_A(i(x),i(y))$, where $i : B \to A$ is an isomorphism. Since any two isomorphisms $B \to A$ differ by an element of the Coxeter group of A and the Coxeter group of A acts by isometries, d_B does not depend on the choice of i.

Let x, y be points in X. Choose an apartment B containing x and y and set $d(x,y) := d_B(x,y)$. If C is another apartment containing x and y, there is an isomorphism $j : B \to C$ fixing x and y. By definition, isomorphisms between apartments are isometries. Hence d is a well defined function on $X \times X$. It is not a priori clear that d satisfies the triangle inequality, whereas positivity and symmetry are obviously satisfied. The proof of the triangle inequality will be based on the following lemma.

5.6. LEMMA. *Let S be a chamber contained in an apartment A of X, and let $r : X \to A$ be the associated retraction as in Lemma 5.5. Then*

$$d(r(x), r(y)) \leq d(x,y),$$

for all $x, y \in X$. If x is a point in S, then

$$d(x, r(y)) = d(x,y).$$

Proof. Let $x \in S$. Let B be an apartment containing S and y. Then $r : B \to A$ is the unique isomorphism fixing S and y and hence an isometry with respect to d_B and d_A. Therefore

$$d(x,y) = d_B(x,y) = d_A(r(x), r(y)) = d(r(x), r(y)).$$

This proves the second assertion since $r(x) = x$.

To show the first assertion, we recall that r is a chamber map, that is, r maps chambers isomorphically onto chambers. In particular, r preserves the length of curves of the standard piecewise smooth Riemannian metric. We will use now that the metrics on the apartments are the length metrics of this Riemannian metric. Let $x, y \in X$. Choose an apartment B in X containing x and y. Let $\sigma : [0,1] \to B$ be a minimal geodesic in B from x to y. Then

$$d(r(x), r(y)) = d_A(r(x), r(y)) \le L(r \circ \sigma) = L(\sigma) = d_B(x, y). \qquad \square$$

5.7. Lemma. *The function d is the length metric associated to the standard piecewise smooth Riemannian metric above. In partcular, d is complete and for any two points of distance $< D_\kappa$, there is a minimal geodesic connecting them.*

Proof. Let x, y, z be points in X. Let A be an apartment containing x and z and $r : X \to A$ be a retraction as in Lemma 5.5. Then

$$\begin{aligned} d(x, z) &= d_A(r(x), r(y)) \\ &\le d_A(r(x), r(y)) + d_A(r(y), r(z)) \\ &= d(r(x), r(y)) + d(r(y), r(z)) \\ &\le d(x, y) + d(y, z), \end{aligned}$$

by Lemma 5.6 and the triangle inequality for d_A. It follows that d is a metric.

Now let x, y in X and $\sigma : [0,1] \to X$ be a curve from x to y. Let A be an apartment containing x and y and $r : X \to A$ be a retraction as in Lemma 5.5. Since r is a chamber map, r preserves the length of curves. Hence

$$L(\sigma) = L(r \circ \sigma) \ge d_A(x, y) = d(x, y).$$

On the other hand, the minimal geodesic in A from x to y has length $d_A(x, y) = d(x, y)$. $\square$

5.8. Theorem (Bruhat-Tits [BT]). *The standard metric d on X is $CAT(\kappa)$ with injectivity radius equal to diameter,*

$$\operatorname{inj} X = \operatorname{diam} X = D_\kappa,$$

where $\kappa = 1, 0$ and -1, respectively.

For the definition of injectivity radius, we refer to the beginning of Subsection 6.2.

Proof of Theorem 5.8. By definition, the diameter of X is equal to the diameter of its apartments, hence $\operatorname{diam} X = D_\kappa$. Note also that geodesics in apartments realize distance in X up to length D_κ.

Let x, y be points in X of distance $< D_\kappa$ and A an apartment containing them. Let $\sigma_0 : [0,1] \to A$ be the unique minimal geodesic in A from x to y. Let $\sigma_1 : [0,1] \to X$ be another geodesic of length $d(x, y)$ from x to y. Let $t \in [0,1]$ and $S \subset A$ be a chamber containing $\sigma_0(t)$.

Let $r : X \to A$ be the retraction associated to S and A as in Lemma 5.5. Now r preserves the lengths of curves, hence $r \circ \sigma_1 : [0,1] \to A$ is a constant speed curve from x to y of length $\le d(x,y)$. Hence $r \circ \sigma_1 = \sigma_0$. On the other hand, the preimage of S under r consists of S only. Hence $\sigma_1(t) = \sigma_0(t)$ and, therefore, $\sigma_1 = \sigma_0$. This does not quite show $\operatorname{inj} X = D_\kappa$ in the sense of our definition. However, together with the CAT property which we show next, the claim about $\operatorname{inj} X$ follows.

Triangles in X of perimeter $< 2D_\kappa$ have sides of length $< D_\kappa$ and hence are determined by their end points. Let $\Delta = (x,y,z)$ be a triangle in X of perimeter $< 2D_\kappa$. Let A be an apartment containing x and y. Now A is isometric to M^n_κ and hence contains a comparison triangle $\bar\Delta = (x,y,\bar z)$ of (x,y,z) (in a copy of $M^2_\kappa \subset A$). Let $S \subset A$ be a chamber containing $\bar z$. Let $r : X \to A$ be the retraction associated to S and A as in Lemma 5.5. Then, by the choice of $\bar z$,

$$d(x, r(z)) \le d(x,z) = d(x,\bar z),$$
$$d(y, r(z)) \le d(y,z) = d(y,\bar z).$$

Hence the comparison triangle $\bar\Delta$ is obtained from the triangle $(x,y,r(z))$ in $A \cong M^n_\kappa$ by increasing the distance of the third vertex to the vertices x and y. Since r does not increase distances, it follows that $\bar\Delta$ is at least as fat as Δ. Hence X is CAT(κ). □

5.1. **Notes.** The book [Bro] contains a nice introduction to Coxeter complexes and Tits buildings. It also contains more on the geometry of Tits buildings. In [CL], [Le], [Ly], there are interesting geometric characterizations of Euclidean buildings. Geodesic flows on Euclidean buildings play a role in the discussion in [BBr].

6. Local to Global

In this section, we present the proof of the Hadamard-Cartan Theorem for simply connected, complete metric spaces with non-positive Alexandrov curvature. We start with an important lemma of Alexander and Bishop which states, in effect, that geodesics segments of length $< D_\kappa$ in spaces of Alexandrov curvature $\le \kappa$ do not have conjugate points, see [AB1].

6.1. AB-Lemma. *Let X be a complete metric space with $K_X \le \kappa$ and $\sigma : [a,b] \to X$ a geodesic segment of length $< D_\kappa$.*

Then there is a $\delta > 0$ such that for all points $x \in B_\delta(\sigma(a))$ and $y \in B_\delta(\sigma(b))$ there is a unique geodesic $\sigma_{xy} : [a,b] \to X$ from x to y close to σ. Moreover, any triangle $(\sigma_{xy}, \sigma_{xz}, \sigma_{yz})$, where $x \in B_\delta(\sigma(a))$, $y, z \in B_\delta(\sigma(b))$, and σ_{yz} is minimal from y to z, is CAT(κ).

Proof. We can assume $[a,b]=[0,1]$. For $0 \le L \le L(\sigma)$ consider the following assertion $A(L)$:

Given $\varepsilon > 0$ small there is $\delta > 0$ such that for any subsegment $\bar\sigma = \overline{x_0 y_0}$ of σ of length at most L and any two points x, y with $d(x,x_0), d(y,y_0) < \delta$ there is a unique geodesic σ_{xy} from x to y whose distance from $\bar\sigma$ is less than ε. Moreover, $|L(\sigma_{xy}) - L(\bar\sigma)| \le \varepsilon$ and any triangle $(\sigma_{xy}, \sigma_{xz}, \sigma_{yz})$, where $d(x,x_0), d(y,y_0), d(z,y_0) < \delta$ and where σ_{yz} is minimizing from y to z, is $\mathrm{CAT}(\kappa)$.

Choose $r > 0$ such that $B_{3r}(z)$ is $\mathrm{CAT}(\kappa)$ for all points z on σ. Then $A(L)$ holds for $L \le r$. We show now that $A(3L/2)$ holds if $A(L)$ holds and $3L/2 \le L(\sigma) < D_\kappa$:

Choose $\alpha > 0$ with $L(\sigma) + 3\alpha < D_\kappa$. Then $L + 2\alpha < 2D_\kappa/3$ and therefore there is a constant $\lambda < 1$ such that for any two geodesics $\bar\sigma_1, \bar\sigma_2 : [0,1] \to M^2_\kappa$ with $\bar\sigma_1(0) = \bar\sigma_2(0)$ and with length at most $L + 2\alpha$ we have

$$(*) \qquad d(\bar\sigma_1(t), \bar\sigma_2(t)) \le \lambda d(\bar\sigma_1(1), \bar\sigma_2(1)),\ 0 \le t \le 1/2.$$

Now let $\bar\sigma = \overline{x_0 y_0}$ be a subsegment of σ of length $3l/2 \le 3L/2$ and let $\varepsilon > 0$ be given. Choose $\varepsilon' > 0$ with $\varepsilon' < \min\{\varepsilon/3, \alpha\}$, and let $\delta < \min\{\delta', \delta'(1-\lambda)/\lambda\}$, where δ' is the value for ε' given by $A(L)$.

Subdivide $\bar\sigma$ into thirds by points p_0 and q_0. Let x, y be points such that $d(x,x_0), d(y,y_0) < \delta$. By $A(L)$ there are unique geodesics σ_{xq_0} from x to q_0 and $\sigma_{p_0 y}$ from p_0 to y of distance at most ε' to $\overline{x_0 q_0}$ and $\overline{p_0 y_0}$, respectively. Furthermore, their lengths are in $[l-\varepsilon', l+\varepsilon']$ and the triangles $(x_0\bar{q}_0, \sigma_{xq_0}, \sigma_{xx_0})$ and $(p_0\bar{y}_0, \sigma_{p_0 y}, \sigma_{yy_0})$ are $\mathrm{CAT}(\kappa)$. Hence we can apply $(*)$ to the midpoints p_1 of σ_{xq_0} and q_1 of $\sigma_{p_0 y}$ and obtain

$$d(p_0,p_1), d(q_0,q_1) \le \lambda \max\{d(x,x_0), d(y,y_0)\} < \lambda\delta < \delta'.$$

By $A(L)$ there are unique geodesics σ_{xq_1} from x to q_1 and $\sigma_{p_1 y}$ from p_1 to y of distance at most ε' to $\overline{x_0 q_0}$ and $\overline{p_0 y_0}$, respectively. Furthermore, their lengths are in $[l-\varepsilon', l+\varepsilon']$ and the triangles $(\sigma_{xq_0}, \sigma_{xq_1}, \sigma_{q_0 q_1})$ and $(\sigma_{p_0 y}, \sigma_{p_1 y}, \sigma_{p_0 p_1})$ are $\mathrm{CAT}(\kappa)$. Hence we can apply $(*)$ to the midpoints p_2 of σ_{xq_1} and q_2 of $\sigma_{p_1 y}$ and obtain

$$d(p_1,p_2), d(q_1,q_2) \le \lambda \max\{d(q_0,q_1), d(p_0,p_1)\} < \lambda^2\delta.$$

Hence by the triangle inequality

$$d(p_0,p_2), d(q_0,q_2) < (\lambda + \lambda^2)\delta < \delta'.$$

Recursively we obtain geodesics σ_{xq_n} from x to the midpoint q_n of $\sigma_{p_{n-1}y}$ and $\sigma_{p_n y}$ from the midpoint p_n of $\sigma_{xq_{n-1}}$ to y of distance at most ε' to $\overline{x_0 q_0}$ and $\overline{p_0 y_0}$, respectively. Their lengths are in $[l-\varepsilon', l+\varepsilon']$ and the triangles $(\sigma_{xq_{n-1}}, \sigma_{xq_n}, \sigma_{q_{n-1}q_n})$ and $(\sigma_{p_{n-1}y}, \sigma_{p_n y}, \sigma_{p_{n-1}p_n})$ are CAT(κ). Furthermore, we have the estimates

$$d(p_{n-1}, p_n), d(q_{n-1}, q_n) < \lambda^n \delta$$

and

$$d(p_0, p_n), d(q_0, q_n) < (\lambda + \ldots + \lambda^n)\delta < \delta'.$$

In particular, the sequences $(p_n), (q_n)$ are Cauchy. Since X is complete, they converge. If $p = \lim p_n$ and $q = \lim q_n$, then

$$d(p_0, p), d(q_0, q) \leq \frac{\lambda}{1-\lambda}\delta < \delta'$$

and hence, by $A(L)$, the geodesics σ_{xq_n} and $\sigma_{p_n y}$ converge to σ_{xq} and σ_{py}. By construction, $p \in \sigma_{xq}$ and $q \in \sigma_{py}$. Therefore, by the uniqueness of σ_{pq}, the geodesics σ_{xq} and σ_{py} overlap in σ_{pq} and combine to give a geodesic σ_{xy} from x to y. The length of σ_{xy} is given by

$$L(\sigma_{xq}) + L(\sigma_{py}) - L(\sigma_{pq}),$$

hence $|L(\sigma_{xy}) - L(\sigma)| < \varepsilon$ by $A(L)$ and since $\varepsilon' < \varepsilon/3$. The last assertion of the lemma follows by subdividing the triangles suitably. $\square$

6.1. **Short Curves and Homotopies.** Let X be a complete metric space. For $l \in [0, \infty]$, we say that a curve $\sigma : [a, b] \to X$ is *l-short* if $L(\sigma) < l$. We say that a homotopy $H : [0,1] \times [a,b] \to X$ is *l-short*, or, for short, that H is an l-homotopy, if $L(H(s,\cdot)) < l$ for all s. Note that ∞-short is the same as rectifiability.

6.2. Definition and Exercise. Say that a curve $\sigma : [a,b] \to X$ is a *broken geodesic* if there is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $c|\,[t_{i-1}, t_i]$ is a geodesic, $1 \leq i \leq k$.

Let X be a complete metric space with $K_X \leq \kappa$. Use Birkhoff curve shortening to show that any curve $\sigma : [a,b] \to X$ is homotopic to a broken geodesic and that for any homotopy $H : [0,1] \times [a,b] \to X$ between broken geodesics $\sigma_0 = H(0,\cdot)$ and $\sigma_1 = H(1,\cdot)$, there is a homotopy $G : [0,1] \times [a,b] \to X$ with $\sigma_0 = G(0,\cdot)$, $\sigma_1 = G(1,\cdot)$, $G(s,a) = H(s,a)$, $G(s,b) = H(s,b)$, and such that $\sigma_s = G(s,\cdot)$ is a broken geodesic, $0 \leq s \leq 1$.

The above exercise shows that any curve is homotopic to a rectifiable curve. Bruce Kleiner observed that Birkhoff curve shortening applies to spaces with upper curvature bounds and that it implies existence results for geodesics. The following result is exemplary.

6.3. Lemma. *Let X be a complete metric space with $K_X \leq \kappa$. Let $\sigma : [0,1] \to X$ be a D_κ-short curve. Then there is a unique (D_κ-short) homotopy $H : [0,1] \times [0,1] \to X$ of $\sigma = H(0,\cdot)$ such that $\sigma_s = H(s,t)$,*

$0 \le t \le s$, is a geodesic from $\sigma(0)$ to $\sigma(s)$ of length $\le L(\sigma|[0,s])$ and such that $H(s,t) = \sigma(t)$ for $t \ge s$.

Proof. We can assume that σ is parameterized by arc length. Let A be the set of $r \in [0,1]$ such that there is a homotopy $H : [0,r] \times [0,1] \to X$ of σ with the asserted properties for all $s \in [0,r]$. For any such homotopy,

$$(*) \qquad L(H(s,\cdot)) \le L(\sigma) < D_\kappa.$$

Hence H is unique, by Lemma 6.1.

We have $0 \in A$, hence $A \neq \emptyset$. We show next that A is open and closed. By $(*)$ and Lemma 6.1, for all $s \ge r$ sufficiently close to r there is a unique geodesic $\sigma_s : [0,s] \to X$ from $\sigma(0)$ to $\sigma(s)$ close to $\sigma_r = H(r,\cdot)$. Moreover, since the triangles spanned by σ_{s_1}, σ_{s_2} and a shortest geodesic between $\sigma_{s_1}(s_1)$ and $\sigma_{s_2}(s_2)$ are CAT(κ), the extension

$$H(s,t) = \begin{cases} \sigma_s(t) & \text{if } t \le s, \\ \sigma(t) & \text{if } t \ge s, \end{cases}$$

for all s sufficiently close to r, is continuous. Hence A is open. Together with uniqueness, the latter argument also implies that A is closed. Hence $A = [0,1]$, and the lemma follows. □

6.4. Lemma. *Let X be a complete metric space with $K_X \le \kappa$. Let R be a topological space and $\sigma : R \times I \to X$ be a continuous map such that the curves $\sigma_r := \sigma(r,\cdot)$ are D_κ-short. For each $r \in R$, let $H_r = H_r(s,t)$ be the homotopy of σ_r as in Lemma 6.3. Then the map*

$$H : R \times I \times I \to X, \quad H(r,s,t) = H_r(s,t),$$

is continuous.

Proof. Since $H_r(s,t) = \sigma(r,t)$ for $t \ge s$ and σ is continuous, it suffices to prove continuity on the set of (r,s,t) with $t \le s$. For such a triple (r_0,s_0,t_0), we apply Lemma 6.1 to the geodesic $H_{r_0}(s_0,t)$, $0 \le t \le s_0$. There is a $\delta > 0$ such that for all points $x \in B_\delta(\sigma_{r_0}(0))$ and $y \in B_\delta(\sigma_{r_0}(s_0))$, there is a unique geodesic $\gamma_{xy} : [0,s_0] \to X$ from x to y and close to $H_{r_0}(s_0,t)$, $0 \le t \le s$. It is then clear that for $t \le s$

$$H_r(s,t) = \gamma_{\sigma(r,0),\sigma(r,s)}(st/s_0)$$

as long as r is sufficiently close to r_0. □

Lemma 6.4 says that a continuous family of D_κ-curves can be deformed to a family of geodesics, keeping end points and bounds on lengths.

6.5. LEMMA. *Let X be a complete metric space with $K_X \le \kappa$. Let x, y be points in X and $H : [0,1] \times [0,1] \to X$ be a D_κ-homotopy of curves from x to y such that $\sigma_0 = H(0, \cdot)$ and $\sigma_1 = H(1, \cdot)$ are geodesics. Then $\sigma_0 = \sigma_1$.*

Proof. ByLemma 6.4 we can assume that, for each $s \in [0,1]$, $H(s, \cdot)$ is a geodesic from x to y. By Lemma 6.1, $H(s,t)$ does not depend on s. Hence $\sigma_0 = \sigma_1$. □

6.6. COROLLARY. *A D_κ-short geodesic σ_0 is the unique curve of minimal length among all curves σ with the same end points and such that there is a D_κ-homotopy from σ_0 to σ keeping end points.* □

6.7. LEMMA. *Let X be a complete metric space with $K_X \le \kappa$. Let $x \in X$ and $H : [0,1] \times [0,1] \to X$ be a D_κ-homotopy with $H(s, 0) \equiv x$. Assume that $\sigma_0 = H(0, \cdot)$, $\sigma_1 = H(1, \cdot)$, and $\sigma_2 = H(\cdot, 1)$ are D_κ-short geodesics with*

$$L(\sigma_0) + L(\sigma_1) + L(\sigma_2) < 2D_\kappa.$$

Then the triangle $(\sigma_0, \sigma_1, \sigma_2)$ is CAT(κ).

Proof. By Lemma 6.4 we can assume that, for each $s \in [0,1]$, $H(s, \cdot)$ is a geodesic. By Lemma 6.1, the triangles $H(s_1, \cdot), H(s_2, \cdot), \sigma_2|[s_1, s_2]$ are CAT(κ) for all $s_1, s_2 \in [0,1]$ sufficiently close. Now the assertion follows from Lemma 3.5. □

6.2. **Global Comparison.** Let X be a complete length space with $K_X \le \kappa$. For $x \in X$, we let $\operatorname{inj}(x) \in (0, \infty]$ be the supremum over all radii r such that for any point $y \in B_r(x)$ there is exactly one minimal geodesic from x to y. We call $\operatorname{inj}(x)$ the *injectivity radius* of X at x and $\operatorname{inj} X = \inf\{\operatorname{inj}(x) \mid x \in X\}$ the *injectivity radius* of X.

6.8. PROPOSITION. *Let X be a complete metric space with $K_X \le \kappa$. Let $x \in X$ and assume* $\operatorname{inj}(x) \ge r$ *for some $r \in (0, D_\kappa]$. Then we have:*
(1) For each $y \in B_r(x)$, there is a unique geodesic $\sigma_y : [0,1] \to X$ from x to y in $B_r(x)$. Moreover, σ_y depends continuously on y and is a shortest curve in X from x to y.
(2) Any triangle in X of perimeter $< 2r$ with one vertex in x is contained in $B_r(x)$ and satisfies CAT(κ).

Proof. Let $y \in B_r(x)$. By the definition of $\operatorname{inj}(x)$, there is exactly one geodesic $\sigma_y : [0,1] \to X$ from x to y of length $< r$. By the triangle inequality, σ_y is contained in $B_r(x)$. To show that σ_y is minimal, let $\sigma : [0,1] \to X$ be another curve from x to y of length $< r$. By Lemma 6.3, there is a D_κ-short homotopy of σ to a geodesicσ_1 with the same end points and at most as long as σ. By the definition of $\operatorname{inj}(x)$ we have

$\sigma_1 = \sigma_y$. Hence σ_y is unique. From Lemma 6.1 we conclude that σ_y is continuous in y.

Let Δ be triangle of perimeter $< 2r$ and with one vertex in x. By the triangle inequality, Δ is contained in $B_r(x)$. Hence Δ is of the form $(\sigma_y, \sigma_z, \sigma)$. Now σ is contained in $B_r(x)$, hence there is a D_κ-short homotopy from σ_y to σ_z keeping the initial point x fixed. Hence Lemma 6.7 applies and shows that Δ is CAT(κ). □

6.9. Proposition. *Let X be a complete length space with $K_X \le \kappa$.*
(1) If X is locally compact and $x \in X$ satisfies $0 < \operatorname{inj}(x) < D_\kappa$, then there is a geodesic loop at x of length $2\operatorname{inj}(x)$.
(2) If X is compact and $0 < \operatorname{inj} X < D_\kappa$, then X contains a closed geodesic of length $2\operatorname{inj} X$.

Proof. Let (y_n) be a sequence of points with $d(x, y_n) \to \operatorname{inj}(x)$ such that there are two different minimal geodesics $\sigma_n, \rho_n : [0,1] \to X$ from x to y_n. Now X is a locally compact and complete length space. Hence the closed ball of radius $2\operatorname{inj}(x)$ is compact. It follows that, after pasing to a subsequence if necessary, $\sigma_n \to \sigma$ and $\rho_n \to \rho$, where σ and ρ are minimal geodesics from x to a point $y \in X$ with $d(x, y) = \operatorname{inj}(x)$. It is immediate from Lemma 6.1 that $\sigma \ne \rho$.

Suppose that σ concatenated with ρ^{-1} is not a geodesic loop, that is, is not a geodesic at y. Then the angle between σ^{-1} and ρ^{-1} at y is $< \pi$. This implies that there is a non-constant geodesic $c : [0, \varepsilon) \to X$ with $c(0) = y$ such that the angles between c and σ^{-1} and between c and ρ^{-1} at y are $< \pi/2$.

By Lemma 6.1, there are geodesics σ_s and ρ_s from x to $c(s)$ close to σ and ρ, respectively. It follows from (3.15) that the angles between $c|[s, \varepsilon)$ and σ_s^{-1} and between $c|[s, \varepsilon)$ and ρ_s^{-1} at $c(s)$ are $< \pi/2$ for all $s > 0$ sufficiently small. But then $L(\sigma_s) < L(\sigma)$ and $L(\rho_s) < L(\rho)$ for all $s > 0$ sufficiently small. On the other hand, $L(\sigma) = L(\rho) = \operatorname{inj}(x)$. This contradicts the uniqueness assertion in Lemma 6.8. The first assertion follows.

Let X be compact and assume that $0 < \operatorname{inj} X < D_\kappa$. By the first part, there is a sequence of geodesic loops $\sigma_n : [0, 2] \to X$ such that $L(\sigma_n) \to 2\operatorname{inj} X$. Now X is compact, hence after passing to a subsequence if necessary $\sigma_n \to \sigma$, where $\sigma : [0, 2] \to X$ is a geodesic loop in X of length $2\operatorname{inj} X$. Let $x = \sigma(0)$ and $y = \sigma(1)$. Then x and y are connected by two different geodesics of length $\operatorname{inj} X$, namely the two legs of σ between them. Hence $\operatorname{inj}(x) = \operatorname{inj}(y) = \operatorname{inj} X$. It follows from Assertion (1) that σ is a geodesic loop at x and at y, hence that σ is a closed geodesic. □

6.10. PROPOSITION. *Let X be a complete metric space with $K_X \le \kappa$. Let $x \in X$ and assume* $\mathrm{inj}(x) \ge r$ *for some* $r \in (0, D_\kappa/2]$. *Then we have:*
(1) For all $y, z \in B_r(x)$, there is a unique geodesic $\sigma_{yz} : [0,1] \to X$ from y to z in $B_r(x)$. Moreover, σ_{yz} depends continuously on y and z and is a shortest curve in $B_r(x)$ from y to z.
(2) Any triangle in $B_r(x)$ of perimeter $< 4r$ is CAT(κ).

Proof. Let $y, z \in B_r(x)$. Let σ_y and σ_z be the geodesics from x to y and x to z, respectively, as in Proposition 6.8. Let $\sigma_0 = \sigma_y^{-1}$. By Lemma 6.3 and Lemma 6.4, there is a D_κ-short homotopy from the curve $\sigma_0 * \sigma_z|[0,s]$ to a geodesic σ_s from y to $\sigma_z(s)$, $0 \le s \le 1$, such that $\sigma_s(t)$ is continuous in (s,t). Let $\sigma_{yz} := \sigma_1$. Then $L(\sigma_1) \le L(\sigma_y^{-1} * \sigma_z) < 2r$. By Lemma 6.7, the triangle

$$(\sigma_y^{-1}, \sigma_{yz}, \sigma_z) = (\sigma_0, \sigma_1, \sigma_z)$$

is CAT(κ). Now balls of radius r in M_κ^2 are convex. Hence σ_{yz} is contained in $B_r(x)$.

Let $(\sigma_0, \sigma_1, \sigma)$ be a triangle in $B_r(x)$ of perimeter $< 4r$, where σ_0 and σ_1 connect y to z_0 and z_1, respectively and σ connects z_0 and z_1. We also assume first that all three geodesics have length $< 2r$. Let $\bar\sigma_s = \sigma_y^{-1} * \sigma_{\sigma(s)}$, a curve of length $< 2r \le D_\kappa$ from y to $\sigma(t)$. By Lemma 6.3 and Lemma 6.4, there is a D_κ-short homotopy from the family $\bar\sigma_s$ to the family of geodesics γ_s in the D_κ-homotopy class of $\bar\sigma_s$. By what we said above, $\gamma_0 = \sigma_0$ and $\gamma_1 = \sigma_1$. By Lemma 6.4, $(\sigma_0, \sigma_1, \sigma)$ is CAT(κ).

It remains to show that geodesics in $B_r(x)$ are minimal in $B_r(x)$ and, therefore, of length $< 2r$. The argument is as in the proof of Lemma 3.6. □

6.11. THEOREM OF HADAMARD-CARTAN. *Let X be a complete and simply connected metric space with $K_X \le 0$. Then X is CAT(0).*

Proof. Let $\sigma : [0,1] \to X$ be a path from x to y. By Exercise 6.2, there is a homotopy of σ to a curve of length $< \infty$ from x to y. Hence by Lemma 6.3, there is a geodesic from x to y.

As for uniqueness, if σ_0 and σ_1 are geodesics from x to y, then since X is simply connected, there is a homotopy between them keeping endpoints. By Exercise Lemma 6.2 we can assume that the curves in such a homotopy have lengths $< \infty$. Hence $\sigma_0 = \sigma_1$, by Lemma 6.5. The rest is immediate from Propositions 6.8 and 6.10. □

6.3. **Notes.** Many of the idea discussed here have their origin in global Riemannian geometry. For example, many of the ideas in Subsections

6.1 and 6.2 already occur in Klingenberg's work on the sphere theorem. It is due to Gromov that their importance in the more general context presented here has been recognized.

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