COMPLETE INTEGRABILITY OF HAMILTONIAN SYSTEMS AFTER ANTON THIMM

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1. Introduction

In these lecture notes, I explain the complete integrability of geodesic flows and other Hamiltonian systems after the method developed by Anton Thimm [7, 8]. I assume that the reader is familiar with basic Riemannian geometry and is willing to delve into some symplectic geometry.

We say that the Hamiltonian system associated to a smooth function on a symplectic manfold M of dimension $2n$ is *completely integrable* if it admits n first integrals which are in involution and functionally independent on a large subset of M. Here the notion of large depends on the situation. We will settle for open subsets of full measure, although our examples give more. (For unknown concepts, search the text.)

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The notes are in a preliminary state. Comments are welcome.

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2. Symplectic manifolds and actions

A bilinear form ω on a real vector space is called a *symplectic form* if it is alternating and nondegenerate. A differential 2-form ω on a manifold M is called a *symplectic form* if ω is closed and ω_m is a symplectic form on T_mM for all $m \in M$. A manifold together with a symplectic form is called a *symplectic manifold*. The model example is $\mathbb{R}^{2n}$ together with the symplectic form

$$\omega_0 = \sum dx_i \wedge dy_i. \tag{2.1}$$

As in this example, any symplectic manifold M is of even dimension, $\dim M = 2n$ for some $n \in \mathbb{N}$, and the n-th power ω^n of its symplectic form ω is an orientation form of M. In Exercise 2.24, we discuss some of the linear algebra related to symplectic forms on finite dimensional real vector spaces.

Let M be a symplectic manifold with symplectic form ω. Then, for any point $m \in M$, there are coordinates z about m such that $\omega = z^*\omega_0$, by the fundamental Theorem of Darboux. Such coordinates will be called *symplectic coordinates*. The existence of such coordinates implies that all symplectic manifolds of the same dimension are locally equivalent. Recall that the situation is different in the case of Riemannian manifolds, where curvature is an invariant of the local structure.

2.1. **Symplectic gradient and Poisson bracket.** For a function $h \in \mathcal{F}(M)$ and vector field $X \in \mathcal{V}(M)$, h is called a *Hamiltonian potential* of X and X the *Hamiltonian vector field* associated to h or the *symplectic gradient* of h, if

$$dh = i_X\omega, \quad \text{that is,} \quad dh(Y) = (i_X\omega)(Y) = \omega(X,Y), \tag{2.2}$$

for all vector fields Y on M. We denote the symplectic gradient of h by X_h. In terms of symplectic coordinates, we have

$$X_h = \sum \left(\frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i} \right). \tag{2.3}$$

The dynamical system associated to the Hamiltonian vector field X_h is called the *Hamiltonian system* associated to h and h the *Hamilton function* of the system. With respect to symplectic coordinates, the ordinary differential equation defined by the Hamiltonian vector field X_h is given by the Hamiltonian equations associated to h as in classical mechanics.

By the nondegeneracy of ω, a function $h \in \mathcal{F}(M)$ determines a unique Hamiltonian vector field X_h. On the other hand, not any vector field X on M has a Hamiltonian potential. Furthermore, Hamiltonian potentials are only unique up to locally constant functions.

In what follows, we will use *Cartan's formula*,

$$L_X = i_X d + d i_X, \tag{2.4}$$

frequently. For example, for a Hamiltonian vector field X_h with Hamiltonian potential h, Cartan's formula gives

$$\begin{aligned}(L_{X_h}\omega)(Y,Z) &= i_{X_h}d\omega(Y,Z) + d(i_{X_h}\omega)(Y,Z)\\ &= d(i_{X_h}\omega)(Y,Z) = (ddh)(Y,Z) = 0.\end{aligned}$$

We conclude that the Lie derivative of ω in the direction of any Hamiltonian vector field vanishes,

$$L_{X_h}\omega = 0. \tag{2.5}$$

The *Poisson bracket* of functions $h_1, h_2 \in \mathcal{F}(M)$ is defined to be

$$\{h_1, h_2\} = \omega(X_{h_1}, X_{h_2}) = dh_1(X_{h_2}). \tag{2.6}$$

By definition, $\{h_1, h_2\}(m)$ only depends on $dh_1(m)$ and $dh_2(m)$. In terms of symplectic coordinates, we have

$$\{h_1, h_2\} = \sum \left(\frac{\partial h_1}{\partial x_i}\frac{\partial h_2}{\partial y_i} - \frac{\partial h_1}{\partial y_i}\frac{\partial h_2}{\partial x_i} \right). \tag{2.7}$$

The Poisson bracket turns $\mathcal{F}(M)$ into a *Poisson algebra*; that is, we have

$$\{h_1, h_2\} = -\{h_2, h_1\}, \tag{2.8}$$

$$\{h_1, h_2h_3\} = \{h_1, h_2\}h_3 + h_2\{h_1, h_3\}, \tag{2.9}$$

$$\{h_1, \{h_2, h_3\}\} = \{\{h_1, h_2\}, h_3\} + \{h_2, \{h_1, h_3\}\}, \tag{2.10}$$

for all $h_1, h_2, h_3 \in \mathcal{F}(M)$. The above identities follow easily from (2.6) and (2.7); the proof is left as Exercise 2.25. We refer to (2.9) and (2.10) as *Leibniz rule* and *Jacobi identity*, respectively. Up to locally constant functions, the Jacobi identity also follows from the next result.

Proposition 2.11. *For all $h_1, h_2 \in \mathcal{F}(M)$, we have $X_{\{h_1,h_2\}} = -[X_{h_1}, X_{h_2}]$.*

Proof. For $X_1 = X_{h_1}$ and $X_2 = X_{h_2}$, we have

$$\begin{aligned}
i_{[X_1,X_2]}\omega &= i_{L_{X_1}X_2}\omega \\
&= L_{X_1} i_{X_2}\omega - i_{X_2} L_{X_1}\omega \\
&= L_{X_1} i_{X_2}\omega \\
&= i_{X_1} d i_{X_2}\omega + d i_{X_1} i_{X_2}\omega \\
&= i_{X_1} d d h_2 + d i_{X_1} i_{X_2}\omega \\
&= d i_{X_1} i_{X_2}\omega = d(\omega(X_2, X_1)) \\
&= -d\{h_1, h_2\},
\end{aligned}$$

where we use, from line to line, $L_{X_1}X_2 = [X_1, X_2]$, the product rule for the Lie derivative, Equation 2.5, Cartan's formula 2.4, the definition of symplectic gradient, $dd = 0$, and the definition of the Poisson bracket. □

2.2. **Cotangent bundles as exact symplectic manifolds.** We consider now the case of the cotangent bundle, $\pi\colon M = T^*N \to N$, of a manifold N. The *canonical one-form* λ on T^*N is given by

$$\lambda_\alpha(v) = \alpha(\pi_* v). \tag{2.12}$$

Associated to coordinates x on an open domain U of N, we obtain *canonical coordinates* (x, a) on $T^*N|_U = T^*U$, where $\alpha = \sum a_i dx^i|_p \in T^*_pU$ corresponds to $(x(p), a)$. Then π corresponds to the projection onto x and therefore

$$\lambda_\alpha = \sum a_i dx^i|_\alpha. \tag{2.13}$$

The negative ω of the differential of λ is given by

$$\omega = -d\lambda = \sum dx^i \wedge da_i. \tag{2.14}$$

Thus ω is a symplectic form and turns T^*N into an *exact symplectic manifold*; that is, the symplectic form ω is exact. Moreover, the natural coordinates of T^*N are symplectic coordinates with respect to ω. Note however that lower and upper position of indices loose their meaning for T^*N as a symplectic manifold.

Frequently it is possible and advisable to consider the tangent instead of the cotangent bundle. Let N be endowed with a semi-Riemannian metric, as usual denoted by $\langle .,.\rangle$. Since $\langle .,.\rangle_p$ is non-degenerate at each $p \in N$,

$$\mathcal{L}\colon TN \to T^*N, \quad \mathcal{L}(v)(w) := \langle v, w\rangle, \tag{2.15}$$

is an isomorphism, called the *Legendre transform*. Under the Legendre transform, we have, for any isometry f of N,

$$\mathcal{L}(f_*v)(w) = \langle f_*v, w\rangle = \langle v, f_*^{-1}w\rangle = (\mathcal{L}(v) \circ f_*^{-1})(w), \tag{2.16}$$

and hence the action by f_*^{-1} on T^*N in the previous section corresponds to the action by f_* on TN here. The one-form λ on T^*N corresponds to

$$\lambda_{X_0}(v) = \langle X_0, \pi_* v\rangle, \quad v \in T_{X_0}TN, \tag{2.17}$$

where the projection $TN \to N$ is again denoted by π.

We will need some preparation to identify TTN conveniently; cf. Section 2.4 in [1]. Let ∇ be a connection on N. Then $TTN = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical distributions of TTN associated to ∇: If $X = X(s)$ is a curve in TM through $X(0) = X_0$ with $v = \dot{X}(0)$, then the horizontal and vertical components of v are given by

$$v = (v_{\mathcal{H}}, v_{\mathcal{V}}) = (\dot{c}(0), X'(0)), \tag{2.18}$$

where $c = \pi \circ X$ is the curve of foot points of X and X' denotes the covariant derivative of the vector field X along the curve c with respect to ∇.

Assume from now on that ∇ is a metric connection on N. To compute the differential $d\lambda$, we consider a map $X = X(s,t)$ to TN with

$$X(0,0) = X_0, \quad \frac{\partial X}{\partial s}(0,0) = u, \quad \text{and} \quad \frac{\partial X}{\partial t}(0,0) = v, \tag{2.19}$$

where $X_0 \in TN$ and $u, v \in T_{X_0}TN$ are given. At $s = t = 0$ (skipped in the notation of the ensuing computation), we obtain

$$\begin{aligned} d\lambda(u,v) &= \frac{\partial}{\partial s}\lambda(\frac{\partial X}{\partial t}) - \frac{\partial}{\partial t}\lambda(\frac{\partial X}{\partial s}) \\ &= \frac{\partial}{\partial s}\langle X, \frac{\partial c}{\partial t}\rangle - \frac{\partial}{\partial t}\langle X, \frac{\partial c}{\partial s}\rangle \\ &= \langle \frac{\nabla X}{\partial s}, \frac{\partial c}{\partial t}\rangle + \langle X, \frac{\nabla}{\partial s}\frac{\partial c}{\partial t}\rangle - \langle \frac{\nabla X}{\partial t}, \frac{\partial c}{\partial s}\rangle - \langle X, \frac{\nabla}{\partial t}\frac{\partial c}{\partial s}\rangle \\ &= \langle \frac{\nabla X}{\partial s}, \frac{\partial c}{\partial t}\rangle - \langle \frac{\partial c}{\partial s}, \frac{\nabla X}{\partial t}\rangle + \langle X, T(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t})\rangle, \end{aligned} \tag{2.20}$$

where T denotes the torsion tensor of ∇. We conclude that

$$\omega(u,v) = -d\lambda(u,v) = \langle u_{\mathcal{H}}, v_{\mathcal{V}}\rangle - \langle u_{\mathcal{V}}, v_{\mathcal{H}}\rangle - \langle X_0, T(u_{\mathcal{H}}, v_{\mathcal{H}})\rangle. \tag{2.21}$$

This formula is well known in the case of the Levi-Civita connection ∇^{LC}, where the T-term vanishes since ∇^{LC} is torsion free.

2.3. **Exercises and some definitions.**

Exercise 2.22. Prove Cartan's formula (2.4).

Exercise 2.23. Consider $\mathbb{R}^{2n}$ with standard Euclidean product $\langle .,. \rangle$, complex structure J defined by $J(x,y) = (-y,x)$, and symplectic form ω_0 as in (2.1). Show that

1) $\langle Ju, Jv \rangle = \langle u, v \rangle$ and $\omega_0(u,v) = \langle Ju, v \rangle$ for all $u, v \in \mathbb{R}^{2n}$.

2) a linear map $A \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves ω_0, that is, $A^*\omega_0 = \omega_0$ if and only if $A^tJA = J$. Express this condition also in terms of matrices.

Exercise 2.24 (Linear algebra of symplectic forms)**.** Let ω be a symplectic form on a finite dimensional real vector space V.

1) Show that V has a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij}$$

for all $1 \le i, j \le n$. Such a basis is called a *symplectic basis* of V. In particular, the dimension of V is even, $\dim V = 2n$.

2) A subspace $U \subseteq V$ is called *isotropic* if $\omega|_U = 0$. Show that an isotropic subspace U of V satisfies $\dim U \le \dim V/2$ and that V has a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that $e_1, \dots, e_m$ span U for some $1 \le m \le n$.

3) Show that V has a positive definite inner product $\langle , \rangle$ and a complex structure J such that $\langle Ju, Jv \rangle = \langle u, v \rangle$ and $\omega(u,v) = \langle Ju, v \rangle$ for all $u, v \in V$. Compare this with Exercise 2.23 and show that with respect to $\langle , \rangle$ and J, the bases in 1) and 2) can be chosen to be orthonormal such that $Je_i = f_i$.

4) A subspace $U \subseteq V$ is called *Lagrangian* if it is isotropic of maximal dimension, $\dim U = \dim V/2$. Represent the space of Lagrangian subspaces of V as the homogeneous space $\mathrm{O}(2n)/U(n)$.

Exercise 2.25. Prove the formulas for the symplectic gradient and the Poisson bracket in (2.3) and (2.7) and show that the Poisson bracket (2.6) turns $\mathcal{F}(M)$ into a Poisson algebra.

Exercise 2.26. Let M be a symplectic manifold with symplectic form ω. Let $(x,y) = (x^1, y^1, \dots, x^n, y^n)$ be smooth functions on an open subset U of M. Show that the following are equivalent:

1) For each $m \in U$ there is a neighborhood $V \subseteq U$ such that (x,y) are symplectic coordinates of M on V.

2) For all $1 \le i, j \le n$, we have $\{x_i, x_j\} = \{y_i, y_j\} = 0$ and $\{x_i, y_j\} = \delta_{ij}$.

3) For each $m \in U$ there is a neighborhood $V \subseteq U$ such that (x,y) are coordinates of M on V. Moreover, for all $1 \le i \le n$, the symplectic gradients of x_i and y_i are $-\partial/\partial y_i$ and $\partial/\partial x_i$, respectively.

3. Action-angle variables

In this section, we discuss action-angle variables after Res Jost's note [5]. Let M be a symplectic manifold with symplectic form ω and of dimension $2n$. We say that smooth functions $f_1, f_2 \colon M \to \mathbb{R}$ are *in involution* or *Poisson commute* if their Poisson bracket vanishes, $\{f_1, f_2\} = 0$. We say that functions $f_1, \ldots, f_k \in \mathcal{F}(M)$ are *functionally independent* at a point $m \in M$ if the map $f = (f_1, \ldots, f_k) \colon M \to \mathbb{R}^k$ has rank k at m.

Lemma 3.1. *Suppose that $f_1, \ldots, f_n$ are Poisson commuting smooth functions, defined in a neighborhood of a point $m \in M$, which are functionally independent at m. Then we have:*
1) (Existence) About m, there are smooth functions $g_1, \ldots, g_n$ such that

$$(f_1, \ldots, f_n, g_1, \ldots, g_n)$$

are symplectic coordinates of M, i.e., such that $\omega = \sum df_i \wedge dg_i$.
2) (Uniqueness) For any two such families $g_1, \ldots, g_n$ and $\tilde{g}_1, \ldots, \tilde{g}_n$ of functions, there is a smooth function $G = G(f_1, \ldots, f_n)$ such that, about m,

$$\tilde{g}_i - g_i = \partial G / \partial f_i.$$

Proof. Choose symplectic coordinates (x, y) about m. Consider $f_1, \ldots, f_n$ as functions of the coordinates and write $u_i = f_i(x, y)$. After exchanging the roles of appropriate x_i and $-y_i$, we can assume that the matrix of partial derivatives $F_x = (\partial f_i / \partial x_j)$ is invertible at m. Then we have $x_i = \varphi_i(u, y)$ with smooth functions φ_i. Since $u_i = f_i(x, y) = f_i(\varphi(u, y), y)$, we obtain

$$F_x \Phi + F_y = 0, \tag{3.2}$$

where $F_y = (\partial f_i / \partial y_j)$ and $\Phi = (\partial \varphi_i / \partial y_j)$. The vanishing $\{f_i, f_j\} = 0$ of the Poisson brackets translates into

$$F_x F_y^t - F_y F_x^t = 0. \tag{3.3}$$

Now (3.2) and (3.3) together imply that

$$0 = F_x \Phi F_x^t + F_y F_x^t = F_x \Phi F_x^t + F_x F_y^t.$$

By the invertibility of F_x (in a neighborhood of m), cancellation of F_x on the right hand side is justified and transposition gives

$$F_x \Phi^t + F_y = 0. \tag{3.4}$$

Again by the invertibilty of F_x, (3.2) and (3.4) imply that Φ is symmetric. Hence the one-form $\varphi_i dy_i$ is closed. Therefore it is exact, i.e., $\varphi_i = \partial S / \partial y_i$. The function $S = S(u, y)$ is a *generating function* in the sense of symplectic geometry; it generates the symplectic transformation from the given symplectic coordinates to the desired ones by

$$x_i = \partial S / \partial y_i \quad \text{and} \quad g_i = \partial S / \partial u_i.$$

In Exercise 3.10, we discuss that $(f_1, \ldots, f_n, g_1, \ldots, g_n)$ are indeed symplectic coordinates about m. This finishes the proof of 1). Now 2) is a general fact about symplectic coordinates which is discussed in Exercise 3.11. $\square$

The proof of the following version of the implicit funtion theorem is left as Exercise 3.12.

Lemma 3.5. *Let $f\colon M \to N$ be a smooth map between manifolds and suppose that $q_0 \in N$ is a regular value of f and that $L = f^{-1}(q_0)$ is compact. Then there are neighborhoods U of L in M and V of q_0 in N and a diffeomorphism $\Psi\colon V \times L \to U$ such that $f(\Psi(q,p)) = q$ for all $(q,p) \in V \times L$.* □

In what follows, let $f_1, \dots, f_n \in \mathcal{F}(M)$ and set $f = (f_1, \dots, f_n)$. For $x \in \mathbb{R}^n$, let $L(x) = \{m \in M \mid f(m) = x\}$ be the level set of f of level x.

Theorem 3.6 (Action-angle variables (a, α))**.** *Suppose that $f_1, \dots, f_n$ are in involution. Let $x_0 \in \mathbb{R}^n$ and suppose that $L(x_0)$ is compact and connected and that $f_1, \dots, f_n$ are functionally independent at each point of $L(x_0)$. Then there is a neighborhood U of $L(x_0)$ in M and a diffeomorphism*

$$(a, \alpha)\colon U \to A \times T^n,$$

where $A \subseteq \mathbb{R}^n$ is an open subset and T^n is the torus $\mathbb{R}^n/\mathbb{Z}^n$, such that

$$\omega = \sum da_i \wedge d\alpha_i$$

and such that the symplectic gradients of the f_i correspond to vector fields which are tangent to the tori $\{a\} \times T^n$ and constant along each such torus.

Proof. By Lemma 3.5, there are neighborhoods U of $L(x_0)$ and V of x_0 and a diffeomorphism $\Psi\colon V \times L(x_0) \to U$ such that $f(\Psi(x,m)) = x$ for all $(x,m) \in V \times L(x_0)$. In particular, the level sets $L_U(x)$, $x \in V$, of f in U are diffeomorphic to $L(x_0)$, and hence they are compact and connected submanifolds of M of dimension n.

Since the Poisson brackets $\{f_i, f_j\}$ vanish, f_j is constant along the flow lines of the symplectic gradients X_i of the f_i. Hence the X_i are tangent to the level sets $L_U(x)$. Furthermore, by the compactness of the $L_U(x)$, the flows Φ_i of the X_i are defined on all of $\mathbb{R}$ along U and hence define 1-parameter groups of diffeomorphisms $\Phi_i\colon \mathbb{R} \times U \to U$. Now the Lie brackets $[X_i, X_j]$ vanish, and hence the flows of the X_i commute. Therefore, they induce an action Φ of the additive group $\mathbb{R}^n$ on U,

$$\Phi\colon U \times \mathbb{R}^n \to U, \quad \Phi(m,t) = (\Phi_1^{t_1} \circ \Phi_2^{t_2} \circ \cdots \circ \Phi_n^{t_n})(m).$$

Since the rank of f is n on all of U, the $X_i(m)$ form a basis of $T_m L_U(x)$ for each $x \in V$ and $m \in L_U(x)$. Hence the orbits of Φ are open subsets of the $L_U(x)$, and hence, by the connectedness of the $L_U(x)$, they are equal to the $L_U(x)$. Hence each $L_U(x)$ is the quotient of $\mathbb{R}^n$ by a discrete subgroup $\Lambda(x) \subseteq \mathbb{R}^n$. Since each $L_U(x)$ is compact, $\Lambda(x) \cong \mathbb{Z}^n$.

Now we introduce symplectic coordinates $(f_1, \dots, f_n, g_1, \dots, g_n)$ in a neighborhood of a point $m_0 \in L(x_0)$ as in Lemma 3.1. Then $X_i = -\partial/\partial g_i$. The symplectic gradients $\partial/\partial f_i$ of the functions g_i are tangent to the Lagrangian submanifold $Q = \{g = g(m_0)\}$; in particular, Q intersects the levels $L_U(x)$ transversally. Hence, by passing to appropriate neighborhoods of m_0 in Q and x_0 in V we can arrange that $f\colon Q \to V$ is a diffeomorphism. Note that here and below, when shrinking V, we shrink U accordingly, replacing the old U by the new $U = f^{-1}(V)$.

With Φ as above, we obtain a local diffeomorphism

$$\Phi_Q\colon Q \times \mathbb{R}^n \to U, \quad \Phi_Q(q,y) = \Phi(q,-y).$$

Since $X_i = -\partial/\partial g_i$, we have $g_i(\Phi_Q(q,y)) = y_i + g_i(m_0)$ for all sufficiently small y. In particular, writing $x_i = f_i \circ \Phi_Q$, we have $\Phi_Q^*\omega = \sum dx^i \wedge dy^i$ in a neighborhood of $Q \times \{0\}$. Now the vector fields X_i are symplectic gradients, hence their flows preserve ω, and therefore Φ preserves ω. Under Φ_Q, the X_i correspond to $-\partial/\partial y_i$. Hence the translations with $t \in \mathbb{R}^n$,

$$Q \times \mathbb{R}^n \to Q \times \mathbb{R}^n, \quad (q,y) \mapsto (q, y+t),$$

preserve $\Phi_Q^*\omega$. It follows that $\Phi_Q^*\omega = \sum dx^i \wedge dy^i$ on all of $Q \times \mathbb{R}^n$.

Now Φ is an $\mathbb{R}^n$-action with compact orbits $L_U(x)$. Hence, by passing to smaller neighborhoods of m_0 in Q and x_0 in V if necessary, there is an $\varepsilon > 0$ such that Φ_Q is a diffeomorphism from $Q \times B(t,\varepsilon)$ onto its image, for all $t \in \mathbb{R}^n$, where $B(t,r)$ denotes the ball of radius r about t in $\mathbb{R}^n$. Since Φ is an $\mathbb{R}^n$-action, it follows that the open subsets $\Phi(Q \times B(t,\varepsilon/2))$ of U are evenly covered by Φ_Q. Hence Φ_Q is a covering map. Since the $L_U(x)$ are tori, the group Γ of covering transformations of Φ_Q is isomorphic to $\mathbb{Z}^n$.

We use the diffeomorphism $f\colon Q \to V$ to identify Q with V. Then

$$V \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \quad \text{and} \quad \Phi_Q^*\omega = \sum dx^i \wedge dy^i = \omega_0$$

with ω_0 as in (2.1). We fix a set $\gamma_1, \dots, \gamma_n$ of generators of the group Γ of covering transformations of Φ_Q. For $1 \le i \le n$, we write the image $\gamma_i(x,0)$ of $(x,0)$ under γ_i as $(x, e_i(x))$. Then $e_i(x) = (e_{i1}(x), \dots, e_{in}(x)) \in \mathbb{R}^n$ and

$$e_1(x), \dots, e_n(x)$$

is a set of generators of the lattice $\Lambda_U(x) \subseteq \mathbb{R}^n$ which depends smoothly on $x \in V$. Since Φ_Q is induced by an $\mathbb{R}^n$-action, γ_i acts on $V \times \mathbb{R}^n$ by

$$\gamma_i(x,y) = (x, y + e_i(x)).$$

Since γ_i acts symplectically, we get $E_i = E_i^t$, where E_i denotes the matrix $(\partial e_{ij}/\partial x_k)$ of partial derivatives of e_i; see Exercise 3.13. Hence there exist functions $S_i = S_i(x)$ such that $e_{ij} = \partial S_i/\partial x_j$. We normalize them by requiring $S_i(x_0) = 0$. By passing to a smaller V if necessary, we obtain new symplectic coordinates (a,α) on $V \times \mathbb{R}^n$ by setting

$$a_i = S_i(x) \quad \text{and} \quad y_i = \sum \frac{\partial S_j}{\partial x^i}(x)\alpha_j;$$

see Exercise 3.14. Under the covering Φ_Q, the α_i count modulo 1. By construction, we obtain a diffeomorphism $(a,\alpha)\colon U \to A \times T^n$ as desired. $\square$

We say that a submanifold $L \subseteq M$ is *Lagrangian* if T_mL is a Lagrangian subspace of T_mM for all $m \in L$; see Exercise 2.24.4. The tori $a = \text{const}$ and the submanifolds $\alpha = \text{const}$ in Theorem 3.6 are Lagrangian submanifolds. The tori $a = \text{const}$ are also called *invariant tori* since they are invariant under the Hamiltonian systems associated to the f_i and under Hamiltonian systems associated to functions which Poisson commute with the f_i; see Corollary 3.8.

With respect to action-angle variables (a,α) as in Theorem 3.6, the symplectic gradient X_h of a function $h \in \mathcal{F}(M)$ is given by

$$X_h = \sum \left(\frac{\partial h}{\partial \alpha_j}\frac{\partial}{\partial a_j} - \frac{\partial h}{\partial a_j}\frac{\partial}{\partial \alpha_j} \right). \tag{3.7}$$

Corollary 3.8. *In the situation of Theorem 3.6, if $h \in \mathcal{F}(M)$ is in involution with $f_1, \dots, f_n$, then the symplectic gradient X_h of h is tangent to the invariant tori and constant along each of them. In particular, if $h_1, h_2 \in \mathcal{F}(M)$ are in involution with $f_1, \dots, f_n$, then $\{h_1, h_2\} = 0$ on U.*

Proof. Recall that $\{h, f_i\} = dh(X_i)$, where X_i denotes the symplectic gradient of f_i. Now the f_i do not depend on α. Hence we have

$$0 = [h, f_i] = -\sum_j \frac{\partial h}{\partial \alpha_j} \frac{\partial f_i}{\partial a_j},$$

by (3.7) applied to f_i. Since the f_i are functionally independent on U, we conclude that the partial derivatives of h in the α-directions vanish. □

Corollary 3.9. *Suppose that $f_1, \dots, f_n \in \mathcal{F}(M)$ are in involution and functionally independent on an open and dense subset of M and that the level sets of $f = (f_1, \dots, f_n)$ are compact. Then the space of functions $h \in \mathcal{F}(M)$ which are in involution with $f_1, \dots, f_n$ is a commutative Poisson algebra.*

Proof. By the theorem of Sard and since f has rank n on an open and dense subset of M, the set of points p in M such that $f(p)$ is a regular value of f is dense in M. Hence the conclusion follows from Corollary 3.8. □

3.1. **Exercises.**

Exercise 3.10. Show that the functions $g_1, \dots, g_n$ in the proof of Lemma 3.1 satisfy $\{g_i, g_j\} = 0$ and $\{f_i, g_j\} = \delta_{ij}$ for all $1 \le i, j \le n$. Show that the symplectic gradients of the g_i are linearly independent at the given point m and conclude that $(f_1, g_1, \dots, f_n, g_n)$ define symplectic coordinates in a neighborhood of m.

Exercise 3.11. Suppose that (x, y) and $(\tilde{x}, \tilde{y})$ are symplectic coordinates about a point m in a symplectic manifold M such that $x = \tilde{x}$. Show that there is a smooth function $G = G(x)$ about $x(m)$ such that $\tilde{y}_i - y_i = \partial G / \partial x_i$.

Exercise 3.12. Use the implicit function theorem to prove Lemma 3.5. Hint: It may be helpful to note first that the normal bundle of L is trivial.

Exercise 3.13. Consider $\mathbb{R}^{2n}$ with symplectic form ω_0 as in (2.1). Show that a transformation of the form $(x, y) \mapsto (x, y + e(x))$ preserves ω_0 if and only if the matrix $E_x = (\partial e_j / \partial x_k)$ of partial derivatives of e is symmetric.

Exercise 3.14. Show that the functions (a, α) in the proof of Theorem 3.6 define symplectic coordinates.

4. Symplectic actions and moment maps

4.1. **Coadjoint orbits as symplectic manifolds.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the adjoint action of G on $\mathfrak{g}$ is $(g, X) \mapsto \operatorname{Ad}_g X$; the coadjoint action of G on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is given by $(g, \alpha) \mapsto \operatorname{Ad}_g^{-*} \alpha$. Here and below we use Ad_g^{-*} as a shorthand for $(\operatorname{Ad}_g^{-1})^*$.

For $X \in \mathfrak{g}$, define a vector field X^* on $\mathfrak{g}^*$ by

$$X^*(\alpha) = \frac{d}{dt}\big(\operatorname{Ad}_{\exp(tX)}^{-*} \alpha\big)\Big|_{t=0}; \tag{4.1}$$

compare with (4.8). We have

$$X^*(\alpha) = \frac{d}{dt}\big((\exp(-t \operatorname{ad}_X))^* \alpha\big)\Big|_{t=0} = -\operatorname{ad}_X^* \alpha = -\alpha \circ \operatorname{ad}_X . \tag{4.2}$$

The flow of X^* is given by the one-parameter group $(\operatorname{Ad}_{\exp(tX)}^{-*})$ of automorphisms of $\mathfrak{g}^*$. In particular, X^* is tangent to the orbits of the coadjoint action. Moreover, the space of X^* spans the tangent spaces of the orbits at each point. Note also that $[X^*, Y^*] = -[X, Y]^*$; compare with (4.9).

Let $A \subseteq \mathfrak{g}^*$ be a *coadjoint orbit*, that is, an orbit of the coadjoint action of G, and let $\alpha \in A$. Define a two-form ω_α on $T_\alpha A$ by

$$\omega_\alpha(X^*(\alpha), Y^*(\alpha)) = (\alpha, [X, Y]), \tag{4.3}$$

where we use the perfect pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ on the right hand side.

Proposition 4.4. *The differential two-form ω is well defined on the coadjoint orbits of G and turns them into symplectic manifolds.*

Proof. If $X_1^*(\alpha) = X_2^*(\alpha)$, then $\operatorname{ad}_{X_1}^* \alpha = \operatorname{ad}_{X_2}^* \alpha$ by (4.2) and hence

$$(\alpha, [X_1, Y]) = (\operatorname{ad}_{X_1}^*(\alpha), Y) = (\operatorname{ad}_{X_2}^*(\alpha), Y) = (\alpha, [X_2, Y])$$

for all $Y \in \mathfrak{g}$. It follows that ω is well defined on the coadjoint orbits.

Let A be a coadjoint orbit and $\alpha \in A$. Suppose that X_α^* lies in the null space of ω_α. Then $(\operatorname{ad}_X^*(\alpha), Y) = (\alpha, [X, Y]) = 0$ for all $Y \in \mathfrak{g}$; that is, we have $X^*(\alpha) = -\operatorname{ad}_X^*(\alpha) = 0$. Hence ω_α is nondegenerate on $T_\alpha A$.

It remains to show that ω is a closed differential form. This follows from the Jacobi identity; the proof is left as an exercise. □

For $\varphi \in \mathcal{F}(\mathfrak{g}^*)$, denote by $\operatorname{grad} \varphi$ the gradient of φ in the sense of the perfect pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$; that is, we have

$$d\varphi|_\alpha(\beta) = (\beta, \operatorname{grad} \varphi(\alpha)) \quad \text{for all } \alpha, \beta \in \mathfrak{g}^*. \tag{4.5}$$

Proposition 4.6. *For all $\varphi, \psi \in \mathcal{F}(\mathfrak{g}^*)$ and $\alpha \in \mathfrak{g}^*$, we have*

$$X_\varphi(\alpha) = X^*(\alpha), \tag{1}$$

$$\{\varphi, \psi\}(\alpha) = \omega_\alpha(X^*(\alpha), Y^*(\alpha)) = (\alpha, [X, Y]). \tag{2}$$

where $X = \operatorname{grad} \varphi(\alpha)$, $Y = \operatorname{grad} \psi(\alpha)$ and where we consider symplectic gradients and Poisson brackets separately along the coadjoint orbits of G.

Proof. For any $Z \in \mathfrak{g}$, we have

$$\begin{aligned}\omega_\alpha(X_\varphi(\alpha), Z^*(\alpha)) &= d\varphi|_\alpha(Z^*(\alpha)) = (Z^*(\alpha), X) = -(\operatorname{ad}_Z^* \alpha, X) \\ &= -(\alpha, \operatorname{ad}_Z X) = (\alpha, [X, Z]) = \omega_\alpha(X^*(\alpha), Z^*(\alpha)),\end{aligned}$$

by (4.2). This implies (1), and (2) is an immediate consequence. □

4.2. **Symplectic actions and moment maps.** Let $G \times M \to M$ be a smooth action by a Lie group G. For $g \in G$ and $m \in M$, let $l_g \colon M \to M$ and $r_m \colon G \to M$ be *left* and *right translation* by g and m, respectively,

$$l_g(m) = gm = r_m(g). \tag{4.7}$$

For any $X \in \mathfrak{g}$, define a vector field X^* on M by

$$X^*(m) = \frac{d}{dt}\big(\exp(tX)m\big)\bigg|_{t=0}; \tag{4.8}$$

compare with (4.1). The flow of X^* is given by the one-parameter group $(\ell_{\exp(tX)})$ of diffeomeorphisms of M. Note that $\mathfrak{g} \to \mathcal{V}(M)$, $X \mapsto X^*$, is an anti-morphism of Lie algebras,

$$[X^*, Y^*] = -[X, Y]^*. \tag{4.9}$$

We also have the following equivariance property:

$$l_{g*}X^* := l_{g*} \circ X^* \circ l_g^{-1} = (\operatorname{Ad}_g X)^*. \tag{4.10}$$

We say that the action of G is *symplectic* if it preserves ω, that is, that we have $l_g^*\omega = \omega$ for all $g \in G$. Then

$$L_{X^*}\omega = 0, \tag{4.11}$$

for all $X \in \mathfrak{g}$. We say that the action of G is *Hamiltonian* if it is symplectic and if any vector field X^* as above has a Hamiltonian potential, f_X. Recall that the latter means that $df_X = i_{X^*}\omega$ respectively $X_{f_X} = X^*$.

Proposition 4.12. *If the action of G is Hamiltonian, then*

$$f_{[X,Y]} = \{f_X, f_Y\} + \textit{a locally constant function}, \tag{1}$$

$$f_X \circ l_g = f_{\operatorname{Ad}_g^{-1} X} + \textit{a locally constant function}, \tag{2}$$

for all $X, Y \in \mathfrak{g}$ and $g \in G$.

Proof. Since X^* and Y^* are the symplectic gradients of f_X and f_Y, we have

$$X_{\{f_X, f_Y\}} = -[X^*, Y^*] = [X, Y]^*,$$

by Proposition 2.11 and (4.9). Hence $df_{[X,Y]} = d\{f_X, f_Y\}$, which is equivalent to (1). The proof of (2) is similar and involves (4.10). □

We now discuss the additional requirement that the locally constant functions on the right hand side of (1) and (2) in Proposition 4.12 vanish. For a further analysis of this, compare Section 26 in [4].

Definition 4.13. A *moment map* for a symplectic action $G \times M \to M$ is a map $F \colon M \to \mathfrak{g}^*$ which satisfies the following three conditions:

(1) For all $X \in \mathfrak{g}$, $f_X = (F, X)$ is a Hamiltonian potential for X^*, where $(\alpha, X) = \alpha(X)$ denotes the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.
(2) For all $X, Y \in \mathfrak{g}$, we have $f_{[X,Y]} = \{f_X, f_Y\}$.
(3) For all $g \in G$ and $X \in \mathfrak{g}$, we have $f_X \circ l_g = f_{\operatorname{Ad}_g^{-1} X}$.

Item 3 says that F is equivariant: For all $g \in G$, we have $F \circ l_g = \operatorname{Ad}_g^{-*} \circ F$.

Proposition 4.14. *If F is a moment map for the action of G on M and $h \in \mathcal{F}(M)$ is G-invariant, then F is a* first integral *of the Hamiltonian system associated to h. That is, F is constant along the flow lines of X_h.*

Proof. For any $X \in \mathfrak{g}$, we have $dh(X^*) = 0$ by the G-invariance of h. Hence

$$df_X(X_h) = \omega(X^*, X_h) = -\omega(X_h, X^*) = -dh(X^*) = 0, \tag{4.15}$$

where X_h is the Hamiltonian vector field associated to h. It follows that f_X, and therefore also F, is constant along the flow lines of X_h. □

Corollary 4.16. *Under the above assumptions, let $\varphi \in \mathcal{F}(\mathfrak{g}^*)$. Then the composition $f = \varphi \circ F$ is a first integral of the Hamiltonian system associated to h. In other words, $\{f, h\} = 0$.* □

The functions $f_X = f_X(m) = (F(m), X)$ further up are of the above kind.

Proposition 4.17. *Let F be a moment map for the action of G on M. For $j = 1, 2$, let $\varphi_j \in \mathcal{F}(\mathfrak{g}^*)$ and set $f_j := \varphi_j \circ F \in \mathcal{F}(M)$. Then we have*

$$\{f_1, f_2\} = \{\varphi_1, \varphi_2\} \circ F,$$

where we take the Poisson bracket on the right hand side orbitwise, that is, with respect to the symplectic structure on the coadjoint orbits.

Proof. Let $m \in M$ and set $\alpha = F(m)$. For $j = 1, 2$, let $X_j = \operatorname{grad} \varphi_j(\alpha)$ in the sense of (4.5). Then we have

$$\begin{aligned} df_j|_m &= d\varphi_j|_\alpha \circ dF|_m = (dF|_m, \operatorname{grad} \varphi_j(\alpha)) \\ &= (dF|_m, X_j) = df_{X_j}|_m. \end{aligned}$$

Therefore

$$\begin{aligned} \{f_1, f_2\}(m) &= \{f_{X_1}, f_{X_2}\}(m) \\ &= f_{[X_1, X_2]}(m) = (\alpha, [X_1, X_2]) = \{\varphi_1, \varphi_2\}(\alpha), \end{aligned}$$

by the second requirement of Definition 4.13 and Proposition 4.6.2. □

Corollary 4.16 together with Proposition 4.17 show that commuting functions in $\mathcal{F}(\mathfrak{g}^*)$ give rise to commuting first integrals of Hamiltonian systems associated to G-invariant Hamilton functions.

4.3. **Moment maps for exact symplectic actions.** Suppose that M is an exact symplectic manifold and let $G \times M \to M$ be a smooth action by a Lie group G. Assume that the action is *exact symplectic*, that is, that $\ell_g^* \lambda = \lambda$ for all $g \in G$. For $X \in \mathfrak{g}$, let f_X be the function on M defined by

$$f_X := i_{X^*} \lambda = \lambda(X^*). \tag{4.18}$$

Proposition 4.19. *The map $F \colon M \to \mathfrak{g}^*$, defined by $(F(m), X) := f_X(m)$, is a moment map in the sense of Definition 4.13.*

Proof. By Cartan's formula 2.4 and since $L_{X^*} \lambda = 0$, we have

$$df_X = d i_{X^*} \lambda = -i_{X^*} d\lambda = i_{X^*} \omega,$$

and hence h_X is a Hamiltonian potential of X^*. We also have

$$\begin{aligned}\{f_X, f_Y\} &= -d\lambda(X^*, Y^*) \\ &= -X^*\lambda(Y^*) + Y^*\lambda(X^*) + \lambda([X^*, Y^*]) \\ &= -(di_{Y^*}\lambda)(X^*) + (di_{X^*}\lambda)(Y^*) + \lambda([X^*, Y^*]) \\ &= (i_{Y^*}d\lambda)(X^*) - (i_{X^*}d\lambda)(Y^*) + \lambda([X^*, Y^*]) \\ &= -2d\lambda(X^*, Y^*) - \lambda([X, Y]^*) \\ &= 2\{f_X, f_Y\} - f_{[X,Y]},\end{aligned}$$

where we use Cartan's formula 2.4 and that $L_{X^*}\lambda = L_{Y^*}\lambda = 0$. Finally, by (4.10) and since the action of G on M preserves λ, we have

$$\begin{aligned}f_X \circ l_g &= \lambda(X^* \circ l_g) \\ &= \lambda(l_{g*} \circ (\operatorname{Ad}_g^{-1} X)^*) \\ &= \lambda((\operatorname{Ad}_g^{-1} X)^*) = f_{\operatorname{Ad}_g^{-1} X}.\end{aligned}$$ □

Example 4.20. Let ℓ be a smooth action of a Lie group G on a manifold N and consider the induced action on the cotangent bundle $M = T^*N$,

$$l_g(\alpha) = \alpha \circ \ell_{g*}^{-1}. \tag{4.21}$$

Now λ is invariant under the differentials of diffeomorphisms of N and hence under the action of G on T^*N. Therefore the induced action of G on T^*N is exact symplectic. Thus we get a moment map F as in Proposition 4.19.

To get variantions of the formulae for the functions f_X and their Poisson brackets, we define vector fields X_N^* on N according to the recipe for the vector fields X^* on $M = T^*N$ in (4.8): For $X \in \mathfrak{g}$ and $p \in N$, let

$$X_N^*(p) = \frac{d}{dt}\Big|_{t=0} \ell_{\exp(tX)}(p). \tag{4.22}$$

Note that $\pi_* \circ X^* = X_N^* \circ \pi$ and that (again) $[X_N^*, Y_N^*] = -[X, Y]_N^*$. For the function f_X as in (4.18) and $\alpha \in T_p^*N$, we obtain

$$f_X(\alpha) = \lambda(X^*(\alpha)) = \alpha(\pi_* X^*(\alpha)) = \alpha(X_N^*(p)). \tag{4.23}$$

For the Poisson bracket of such functions, we get

$$\{f_X, f_Y\}(\alpha) = f_{[X,Y]}(\alpha) = \alpha([X, Y]_N^*(p)). \tag{4.24}$$

4.4. **Exercises.**

Exercise 4.25. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{g}$ carries a nondegenerate bilinear form which is invariant under the adjoint action of G; i.e., $\langle \operatorname{Ad}_g X, \operatorname{Ad}_g Y\rangle = \langle X, Y\rangle$ for all $g \in G$ and $X, Y \in \mathfrak{g}$. Use the isomorphism $L\colon \mathfrak{g} \to \mathfrak{g}^*$, $L(X)(Y) = \langle X, Y\rangle$, to identify $\mathfrak{g}$ with $\mathfrak{g}^*$.

1) For $\psi \in \mathcal{F}(\mathfrak{g})$ and $X \in \mathfrak{g}$, let $\operatorname{grad}\psi(X) \in \mathfrak{g}$ be the gradient of ψ at X:

$$\langle \operatorname{grad}\psi(X), Y\rangle = d\psi|_X(Y) \quad \text{for all } Y \in \mathfrak{g}.$$

For $\varphi \in \mathcal{F}(\mathfrak{g}^*)$, show that $\operatorname{grad}(\varphi \circ L) = (\operatorname{grad}\varphi) \circ L$, where the gradient on the right is defined as in (4.5).

2) Show that L intertwines the adjoint action of G on $\mathfrak{g}$ with the coadjoint action of G on $\mathfrak{g}^*$. Determine the symplectic structure on the adjoint orbits of G which corresponds, under L, to the symplectic structure on the coadjoint orbits of G.

5. Normal homogeneous semi-Riemannian spaces.

In this section we discuss the setup for the examples for which we will discuss geodesic flows. We change the notation and denote the Hamiltonian vector field of a smooth function f by V_f.

5.1. **Homogeneous spaces.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $K \subseteq G$ be a closed subgroup with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Consider the homogeneous space $N = G/K$ together with the natural projection

$$\bar\pi\colon G \to N, \quad \bar\pi(g) = gK \in N, \tag{5.1}$$

a principal bundle with structure group K. We also write $[g]$ for $\bar\pi(g)$. The natural action of G on N is given by

$$\ell(g,[h]) = \ell_g(p) = [gh]. \tag{5.2}$$

For $g \in G$ and $p \in N$, we also use the shorthand gp for $\ell(g,p)$. To simplify notation, we write $\bar\pi_*$ instead of $\bar\pi_{*e}$.

We let $p_0 = \bar\pi(e) = [e]$ be the distinguished point of N and use $\bar\pi_*\colon \mathfrak{p} \to T_{p_0}N$ to identify $\mathfrak{p}$ with $T_{p_0}N$. We signify this by $\mathfrak{p} \equiv T_{p_0}N$.

The bracket notation used for points of N and, in adapted ways, for other objects assciated to N, is in conflict with the notation for the Lie bracket in $\mathfrak{g}$; an instance of this can be seen in (5.9). We hope that the context clarifies which bracket is in respective use.

We assume that the homogeneous space G/K is *reductive*: There is a complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{5.3}$$

which is *K-invariant*, that is, such that

$$\mathrm{Ad}_k \mathfrak{p} = \mathfrak{p} \quad \text{for all } k \in K. \tag{5.4}$$

We fix such a K-invariant complement $\mathfrak{p}$. For $X \in \mathfrak{g}$, we write

$$X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} + \mathfrak{p} \tag{5.5}$$

and call $X_{\mathfrak{k}}$ and $X_{\mathfrak{p}}$ the $\mathfrak{k}$- and $\mathfrak{p}$-component of X, respectively.

Proposition 5.6. *1) Under $\mathfrak{p} \equiv T_{p_0}N$, the isotropy representation of K on $T_{p_0}N$ corresponds to the adjoint representation of K on $\mathfrak{p}$.*
2) The map $G \times \mathfrak{p} \to TN$, $(g,X) \mapsto \ell_{g*}\bar\pi_*X$, is surjective. Moreover, pairs (g,X) and (h,Y) have the same image if and only if $k := g^{-1}h$ is contained in K and $Y = \mathrm{Ad}_k^{-1} X$. In other words, TN is isomorphic to the vector bundle associated to the adjoint representation of K on $\mathfrak{p}$.
3) Via evaluation in p_0 and $\mathfrak{p} \equiv T_{p_0}N$, G-invariant tensor fields on N correspond one to one to K-invariant tensors on $\mathfrak{p}$. □

Following Proposition 5.6.2, we write tangent vectors of N as pairs $[g,X]$, where $g \in G$ and $X \in \mathfrak{p}$, with the understanding that $[g,X]$ represents the tangent vector $\ell_{g*}\bar\pi_*X \in TN$. For all $k \in K$, we have $[gk,X] = [g,\mathrm{Ad}_k X]$. Moreover, the differential of ℓ_g is given by $\ell_{g*}([h,X]) = [gh,X]$.

For $g \in G$, the *vertical* and *horizontal space* at g are defined to be

$$\mathcal{V}_g = L_{g*}\mathfrak{k} = \ker \bar\pi_{g*} \quad \text{and} \quad \mathcal{H}_g = L_{g*}\mathfrak{p}, \tag{5.7}$$

respectively, where L_g denotes left translation by g in G. For each $g \in G$, we have $T_gG = \mathcal{V}_g \oplus \mathcal{H}_g$. We call the families $\mathcal{V}$ and $\mathcal{H}$ of vertical and horizontal spaces the *vertical* and *horizontal distribution* of G, respectively. A piecewise smooth curve $g\colon I \to G$ is called *horizontal* if $\dot g(t) \in \mathcal{H}_{g(t)}$ for all $t \in I$. Note that curves of the form $g(t) = g_0 \exp(tX)$ with $X \in \mathfrak{p}$ are horizontal.

Proposition 5.8. *The distributions $\mathcal{V}$ and $\mathcal{H}$ are right invariant under K,*

$$R_{k*}\mathcal{V}_g = \mathcal{V}_{gk} \quad \textit{and} \quad R_{k*}\mathcal{H}_g = \mathcal{H}_{gk}$$

for all $k \in K$ and $g \in G$. In particular, $\mathcal{H}$ defines a K-connection for the principal bundle $\bar\pi\colon G \to N$. □

The horizontal distribution $\mathcal{H}$ defines a G-invariant connection on each vector bundle associated to the principal bundle $\bar\pi\colon G \to N$. In particular, it induces a G-invariant connection ∇ on TN, the vector bundle associated to the adjoint representation of K on $\mathfrak{p}$ as explained above. If $V\colon I \to TN$ is a smooth vector field along a smooth curve $c\colon I \to N$ and if we write $V(t) = [g(t), X(t)]$ with smooth maps $g\colon I \to G$ and $X\colon I \to \mathfrak{p}$, then the covariant derivative of V with respect to ∇ is given by

$$\nabla_t V = [g, \dot X + \operatorname{ad}_{\kappa(\dot g)} X] = [g, \dot X + [\kappa(\dot g), X]], \tag{5.9}$$

where κ denotes the connection form associated to the above K-connection of $G \to N$; that is, κ is the one-form on G with values in $\mathfrak{k}$ which associates to a tangent vector $v \in T_gG$ the $\mathfrak{k}$-component of the left invariant vector field V with $V(g) = v$. In other words,

$$\kappa(v) = (L_{g*}^{-1} v)_{\mathfrak{k}}, \tag{5.10}$$

where we identify the space of left invariant vector fields on G with T_eG. The covariant derivative of $V = [g, X]$ as above reduces to

$$\nabla_t V = [g, \dot X] \quad \text{if } g\colon I \to G \text{ is horizontal.} \tag{5.11}$$

Proposition 5.12. *With respect to the above connection ∇, we have:*

1) *For each $X \in \mathfrak{p}$, $\gamma_X = \gamma_X(t) = \ell_{\exp(tX)}(p_0)$ is the geodesic through p_0 with initial velocity $[e, X]$;*
2) *The one-parameter group $(\ell_{\exp(tX)})_{t\in\mathbb{R}}$ of diffeomorphisms of N induces parallel translation along γ_X;*
3) *G-invariant tensor fields on N are parallel with respect to ∇.* □

For a piecewise smooth vector field $V = [g, X]$ along a piecewise smooth curve $c = \bar\pi \circ g$ in N, the horizontal and vertical part of $\dot V$ in the sense of (2.18) are given by

$$[g, (L_{g*}^{-1}\dot g)_{\mathfrak{p}}] \quad \text{and} \quad [g, \dot X + \operatorname{ad}_{\kappa(\dot g)}, X] = [g, \dot X + [(L_{g*}^{-1}\dot g)_{\mathfrak{k}}, X]], \tag{5.13}$$

respectively. Thus we arrive at the natural identification

$$TTN \cong G \times_K (\mathfrak{p} \times \mathfrak{p} \times \mathfrak{p}), \tag{5.14}$$

where the right hand side consists of all quadruples $[g, X, Y, Z]$, where $g \in G$ and $X, Y, Z \in \mathfrak{p}$, with the relation $[gk, X, Y, Z] = [g, \operatorname{Ad}_k X, \operatorname{Ad}_k Y, \operatorname{Ad}_k Z]$ for all $k \in K$. Here $[g, X]$ is the footpoint in TN, $[g, Y]$ the horizontal component, and $[g, Z]$ the vertical component of the tangent vector $[g, X, Y, Z]$

of TTN. We also call Y and Z the horizontal and vertical component of $[g, X, Y, Z]$, respectively, although they depend on the choice of g.

Proposition 5.15. *Under $\mathfrak{p} \equiv T_{p_0}N$, torsion and curvature tensor of ∇ are given by*

$$T(X,Y) = -[X,Y]_{\mathfrak{p}} \quad \textit{and} \quad R(X,Y)Z = -[[X,Y]_{\mathfrak{k}}, Z],$$

where $X, Y, Z \in \mathfrak{p}$. Moreover, T and R are parallel with respect to ∇. □

We assume from now on that $N = G/K$ is a *normal homogeneous* semi-Riemannian space; that is, we require that N is reductive as above and that there is a non-degenerate Ad_G-invariant bilinear form $\langle .,. \rangle$ on $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{k}^{\perp}$ with respect to $\langle .,. \rangle$. This implies that $\mathfrak{k}$ and $\mathfrak{p}$ are *nondegenerate subspaces* of $\mathfrak{g}$ in the sense that the restriction of $\langle .,. \rangle$ to $\mathfrak{k}$ and $\mathfrak{p}$ is nondegenerate. By the Ad_G-invariance of $\langle .,. \rangle$, we have

$$\langle [X,Y], Z \rangle = \langle X, [Y,Z] \rangle \tag{5.16}$$

for all $X, Y, Z \in \mathfrak{g}$. We endow N with the G-invariant semi-Riemannian metric, also denoted $\langle .,. \rangle$, which corresponds to $\langle .,. \rangle$ on $\mathfrak{p}$ under $\mathfrak{p} \equiv T_{p_0}N$; see Proposition 5.6.3.

The G-invariant connection ∇ from Section 5.1 is metric with respect to $\langle .,. \rangle$; compare with (5.11). By (5.16), the trilinear form

$$\tau = \tau(X,Y,Z) = \langle X, T(Y,Z) \rangle = -\langle X, [Y,Z] \rangle \tag{5.17}$$

on $\mathfrak{p}$ is alternating. It follows that the difference $\nabla^{LC} - \nabla$ is skewsymmetric, where ∇^{LC} denotes the Levi-Civita connection on N. Therefore we obtain

$$\nabla^{LC}_X Y = \nabla_X Y - \frac{1}{2} T(X,Y) = \nabla_X Y + \frac{1}{2}[X,Y]_{\mathfrak{p}}. \tag{5.18}$$

In particular, ∇ and ∇^{LC} have the same geodesics. However, ∇ has the advantage of computational simplicity.

Remark 5.19. In the case of symmetric spaces, we have $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Hence the torsion tensor field T of ∇ vanishes in this case, and hence $\nabla = \nabla^{LC}$.

5.2. **Hamiltonians on TN.** We consider a normal homogeneous space $N = G/K$ as in Section 5.1 above and use the Legendre transform to identify TN with T^*N. We also use $\langle .,. \rangle$ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then the orbits of the adjoint action of G in $\mathfrak{g}$ correspond to the coadjoint orbits in $\mathfrak{g}^*$ and inherit the corresponding symplectic structure.

Proposition 5.20. *For $v_1 = [g, X, Y_1, Z_1], v_2 = [g, X, Y_2, Z_2] \in T_{[g,X]}TN$, we have*

$$\omega_{[g,X]}(v_1, v_2) = \langle Y_1, Z_2 \rangle - \langle Z_1, Y_2 \rangle + \langle X, [Y_1, Y_2] \rangle.$$

Proof. This is immediate from (2.21), the second equality in Proposition 5.15, and since $\mathfrak{k}$ is perpendicular to $\mathfrak{p}$,

$$\langle X, [Y_1, Y_2] \rangle = \langle X, [Y_1, Y_2]_{\mathfrak{p}} \rangle = -\langle X, T(Y_1, Y_2) \rangle.$$ □

Via $f([g, X]) = \varphi(X)$, G-invariant functions f on TN correspond to K-invariant functions φ on $\mathfrak{p}$, that is, functions φ on $\mathfrak{p}$ which are invariant under all Ad_k, $k \in K$.

Proposition 5.21. *For a K-invariant function $\varphi\colon \mathfrak{p} \to \mathbb{R}$, the G-invariant extension $f\colon TN \to \mathbb{R}$ has associated Hamiltonian vector field*

$$V_f = V_f([g,X]) = [g, X, \operatorname{grad}\varphi(X), [X, \operatorname{grad}\varphi(X)]_{\mathfrak{p}}].$$

For K-invariant functions $\varphi_1, \varphi_2\colon \mathfrak{p} \to \mathbb{R}$, the Poisson bracket of their G-invariant extensions $f_1, f_2\colon TN \to \mathbb{R}$ is given by

$$\{f_1, f_2\} = \{f_1, f_2\}([g,X]) = -\langle X, [\operatorname{grad}\varphi_1(X), \operatorname{grad}\varphi_2(X)]\rangle.$$

Proof. Recall that $\ell_{g*}([h,X]) = [gh, X]$ for all $g, h \in G$ and $X \in \mathfrak{p}$. Hence

$$\begin{aligned} df_{[g,X]}([g,X,Y,Z]) &= \frac{d}{dt} f([g\exp(tY), X + tZ]))\big|_{t=0} \\ &= \frac{d}{dt}\varphi(X + tZ)\big|_{t=0} \\ &= \langle \operatorname{grad}\varphi(X), Z\rangle. \end{aligned}$$

By definition, the left hand side is equal to $\omega(V_f([g,X]), [g,X,Y,Z])$, and hence $V_f([g,X]) = [g, X, \operatorname{grad}\varphi(X), [X, \operatorname{grad}\varphi(X)]_{\mathfrak{p}}]$ by (5.16) and Proposition 5.20. We also conclude that

$$\begin{aligned} \{f_1, f_2\}([g,X]) &= \langle \operatorname{grad}\varphi_1(X), [X, \operatorname{grad}\varphi_2(X)]) \\ &= -\langle X, [\operatorname{grad}\varphi_1(X), \operatorname{grad}\varphi_2(X)]\rangle. \end{aligned} \qquad \square$$

Proposition 5.22. *For $X \in \mathfrak{g}$, we have*

$$X_N^*([g]) = [g, (\operatorname{Ad}_g^{-1} X)_{\mathfrak{p}}] \quad \text{and} \quad f_X([g,Y]) = \langle \operatorname{Ad}_g Y, X\rangle.$$

Proof. We compute

$$X_N^*([g]) = \frac{d}{dt}\ell_{\exp(tX)}(\bar\pi(g))\big|_{t=0} = \ell_{g*}\bar\pi_* \operatorname{Ad}_g^{-1} X = [g, (\operatorname{Ad}_g^{-1} X)_{\mathfrak{p}}],$$

which proves the first equality. As for the second, we have, by (4.23),

$$f_X([g,Y]) = \langle Y, (\operatorname{Ad}_g^{-1} X)_{\mathfrak{p}}\rangle = \langle Y, \operatorname{Ad}_g^{-1} X\rangle = \langle \operatorname{Ad}_g Y, X\rangle. \qquad \square$$

Proposition 5.23. *The moment map $F\colon TN \to \mathfrak{g}$ and its derivative are given by*

$$F([g,X]) = \operatorname{Ad}_g X \quad \text{and} \quad dF_{[g,X]}([g,X,Y,Z]) = \operatorname{Ad}_g(Z - [X,Y]).$$

Proof. The formula for the moment map is immediate from the second equality in Proposition 5.22. Furthermore,

$$\begin{aligned} dF_{[g,X]}([g,X,Y,Z]) &= \frac{d}{dt} F([g\exp(tY), X + tZ])\big|_{t=0} \\ &= \frac{d}{dt}\operatorname{Ad}_g \operatorname{Ad}_{\exp(tY)}(X + tZ)\big|_{t=0} \\ &= \operatorname{Ad}_g(Z - [X,Y]). \end{aligned} \qquad \square$$

We know from Section 4 that any smooth function of the form $f = \varphi \circ F$ on TN is an integral for any G-invariant Hamiltonian system on TN.

Proposition 5.24. *Let $\varphi\colon \mathfrak{g} \to \mathbb{R}$ be a smooth function and $f = \varphi \circ F$. Then the Hamiltonian vector field V_f associated to f is given by*

$$V_f([g,X]) = [g, X, (\operatorname{Ad}_g^{-1} \operatorname{grad}\varphi(\operatorname{Ad}_g X))_{\mathfrak{p}}, [(\operatorname{Ad}_g^{-1} \operatorname{grad}\varphi(\operatorname{Ad}_g X))_{\mathfrak{k}}, X]].$$

For smooth functions $\varphi_1, \varphi_2 \colon \mathfrak{g} \to \mathbb{R}$, the Poisson bracket of $f_1 = \varphi_1 \circ F$ and $f_2 = \varphi_2 \circ F$ is given by $\{f_1, f_2\} = \{\varphi_1, \varphi_2\} \circ F$, where we use the symplectic structure on the orbits of the adjoint action of G,

$$\{\varphi_1, \varphi_2\}(X) = \langle X, [\operatorname{grad}\varphi_1(X), \operatorname{grad}\varphi_2(X)]\rangle.$$

Proof. By the chain rule and Proposition 5.23,

$$\begin{aligned} df_{[g,X]}([g,X,Y,Z]) &= d\varphi_{F([g,X])}(dF_{[g,X]}([g,X,Y,Z])) \\ &= \langle \operatorname{grad}\varphi(F([g,X])), dF_{[g,X]}([g,X,Y,Z])\rangle \\ &= \langle \operatorname{grad}\varphi(\operatorname{Ad}_g X), \operatorname{Ad}_g(Z - [X,Y])\rangle. \end{aligned}$$

Now using Proposition 5.20 and the claimed expression, the first claim follows by comparing components, recalling that $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$. The second claim is a special case of Proposition 4.17, and the formula for the Poisson bracket of φ_1 with φ_2 follows from Proposition 4.6. ☐

Corollary 5.25. *For $f = \varphi \circ F$ as above and $[g_0, X_0] \in TN$, let $g = g(t)$ be the solution of the ordinary differential equation*

$$\dot g = R_{g*}(\operatorname{grad}\varphi(\operatorname{Ad}_g X_0)), \quad g(0) = g_0.$$

Then $[g, X_0]$ is the trajectory of V_f with initial value $[g_0, X_0]$.

Proof. By Proposition 5.24, the trajectories $[g,X]$ of V_f are solutions of

$$\begin{aligned} (L_{g*}^{-1}\dot g)_{\mathfrak{p}} &= (\operatorname{Ad}_g^{-1}\operatorname{grad}\varphi(\operatorname{Ad}_g X))_{\mathfrak{p}} \\ \dot X + [(L_{g*}^{-1}\dot g)_{\mathfrak{k}}, X] &= [(\operatorname{Ad}_g^{-1}\operatorname{grad}\varphi(\operatorname{Ad}_g X))_{\mathfrak{k}}, X]. \end{aligned}$$

Hence, if g solves

$$\dot g = R_{g*}\operatorname{grad}\varphi(\operatorname{Ad}_g X_0)$$

with $g(0) = g_0$, then g solves the first of the above differential equations with $X = X_0$. Since then also $[(L_{g*}^{-1}\dot g)_{\mathfrak{k}}, X_0] = [(\operatorname{Ad}_g^{-1}\operatorname{grad} h(\operatorname{Ad}_g X))_{\mathfrak{k}}, X_0]$, the second differential equation turns into the consistent $\dot X = 0$. ☐

Remarks 5.26. Let $\varphi \colon \mathfrak{g} \to \mathbb{R}$ be G-invariant. (See also Lemma 6.1.)
1) The restriction of φ to $\mathfrak{p}$ is K-invariant. The corresponding G-invariant extension f to TN is given by $f = \varphi \circ F$ since

$$f([g,X]) = \varphi(X) = \varphi(\operatorname{Ad}_g X) = \varphi(F([g,X])).$$

2) By the G-invariance of φ, we have $\operatorname{grad}\varphi(\operatorname{Ad}_g X) = \operatorname{Ad}_g \operatorname{grad}\varphi(X)$ for all $X \in \mathfrak{g}$. Hence the differential equation in Corollary 5.25 turns into the simpler differential equation $\dot g = L_{g*}\operatorname{grad}\varphi(X_0)$ with solution

$$g = g(t) = g_0 \exp(t \operatorname{grad}\varphi(X_0)) = \exp(t\operatorname{grad}\varphi(\operatorname{Ad}_{g_0} X_0))g_0.$$

5.3. **Exercises.**

Exercise 5.27. Let $N = G/K$ be a reductive space with associated projection $\bar\pi \colon G \to N$ and vertical and horizontal distributions $\mathcal{V}$ and $\mathcal{H}$ as in Section 5.1. Show that, for any piecewise smooth curve $c \colon I \to N$, $t_0 \in I$ and $g_0 \in G$ with $\bar\pi(g_0) = c(t_0)$, there is a unique horizontal lift $g \colon I \to G$ with $g(t_0) = g_0$.

Exercise 5.28. Curves in G of the form $g = g(t) = g_0 \exp(tX)$ with $X \in \mathfrak{g}$ are horizontal if and only if $X \in \mathfrak{p}$.

Exercise 5.29. If $V\colon I \to TN$ is a smooth vector field along a smooth curve $c\colon I \to N$, then there are smooth maps $g\colon I \to G$ and $X\colon I \to \mathfrak{p}$ such that $c = \bar\pi \circ g$ and $V(t) = [g(t), X(t)]$. Conversely, for any two smooth maps $g\colon I \to G$ and $X\colon I \to \mathfrak{p}$, $V = [g, X]$ is a smooth vector field along $c = \bar\pi \circ g$.

Exercise 5.30. For any smooth vector field $V\colon I \to TN$ along a smooth curve $c\colon I \to N$ and lift $g\colon I \to G$ of c, there is a unique map $X\colon I \to \mathfrak{p}$ such that $V(t) = [g(t), X(t)]$, and V is smooth.

Exercise 5.31. Show that the covariant derivative $\nabla_t V$ in (5.9) is well defined; that is, it is independent of the choice of the maps g and X such that $V = [g, X]$. Show also that ∇ defines a G-invariant connection on N.

Exercise 5.32. Discuss that, for $[g, X, Y, Z] \in TTN$,
1) $\gamma = \gamma(t) = \bar\pi(g\exp(tX))$ is the geodesic with $\dot\gamma(0) = [g, X]$;
2) the vector field $V = V(t) = [g\exp(tY), X]$ is parallel along γ;
3) the vector field $W = W(t) = [g\exp(tY), X + tZ]$ along γ has initial velocity $\dot W(0) = [g, X, Y, Z]$.

6. Complete integrability after Anton Thimm

We discuss complete integrabilty of Hamiltonian systems after Thimm [7, 8]. Our aim is to obtain conditions which ensure that the geodesic flow of a Riemannian manifold is completely integrable. Thimm's method shows in fact the integrability of more general Hamiltonian systems, given that his conditions are satisfied.

Lemma 6.1. *Let G be a Lie group with a G-invariant nondegenerate inner product $\langle .,. \rangle$ on $\mathfrak{g}$, and let $\varphi\colon \mathfrak{g} \to \mathbb{R}$ be a smooth function. Then $[X, \operatorname{grad}\varphi(X)] = 0$ for all $X \in \mathfrak{g}$ if φ is G-invariant. Conversely, if G is connected and $[X, \operatorname{grad}\varphi(X)] = 0$ for all $X \in \mathfrak{g}$, then φ is G-invariant.* □

Proof. For all $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}\frac{d}{dt}\varphi(\operatorname{Ad}_{\exp(tY)} X)\big|_{t=0} &= d\varphi|_X([Y,X]) \\ &= \langle \operatorname{grad}\varphi(X), [Y,X]\rangle = \langle [X, \operatorname{grad}\varphi(X)], Y\rangle.\end{aligned}$$

Since $\langle .,. \rangle$ is non-degenerate, we conclude that the differential of φ in the direction of the adjoint orbits of G vanishes if and only if $[X, \operatorname{grad}\varphi(X)] = 0$ for all $X \in \mathfrak{g}$. □

We return now to the setup –and the notation– as in the previous sections and consider a semi-Riemannian normal homogeneous space $M = G/K$.

We say that a subspace $V \subseteq \mathfrak{g}$ is *non-degenerate* if the restriction of $\langle .,. \rangle$ to V is non-degenerate and that a Lie subgroup $G' \subseteq G$ is *non-degenerate* if $\mathfrak{g}' \subseteq \mathfrak{g}$ is non-degenerate.

Proposition 6.2. *Let G' be a nondegenerate Lie subgroup of G and $\pi'\colon \mathfrak{g} \to \mathfrak{g}'$ be the orthogonal projection. If $\psi\colon \mathfrak{g}' \to \mathbb{R}$ is an $\operatorname{Ad}_{G'}$-invariant function, then 1) $f = \psi \circ \pi' \circ F$ is G'-invariant and*
2) the V_f-trajectory through $[g_0, X_0]$ is given by

$$[g, X_0] = [\exp(tX_1)g_0, X_0] \quad \textit{with } X_1 = \operatorname{grad}\psi(\pi' \operatorname{Ad}_{g_0} X_0).$$

Proof. 1) For all $g' \in G'$, we have $\pi' \circ \operatorname{Ad}_{g'} = \operatorname{Ad}_{g'} \circ \pi'$.

2) The curve g has derivative $\dot g = R_{g_0*}L_{g_t*}X_1$, where $g_t = \exp(tX_1)$. On the other hand, since ψ is $\operatorname{Ad}_{G'}$-invariant and $\operatorname{grad}(\psi\circ\pi')(X) = \operatorname{grad}\psi(\pi' X)$ for all $X \in \mathfrak{g}$, we have

$$R_{g_0*}L_{g_t*}(\operatorname{grad}(\psi(\pi' \operatorname{Ad}_{g_0} X_0)) = R_{g_0*}R_{g_t*}(\operatorname{grad}(\psi(\pi' \operatorname{Ad}_{g_t} \operatorname{Ad}_{g_0} X_0)).$$

Now the claim follows from Corollary 5.25. □

Proposition 6.3. *Let G_1 and G_2 be nondegenerate Lie subgroups of G with corresponding orthogonal projections $\pi_1\colon \mathfrak{g} \to \mathfrak{g}_1$ and $\pi_2\colon \mathfrak{g} \to \mathfrak{g}_2$. Suppose that $\psi_1\colon \mathfrak{g}_1 \to \mathbb{R}$ and $\psi_2\colon \mathfrak{g}_2 \to \mathbb{R}$ are invariant under Ad_{G_1} and Ad_{G_2}, respectively. Then the Poisson bracket $\{\psi_1 \circ \pi_1, \psi_2 \circ \pi_2\}$ is given by*

$$\{\psi_1 \circ \pi_1, \psi_2 \circ \pi_2\}(X) = \langle X', [\operatorname{grad}\psi_1(\pi_1 X), \operatorname{grad}\psi_2(\pi_2 X)]\rangle,$$

for any linear combination $X' = X + \alpha_1\pi_1 X + \alpha_2\pi_2 X$.

Proof. Since ψ_1 and ψ_2 are invariant under Ad_{G_1} and Ad_{G_2}, the claim follows immediately from (5.16) and Lemma 6.1. □

We are interested in conditions which guarantee the vanishing of the Poisson brackets $\{\psi_1 \circ \pi_1, \psi_2 \circ \pi_2\}$.

Corollary 6.4. *Under the above assumptions, if* $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_2$*, for example, if* $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$ *or* $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$*, then*

$$\{\psi_1 \circ \pi_1, \psi_2 \circ \pi_2\} = 0.$$

Proof. Choose $X' = X - \pi_2 X$, the component of X perpendicular to $\mathfrak{g}_2$, and apply Proposition 6.3 $\square$

The strategy is now to find appropriate chains $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ of nondegenerate subgroups and associated families of invariant functions to obtain families of functions on $\mathfrak{g}$ which have vanishing Poisson brackets. As for the independence of these functions $f = \varphi \circ F$, we aim to show that there is an $X \in \mathfrak{g}$ such that the corresponding

$$((\operatorname{grad}\varphi(X))_{\mathfrak{p}}, [(\operatorname{grad}\varphi(X))_{\mathfrak{k}}, X]) \in \mathfrak{p} \times \mathfrak{p} \tag{6.5}$$

are linearly independent. By Proposition 5.24, this is equivalent to the linear independence of the Hamiltonian vector fields H_f at $\{e, X\}$. By analyticity of the system, this then implies the linear independence of the Hamiltonian vector fields at all points of an open and dense subset in TM of full measure.

6.1. **Real Grassmannians.** Let $N = G_{p,q}(\mathbb{R})$ be the real Grassmannian of p-planes in $\mathbb{R}^{p+q}$, where $p, q \geq 1$. Let $I_{p,q}$ be the orthogonal transformation of $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$ with $I_{p,q}(u, v) = (u, -v)$. Then conjugation with $I_{p,q}$ is an involution σ of $G = \mathrm{O}(p+q)$ with fixed point set $K = \mathrm{O}(p) \times \mathrm{O}(q)$ such that $G_{p,q}(\mathbb{R})$ is the associated symmetric space G/K. We may also choose $G = \mathrm{SO}(p+q)$ with $K = S(\mathrm{O}(p) \times \mathrm{O}(q))$ to represent $G_{p,q}(\mathbb{R})$ as the associated symmetric space. We use

$$\langle X, Y \rangle = \operatorname{tr}(X^t Y) \tag{6.6}$$

as G-invariant positive definite inner product on $\mathfrak{g} = \mathfrak{so}(p+q)$.

As a normalization, assume that $p \leq q$ and consider the chain

$$\begin{aligned} &\mathbb{R}^1 \times \mathbb{R}^1 \subseteq \mathbb{R}^1 \times \mathbb{R}^2 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^{p-1} \times \mathbb{R}^p \subseteq \mathbb{R}^p \times \mathbb{R}^p \\ &\qquad \subseteq \mathbb{R}^p \times \mathbb{R}^{p+1} \subseteq \cdots \subseteq \mathbb{R}^p \times \mathbb{R}^{q-1} \subseteq \mathbb{R}^p \times \mathbb{R}^q, \end{aligned} \tag{6.7}$$

where we view $\mathbb{R}^k \times \mathbb{R}^l$ as the subspace of vectors $(u, v) \in \mathbb{R}^p \times \mathbb{R}^q$ such that the respective last $p-k$ and $q-l$ coordinates of u and v vanish. We obtain a chain of $p+q-1$ associated non-degenerate and σ-invariant subgroups,

$$\begin{aligned} &\mathrm{O}(1+1) \subseteq \mathrm{O}(1+2) \subseteq \mathrm{O}(2+2) \subseteq \cdots \subseteq \mathrm{O}(p-1+p) \subseteq \mathrm{O}(p+p) \\ &\qquad \subseteq \mathrm{O}(p+p+1) \subseteq \cdots \subseteq \mathrm{O}(p+q-1) \subseteq \mathrm{O}(p+q). \end{aligned} \tag{6.8}$$

The fixed point set of σ in each occuring $\mathrm{O}(k+l)$ is $\mathrm{O}(k) \times \mathrm{O}(l)$. Thus we obtain totally geodesically and isometrically embedded submanifolds

$$G_{k,l}(\mathbb{R}) = \mathrm{O}(k+l)/\,\mathrm{O}(k) \times \mathrm{O}(l) \subseteq G_{p,q}(\mathbb{R}).$$

The Lie algebra of $\mathrm{O}(k+l)$, where we view $\mathrm{O}(k+l) \subseteq \mathrm{O}(p+q)$ as above, consists of all matrices $A \in \mathfrak{so}(p+q)$ such that the rows and columns with respective numbers $k+1, \dots, p$ and $p+l+1, \dots, p+q$ vanish.

There are two types of inclusions in the above chain of subspaces,

$$\mathbb{R}^{k-1} \times \mathbb{R}^k \subseteq \mathbb{R}^k \times \mathbb{R}^k \quad \text{and} \quad \mathbb{R}^k \times \mathbb{R}^l \subseteq \mathbb{R}^k \times \mathbb{R}^{l+1},$$

respectivley, with $k \leq l$ in the latter case. There are the corresponding inclusions

$$G_1 = \mathrm{O}(k-1+k) \subseteq G_2 = \mathrm{O}(k+k),$$
$$G_1 = \mathrm{O}(k+l) \subseteq G_2 = \mathrm{O}(k+l+1).$$

In both cases, we denote the fixed point set of σ in G_1 and G_2 by K_1 and K_2, respectively, and obtain corresponding decompositions of the Lie algebras, $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$. In both cases, we have the special feature that

$$\dim \mathfrak{p}_2 = \dim \mathfrak{p}_1 + r_2, \tag{6.9}$$

where r_2 is the rank of the symmetric space $N_2 = G_2/K_2$, that is, the dimension of a maximal Abelian subspaces of $\mathfrak{p}_2$.

Lemma 6.10. *In the above situation, for any regular vector $X_1 \in \mathfrak{p}_1$, there is a regular vector $X_2 \in \mathfrak{p}_2$ with $\pi_1 X_2 = X_1$, where $\pi_1 \colon \mathfrak{so}(p+q) \to \mathfrak{g}_1$ denotes the orthogonal projection, such that the maximal Abelian subspace $\mathfrak{a}_2$ of $\mathfrak{p}_2$ containing X_2 satisfies $\mathfrak{p}_1 \oplus \mathfrak{a}_2 = \mathfrak{p}_2$.*

Proof. We prove Lemma 6.10 in the first case; the proof in the second case is analogous. In the first case, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ consist of all matrices of the form

$$\begin{pmatrix} 0 & 0 & -x^t \\ 0 & 0 & -a^t \\ x & a & 0 \end{pmatrix}$$

with $x \in \mathbb{R}^{k \times (k-1)}$ and $a \in \mathbb{R}^k$, where $a = 0$ for $\mathfrak{p}_1$ and where we delete the vanishing columns and rows which arise from the inclusion into $\mathbb{R}^p \times \mathbb{R}^q$. We will identify any such matrix with the corresponding pair (x, a) in $\mathbb{R}^{k \times (k-1)} \times \mathbb{R}^k$. There is a $k \in K_1$ such that $\mathrm{Ad}_k X_1$ is contained in the distinguished maximal Abelian subspace $\mathfrak{d}_1 \subseteq \mathfrak{p}_1$ which consists of matrices $(x, 0)$ such that $x_{ij} = 0$ for all $i \neq j$. Therefore we may assume without loss of generality that

$$X_1 = (x, 0) \in \mathfrak{d}_1.$$

To find an X_2 as asserted, we use the Ansatz $X_2 = (x, a)$. Then the commutator of X_2 with $Y = (y, 0) \in \mathfrak{p}_1$ is given by

$$[X_2, Y] = \begin{pmatrix} y^t x - x^t y & y^t a & 0 \\ -a^t y & 0 & 0 \\ 0 & 0 & y x^t - x y^t \end{pmatrix}.$$

The upper left and lower right entries correspond to the commutator $[X_1, Y]$. Now X_1 is regular in $\mathfrak{p}_1$. Hence $[X_1, Y] = 0$ implies that $Y \in \mathfrak{d}_1$ and then $a^t y = (a_1 y_{11}, \dots, a_{k-1} y_{k-1,k-1})$. Hence, if we choose the coordinates $a_1, \dots, a_{k-1}$ of a to be nonzero, then the commutator of $X_2 = (x, a)$ with $Y = (y, 0)$ does not vanish for $y \neq 0$, and then the maximal Abelian subspace $\mathfrak{a}_2$ in $\mathfrak{p}_2$ containing X_2 satisfies $\mathfrak{a}_2 \cap \mathfrak{p}_1 = \{0\}$. Thus $\mathfrak{a}_2$ is a complement of $\mathfrak{p}_1$ in $\mathfrak{p}_2$, by (6.9). □

Now we enumerate the chains of groups $\mathrm{O}(k+l)$ and $\mathrm{O}(k) \times \mathrm{O}(l)$ further up consecutively by G_i and K_i and denote their Lie algebras by $\mathfrak{g}_i$ and $\mathfrak{k}_i$, for $1 \leq i \leq p+q-1$. We have the corresponding decompositions $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. Starting with a nonzero $X_1 \in \mathfrak{a}_1 = \mathfrak{p}_1 \cong \mathbb{R}$, we obtain from Lemma 6.10

a chain of regular vectors $X_i \in \mathfrak{p}_i$ such that $\pi_i X_j = X_i$ for all $i \le j$ and such that the maximal Abelian subspace $\mathfrak{a}_i$ of $\mathfrak{p}_i$ containing X_i satisfies $\mathfrak{p}_i = \mathfrak{p}_{i-1} \oplus \mathfrak{a}_i$. In particular, we obtain a direct sum decomposition

$$\mathfrak{p}_{p+q-1} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_{p+q-1}. \tag{6.11}$$

The proof of the following Lemma 6.12 is Exercise 6.24.

Lemma 6.12. *Let $G = \mathrm{O}(k+l)$ and $K = \mathrm{O}(k) \times \mathrm{O}(l)$ with $k \le l$ and write $\mathfrak{so}(k+l) = \mathfrak{k} \oplus \mathfrak{p}$ as above. Then the polynomials*

$$\psi_j \colon \mathfrak{so}(k+l) \to \mathbb{R}, \quad \psi_j(X) = \operatorname{tr} X^{2j},$$

where $1 \le j \le k$, are G-invariant with gradient $-2jX^{2j-1}$. Moreover, at any point X in any maximal Abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, their gradients are tangent to $\mathfrak{a}$ and are linearly independent if X is regular. $\square$

Now Corollary 6.4 together with (6.11) and Lemma 6.12 implies that there are pq smooth functions $f_{ji} = \psi_j \circ \pi_i \circ F$ on $TG_{p,q}(\mathbb{R})$ which are in involution and whose symplectic gradients are linearly independent (and horizontal) at $[e, X_{p+q-1}]$; cf. Proposition 5.24. Since the $\psi_j \circ \pi_i$ are polynomials, the f_{ji} are real analytic. We conclude that their symplectic gradients are independent on an open and dense subset of $TG_{p,q}(\mathbb{R})$ of full measure. With Corollary 4.16 we arrive at

Theorem 6.13. *For any $\mathrm{SO}(p+q)$-invariant smooth function on $TG_{p,q}(\mathbb{R})$, the associated Hamiltonian system on $TG_{p,q}(\mathbb{R})$ is completely integrable with the above functions f_{ji} as a complete and involutive family of real analytic first integrals.* $\square$

Remark 6.14. The proof of Theorem 6.13 shows that the polynomials $\psi_j \circ \pi_i$ also give rise to a complete and involutive family of real analytic first integrals for Hamiltonian systems associated to $\mathrm{SO}(p+q)$-invariant functions on the tangent bundle of the Grassmannian

$$G^o_{k,l}(\mathbb{R}) = \mathrm{SO}(p+q)/\,\mathrm{SO}(p) \times \mathrm{SO}(q)$$

of oriented p-planes in $\mathbb{R}^{p+q}$ and, analogously, for Hamiltonian systems associated to $\mathrm{SO}(p,q)$-invariant functions on the tangent bundle of the dual Grassmannian of negative p-planes in $\mathbb{R}^{p,q}$,

$$G^-_{p,q}(\mathbb{R}) = \mathrm{SO}(p,q)/\,\mathrm{SO}(p) \times \mathrm{SO}(q).$$

Recall that $G^o_{1,q}(\mathbb{R})$ and $G^-_{1,q}$ are the round sphere and the real hyperbolic space of dimension q, respectively. (See also Exercise 6.25.)

6.2. **Complex Grassmannians.** Up to (6.18), the ensuing discussion is completely parallel to the one for the real Grassmannians; up to Lemma 6.21, the changes are easy modifications.

Let $N = G_{p,q}(\mathbb{C})$ be the complex Grassmannian of p-planes in $\mathbb{C}^{p+q}$, where $p, q \ge 1$. Let $I_{p,q}$ be the orthogonal transformation of $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$ with $I_{p,q}(u,v) = (u,-v)$. Then conjugation with $I_{p,q}$ is an involution σ of $G = \mathrm{U}(p+q)$ with fixed point set $K = \mathrm{U}(p) \times \mathrm{U}(q)$ such that $G_{p,q}(\mathbb{C})$ is the associated symmetric space G/K. We may also choose $G = \mathrm{SU}(p+q)$

with $K = S(\mathrm{U}(p) \times \mathrm{U}(q))$ to represent $G_{p,q}(\mathbb{C})$ as the associated symmetric space. We use

$$\langle X, Y \rangle = \operatorname{tr}(X^* Y) \tag{6.15}$$

as G-invariant positive definite inner product on $\mathfrak{g} = \mathfrak{u}(p+q)$, where the star indicates transposition together with complex conjugation.

As a normalization, assume that $p \leq q$ and consider the chain

$$\begin{aligned} &\mathbb{C}^1 \times \mathbb{C}^1 \subseteq \mathbb{C}^1 \times \mathbb{C}^2 \subseteq \mathbb{C}^2 \times \mathbb{C}^2 \subseteq \cdots \subseteq \mathbb{C}^{p-1} \times \mathbb{C}^p \subseteq \mathbb{C}^p \times \mathbb{C}^p \\ &\qquad\qquad \subseteq \mathbb{C}^p \times \mathbb{C}^{p+1} \subseteq \cdots \subseteq \mathbb{C}^p \times \mathbb{C}^{q-1} \subseteq \mathbb{C}^p \times \mathbb{C}^q, \end{aligned} \tag{6.16}$$

where we view $\mathbb{C}^k \times \mathbb{C}^l$ as the subspace of vectors $(u, v) \in \mathbb{C}^p \times \mathbb{C}^q$ such that the respective last $p-k$ and $q-l$ coordinates of u and v vanish. We obtain a chain of $p+q-1$ associated non-degenerate and σ-invariant subgroups,

$$\begin{aligned} &\mathrm{U}(1+1) \subseteq \mathrm{U}(1+2) \subseteq \mathrm{U}(2+2) \subseteq \cdots \subseteq \mathrm{U}(p-1+p) \subseteq \mathrm{U}(p+p) \\ &\qquad\qquad \subseteq \mathrm{U}(p+p+1) \subseteq \cdots \subseteq \mathrm{U}(p+q-1) \subseteq \mathrm{U}(p+q). \end{aligned} \tag{6.17}$$

The fixed point set of σ in each occuring $\mathrm{U}(k+l)$ is $\mathrm{U}(k) \times \mathrm{U}(l)$. Thus we obtain totally geodesically and isometrically embedded submanifolds

$$G_{k,l}(\mathbb{C}) = \mathrm{U}(k+l)/\,\mathrm{U}(k) \times \mathrm{U}(l) \subseteq G_{p,q}(\mathbb{C}).$$

The Lie algebra of $\mathrm{U}(k+l)$, where we view $\mathrm{U}(k+l) \subseteq \mathrm{U}(p+q)$ as above, consists of all matrices $A \in \mathfrak{u}(p+q)$ such that the rows and columns with respective numbers $k+1, \dots, p$ and $p+l+1, \dots, p+q$ vanish.

There are two types of inclusions in the above chain of subspaces,

$$\mathbb{C}^{k-1} \times \mathbb{C}^k \subseteq \mathbb{C}^k \times \mathbb{C}^k \quad \text{and} \quad \mathbb{C}^k \times \mathbb{C}^l \subseteq \mathbb{C}^k \times \mathbb{C}^{l+1},$$

respectivley, with $k \leq l$ in the latter case. There are the corresponding inclusions

$$\begin{aligned} G_1 &= \mathrm{U}(k-1+k) \subseteq G_2 = \mathrm{U}(k+k), \\ G_1 &= \mathrm{U}(k+l) \subseteq G_2 = \mathrm{U}(k+l+1). \end{aligned}$$

In both cases, we denote the fixed point set of σ in G_1 and G_2 by K_1 and K_2, respectively, and obtain correspnding decompositions of the Lie algebras, $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$. In both cases, we have

$$\dim \mathfrak{p}_2 = \dim \mathfrak{p}_1 + 2r_2, \tag{6.18}$$

where r_2 is the rank of the symmetric space $N_2 = G_2/K_2$, that is, the dimension of a maximal Abelian subspaces of $\mathfrak{p}_2$.

Lemma 6.19. *In the above situation, for any regular vector $X_1 \in \mathfrak{p}_1$, there is a regular vector $X_2 \in \mathfrak{p}_2$ with $\pi_1 X_2 = X_1$, where $\pi_1 \colon \mathfrak{u}(p+q) \to \mathfrak{g}_1$ denotes the orthogonal projection, such that the maximal (real) Abelian subspace $\mathfrak{a}_2$ of $\mathfrak{p}_2$ containing X_2 satisfies $\mathfrak{p}_1 \oplus \mathbb{C}\mathfrak{a}_2 = \mathfrak{p}_2$.*

Proof. We prove Lemma 6.19 in the first case; the proof in the second case is analogous. In the first case, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ consist of all matrices of the form

$$\begin{pmatrix} 0 & 0 & -x^* \\ 0 & 0 & -a^* \\ x & a & 0 \end{pmatrix}$$

with $x \in \mathbb{C}^{k\times(k-1)}$ and $a \in \mathbb{C}^k$, where $a = 0$ for $\mathfrak{p}_1$ and where we delete the vanishing columns and rows which arise from the inclusion into $\mathbb{C}^p \times \mathbb{C}^q$. We will identify any such matrix with the corresponding pair (x, a) in $\mathbb{C}^{k\times(k-1)} \times \mathbb{C}^k$. There is a $k \in K_1$ such that $\mathrm{Ad}_k X_1$ is contained in the distinguished maximal Abelian subspace $\mathfrak{d}_1 \subseteq \mathfrak{p}_1$ which consists of matrices $(x, 0)$ such that $x_{ij} = 0$ for all $i \neq j$ and such that $x_{ii} \in \mathbb{R}$. Hence we can assume without loss of generality that $X_1 = (x, 0) \in \mathfrak{d}_1$.

To find an X_2 as asserted, we use the Ansatz $X_2 = (x, a)$. Then the commutator of X_2 with $Y = (y, 0) \in \mathfrak{p}_1$ is given by

$$[X_2, Y] = \begin{pmatrix} y^*x - x^*y & y^*a & 0 \\ -a^*y & 0 & 0 \\ 0 & 0 & yx^* - xy^* \end{pmatrix}.$$

The upper left and lower right entries correspond to the commutator $[X_1, Y]$. Now X_1 is regular in $\mathfrak{p}_1$. Hence $[X_1, Y] = 0$ implies that $Y \in \mathfrak{d}_1$ and then $a^*y = (\bar{a}_1 y_{11}, \dots, \bar{a}_{k-1} y_{k-1,k-1})$. Hence, if we choose the coordinates $a_1, \dots, a_{k-1}$ of a to be nonzero, then the maximal Abelian subspace $\mathfrak{a}_2$ in $\mathfrak{p}_2$ containing X_2 satisfies $\mathbb{C}\mathfrak{a}_2 \cap \mathfrak{p}_1 = \{0\}$. Thus $\mathbb{C}\mathfrak{a}_2$ is a complement of $\mathfrak{p}_1$ in $\mathfrak{p}_2$, by (6.18). $\square$

Now we enumerate the chains of groups $\mathrm{U}(k+l)$ and $\mathrm{U}(k) \times \mathrm{U}(l)$ further up consecutively by G_i and K_i and denote their Lie algebras by $\mathfrak{g}_i$ and $\mathfrak{k}_i$, for $1 \leq i \leq p+q-1$. We have the corresponding decompositions $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. Starting with a nonzero $X_1 \in \mathfrak{p}_1 = \mathbb{C}\mathfrak{a}_1 \cong \mathbb{C}$, we obtain from Lemma 6.10 a chain of regular vectors $X_i \in \mathfrak{p}_i$ such that $\pi_i X_j = X_i$ for all $i \leq j$ and such that the maximal Abelian subspace $\mathfrak{a}_i$ of $\mathfrak{p}_i$ containing X_i satisfies $\mathfrak{p}_i = \mathfrak{p}_{i-1} \oplus \mathbb{C}\mathfrak{a}_i$. In particular, we obtain a direct sum decomposition

$$\mathfrak{p}_{p+q-1} = \mathbb{C}\mathfrak{a}_1 \oplus \cdots \oplus \mathbb{C}\mathfrak{a}_{p+q-1}. \tag{6.20}$$

The proof of the following Lemma 6.12 is Exercise 6.24.

Lemma 6.21. *Let $G = \mathrm{U}(k+l)$ and $K = \mathrm{U}(k) \times \mathrm{U}(l)$ with $k \leq l$ and write $\mathfrak{u}(k+l) = \mathfrak{k} \oplus \mathfrak{p}$ as above. Then the polynomials*

$$\psi_j \colon \mathfrak{u}(k+l) \to \mathbb{R}, \quad \psi_j(X) = \mathrm{tr}((\sqrt{-1}X)^j),$$

where $1 \leq j \leq 2k$, are G-invariant with gradient $-j\sqrt{-1}(\sqrt{-1}X)^{j-1}$. Their gradients at any point X in any maximal Abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ are tangent to $\mathfrak{a}$ if j is even and to $\mathfrak{k}$ if j is odd. Moreover, the gradients are linearly independent at any regular X. $\square$

Corollary 6.4 together with (6.20) and Lemma 6.21 implies that there are pq smooth functions $f_{2j,i} = \psi_{2j} \circ \pi_i \circ F$ on $TG_{p,q}(\mathbb{C})$ which are in involution and whose symplectic gradients are linearly independent and horizontal at $[e, X_{p+q-1}]$; cf. Proposition 5.24. We will call them *functions of the first kind*; they correspond to the functions in the case of the real Grassmannians.

Now the dimension of $G_{pq}(\mathbb{C})$ is $2pq$ and hence we have only half of the number of functions we need for complete integrability. For the other half, we will consider the odd powers ψ_{2j+1}. They will give rise to *functions of* *the second kind* with vertical symplectic gradients. We write

$$X_{p+q-1} = \begin{pmatrix} 0 & 0 & -x^* & -z^* \\ 0 & 0 & -y^* & -u^* \\ x & y & 0 & 0 \\ z & u & 0 & 0 \end{pmatrix},$$

where $x \in \mathbb{C}^{k\times(k-1)}$ in the first case and $x \in \mathbb{C}^{l\times k}$ in the second, where y, z, and u are matrices with complex entries of appropriate sizes, and where X_i is obtained by setting y, z, and u equal to 0. The horizontal part of the symplectic gradient of $\psi_{2j+1} \circ \pi_i \circ F$ at $[e, X_{p+q-1}]$ vanishes and the vertical part is given by

$$\begin{aligned} &(-1)^j\sqrt{-1}\left[\begin{pmatrix} (x^*x)^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (xx^*)^j & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -x^* & -z^* \\ 0 & 0 & -y^* & -u^* \\ x & y & 0 & 0 \\ z & u & 0 & 0 \end{pmatrix}\right] \\ &\quad = (-1)^j\sqrt{-1}\begin{pmatrix} 0 & 0 & 0 & -(x^*x)^j z^* \\ 0 & 0 & y^*(xx^*)^j & 0 \\ 0 & (xx^*)^j y & 0 & 0 \\ -z(x^*x)^j & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now suppose that we are in the first case. Then perturbing X_{p+q-1} slightly if necessary, more precisely, perturbing x and y slightly if necessary, we get that xx^* is a Hermitian $(k \times k)$ matrix with one vanishing and $k-1$ pairwise different positive eigenvalues and that the k vectors $(xx^*)^j y_1$, $0 \le j \le k-1$, are linearly independent, where y_1 denotes the first column of y and where we note that $(xx^*)^0 = \mathrm{id}$ has rank k. Thus the first case gives rise to k further functions with vertical symplectic gradients at $[e, X_{p+q-1}]$ which lie in the kernel of π_i and such that their images under π_{i+1} are linearly independent. In the second case we also get k such functions. All in all, we get pq functions of the second kind, starting with the function we get from $\{1\} \times \mathrm{U}(1) \subseteq \mathrm{U}(1) \times \mathrm{U}(1)$ and ending with the p functions we get from $\mathrm{U}(p+q-1) \subseteq \mathrm{U}(p+q)$. Moreover, their symplectic gradients are vertical and linearly independent at $[e, X_{p+q-1}]$, and they are in involution among themselves and with the pq functions of the first kind.

Since the $\psi_j \circ \pi_i$ are polynomials, the $2pq$ functions $f_{ji} = \psi_j \circ \pi_i \circ F$ are real analytic. We conclude that their symplectic gradients are independent on an open and dense subset of $TG_{p,q}(\mathbb{C})$ of full measure. Together with Corollary 4.16 we arrive at

Theorem 6.22. *For any* $\mathrm{SU}(p+q)$*-invariant smooth function on* $TG_{p,q}(\mathbb{C})$*, the associated Hamiltonian system on* $TG_{p,q}(\mathbb{C})$ *is completely integrable with the above* $2pq$ *functions as a complete and involutive family of real analytic first integrals.* □

Remarks 6.23. 1) The proof of Theorem 6.22 shows that the polynomials $\psi_j \circ \pi_i$ also give rise to a complete and involutive family of real analytic first integrals for Hamiltonian systems associated to $\mathrm{SU}(p,q)$-invariant functions on the tangent bundle of the dual Grassmannian of negative p-planes in $\mathbb{C}^{p,q}$,

$$G^-_{p,q}(\mathbb{C}) = \mathrm{SU}(p,q)/S(\mathrm{U}(p) \times \mathrm{U}(q)).$$

Recall that $G_{1,q}$ and $G^-_{1,q}$ are the complex projective and the complex hyperbolic space of complex dimension q, respectively.

2) Note the shift in the retrieval of functions of the first and second kind. For functions of the first kind, they arise from invariant polynomials on $\mathfrak{u}(1+1), \dots, \mathfrak{u}(p+q)$, for functions of the second kind from invariant polynomials on $\mathfrak{u}(0+1), \dots, \mathfrak{u}(p+q-1)$. In fact, since the projective line $G_{1,1}(\mathbb{C}) \cong S^2$ has real dimension two, we already need two functions on that level. We retrieve one of the first kind from $\mathfrak{u}(1+1)$ and one of the second kind from $\mathfrak{u}(0+1)$. Note also that functions of the second kind from $\mathfrak{u}(p+q)$ would be constant since their symplectic gradients would vanish identically.

6.3. **Exercises.**

Exercise 6.24. Prove Lemma 6.12 and Lemma 6.21. Hint: For the claims about maximal Abelian subspaces of $\mathfrak{p}$, argue that it suffices to prove them for one such subspace and discuss your prefered one. For the claim about the linear independence, a Vandermonde determinant will come into play.

Exercise 6.25. Determine the sectional curvature of the sphere $G^o_{1,q}(\mathbb{R})$ and the real hyperbolic space $G^-_{1,q}(\mathbb{R})$ as in Remark 6.14 with respect to the inner product (6.6).

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