GEOMETRIC STRUCTURES

WERNER BALLMANN

This text is based on lectures at the University of Bonn. I discuss geometric structures with the main aim of proving a theorem of Singer on the local homogeneity of Riemannian manifolds [Si] and Gromov's Open Orbit Theorem [Gr]. Recall that the latter theorem is crucial in the celebrated work of Benoist, Foulon and Labourie on the regularity of stable foliations of contact Anosov flows, see [BFL]. This work was one of my main motivations for discussing geometric structures.

The size of the notes indicates that they do not contain a comprehensive introduction into the field. The section on conformal structures is not complete yet. I also intend to add, at some later point, a discussion of projective structures, Cartan connections, and of Killing fields of geometric structures.

Anyway, much more can be said about geometric structures than is revealed here. Besides Gromov's article [Gr] and the survey [DG] of this article, the papers [Be] by Benoist and [Ze] by Zeghib contain discussions of Gromov's work in [Gr]. In particular, [Be] contains a proof of the Open Orbit Theorem and [Ze] a proof of this theorem in the case needed in [BFL]. References for further reading are Chern's article [Ch] and Kobayashi's monograph [Ko]. The recent paper [We] of Weingart contains interesting new ideas.

I would like to thank Anna Wienhard for helpful remarks. Further criticism is welcome.

Date: July 26, 2000.

Contents

1. Preliminaries 2
Smooth Structure 3
Holonomic Sections 4
$\mathrm{Gl}^k(m)$ and Frames 5
Translations 6
2. Geometric Structures 7
Local Form 8
Higher Order 8
3. Rigid Geometric Structures 10
The Case $r \geq k$ 12
Cross Sections 14
4. The Automorphism Relation 17
5. Singer's Homogeneity Theorem 21
6. Gromov's Open Orbit Theorem 22
7. Example: Conformal Structures 25
8. Appendix: Partial Differential Relations 27
Completeness 27
Consistence 28
References 31

1. Preliminaries

Let M, N be smooth manifolds. We say that a map f is a *local map from M to N* if the domain of f is an open subset of M and the target of f is N or an open subset of N. Let $p \in M$ and $f : U \to N$, $g : V \to N$ be local maps from M to N with $p \in U \cap V$. We say that f and g have *contact of order* $k \geq 0$ at p if $f(p) = g(p)$ and if, with respect to local coordinates x about p and y about $f(p)$, the first k derivatives of $y \circ f \circ x^{-1}$ and $y \circ g \circ x^{-1}$ coincide in $x(p)$. This is independent of the choice of coordinates and defines an equivalence relation on the space of local maps f containing p in their domain of definition. The equivalence class of f is denoted $j^k(f, p)$ and is called the *k-jet of f at p*. The space of all equivalence classes is denoted $J^k_p(M, N)$ and we set

$$J^k(M, N) = \cup_{p \in M} J^k_p(M, N) \,.$$

We keep the notation somewhat flexible in accordance with readability. For example, the k–jet of f at p is also denoted $j^k_p(f)$ or $j^k_f(p)$. We call p the *foot point* of $j^k_p(f)$.

Let $0 \leq l \leq k$ and $p \in M$. It is plain that $j^l(f,p) = j^l(g,p)$ if $j^k(f,p) = j^k(g,p)$. Hence there is a well defined projection

$$J^k(M,N) \to J^l(M,N)\,, \quad j^k(f,p) \mapsto j^l(f,p)\,. \tag{1.1}$$

There are also the following natural projections,

$$\begin{aligned} J^k(M,N) &\to M\,, \quad j^k(f,p) \mapsto p\,, \\ J^k(M,N) &\to N\,, \quad j^k(f,p) \mapsto f(p)\,. \end{aligned} \tag{1.2}$$

Via $j^0(f,p) = (p, f(p))$ we identify $J^0(M,N)$ with $M \times N$. Via $j^1(f,p) = (p, f(p), f_*(p))$ we identify $J^1(M,N)$ with the space of all triples (p,q,L), where p is a point in M, q is a point in N and $L : T_pM \to T_qN$ is a linear map.

Smooth Structure. We now show that $J^k(M,N)$ has a natural smooth structure. The idea is the same as that in the definition of the smooth structure on the tangent bundle. Let $T_0^k(m,n)$ be the vector space of all polynomial maps T of degree at most k from $\mathbb{R}^m$ to $\mathbb{R}^n$ with $T(0) = 0$. Let (x,U) and (y,V) be local coordinates in M and N respectively. Then $J^k(U,V) \subset J^k(M,N)$ consists of all k-jets $j^k(f,p)$ with $p \in U$ and $f(p) \in V$. Define (candidates for) coordinates

$$\begin{aligned} &J^k(U,V) \to x(U) \times y(V) \times T_0^k(m,n)\,, \\ &j^k(f,p) \mapsto (u, \varphi(u), T_u^k(\varphi))\,, \end{aligned} \tag{1.3}$$

where $u = x(p)$, $\varphi = y \circ f \circ x^{-1}$ and $T = T_u^k(\phi)$ is the Taylor polynomial of order k of ϕ in u. That is, we have

$$\varphi(u+h) = \varphi(u) + T(h)$$

up to terms of order $> k$. The chain rule implies that coordinate changes between coordinates of the above type are smooth. It follows easily that $J^k(M,N)$ has precisely one topology such that the coordinates of the above type induce a smooth structure. As a rule, claims about local properties of maps and sets are easy to check in these coordinates. It is usually also helpful to adapt the coordinates x on M and y on N to the situation.

The projections in (1.1) and (1.2) are smooth fiber bundles. For all points $p \in M$, $J_p^k(M,N)$ is a smooth submanifold of $J^k(M,N)$. Similarly, for all points $p \in M$ and $q \in N$, the subspace $J_{pq}^k(M,N) \subset J_p^k(M,N)$ of jets $j_f^k(p)$ with $f(p) = q$ is a smooth submanifold of $J_p^k(M,N)$ and of $J^k(M,N)$.

1.4. Example. As a manifold, $TM = J_0^1(\mathbb{R}, M)$.

Sometimes it is also helpful to consider the direct sum of vector spaces $S^i(m,n)$ of symmetric i–linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ instead of the space $T_0^k(m.n)$ of polynomials as above. This amounts to a different view of the local coordinates in (1.3),

$$\begin{aligned} &J^k(U,V) \to x(U) \times y(V) \times \oplus_{1\le i\le k} S^i(m,n)\,, \\ &j^k(f,p) \mapsto (u, \varphi(u), \varphi'(u), \dots, \varphi^{(k)}(u))\,, \end{aligned} \tag{1.5}$$

where $u = x(p)$ and $\varphi = y \circ f \circ x^{-1}$. Switching from the first type of coordinates to this type amounts to polarization of $T_u^k(\varphi)$. With respect to such coordinates, elements of $J^k(U,V) \subset J^k(M,N)$ are of the form

$$(u, v, \varphi_1, \dots, \varphi_k)$$

with $u \in x(U)$, $v \in y(V)$ and $\varphi_i \in \mathrm{Sym}^i(m,n)$, $1 \le i \le k$.

Many of our considerations are of a local nature. In such considerations we simply view M as an open subset of $\mathbb{R}^m$, N as an open subset of $\mathbb{R}^n$ and denote points of M by the letter u, points of N by the letter v. We indicate the change to this point of view by writing “in terms of local coordinates ”.

Holonomic Sections. We let $Z = J^k(M,N)$ and $\pi : Z \to M$ be the projection to the foot point. A local section s of $\pi : Z \to M$ is called *holonomic* if $s = j_f^k$ for some local map f from M to N. Similarly, a linear map $L : T_pM \to T_zZ$ is called *holonomic* if $L = s'(p)$, where $s = j_f^k$ is a holonomic section. Then $\pi(z) = p$ and $z = j_f^k(p)$.

In terms of local coordinates, local sections of π are local maps of the form

$$s(u) = (u, \varphi_0(u), \varphi_1(u), \dots, \varphi_k(u)) \tag{1.6}$$

with $\varphi_0(u) \in \mathbb{R}^n$ and $\varphi_i(u) \in S^i(m,n)$, $1 \le i \le k$. If $s = j_\varphi^k$ is a holonomic section, then we have

$$\begin{aligned} s(u) &= (u, \varphi_0(u), \varphi_1(u), \dots, \varphi_k(u)) \\ &= (u, \varphi(u), \varphi'(u), \dots, \varphi^{(k)}(u)) = j_\varphi^k(u)\,. \end{aligned} \tag{1.7}$$

Similarly, suppose that $L : T_uM \to T_zZ$ is a holonomic linear map,

$$z = j_\varphi^k(u) = (u, \varphi(u), \varphi'(u), \dots, \varphi^{(k)}(u))$$

and

$$L = (j_\varphi^k)'(u) = (\mathrm{id}, \varphi'(u), \dots, \varphi^{(k+1)}(u))\,.$$

The latter equation has to be understood in the usual way,

$$L(\xi) = (\xi, \xi \llcorner \varphi'(u), \dots, \xi \llcorner \varphi^{(k+1)}(u))\,,$$

where, for $\xi \in \mathbb{R}^m$ and $\psi \in \mathrm{Sym}^{i+1}(m,n)$, $\xi \llcorner \psi \in \mathrm{Sym}^i(m,n)$ is given by

$$(\xi \llcorner \psi)(\xi_1, \dots, \xi_i) = \psi(\xi, \xi_1, \dots, \xi_i)\,.$$

Since the information on L also includes the foot point z, we see that the holonomic map $L = (j^k_\varphi)'(u)$ is nothing else but an interpretation of the $(k+1)$–jet $\tilde{z} = j^{k+1}_\varphi(u)$ of the local map φ in question. In other words, we may view a $(k+1)$–jet $\tilde{z}$ over z as a linear map $T_pM \to T_zZ$. In terms of local coordinates, if

$$\tilde{z} = (u, v, \varphi_1, \dots, \varphi_k, \varphi_{k+1})$$

is a $(k+1)$–jet over

$$z = (u, v, \varphi_1, \dots, \varphi_k)\,,$$

then the corresponding holonomic linear map $T_uM \to T_zZ$ is

$$\xi \mapsto (\xi, \xi \llcorner \varphi_1, \xi \llcorner \varphi_2, \dots, \xi \llcorner \varphi_k, \xi \llcorner \varphi_{k+1})\,.$$

This point of view will be relevant in some instances.

1.8. Lemma. *If s is a local section of π and $s'(p)$ is holonomic for all p in the domain of s, then s is holonomic.*

Proof. In terms of local coordinates, suppose that

$$s(u) = (u, \varphi_0(u), \varphi_1(u), \dots, \varphi_k(u))\,.$$

Then

$$s'(u)(\xi) = (\xi, \varphi_0'(u)(\xi), \dots, \varphi_k'(u)(\xi))\,.$$

Now $s'(u)$ is holonomic for all u, hence by what we said above, $\varphi_i'(u)(\xi) = \xi \llcorner \varphi_{i+1}(u)$ for all u and i, $0 \le i \le k-1$. Hence s is holonomic. □

$\mathrm{Gl}^k(m)$ and Frames. Let $k \ge 0$ and $\mathrm{Gl}^k(m)$ be the space of all k–jets of local diffeomorphisms of $\mathbb{R}^m$ fixing the origin $0 \in \mathbb{R}^m$. Composition turns $\mathrm{Gl}^k(m)$ into a Lie group,

$$j^k_0(\varphi) \cdot j^k_0(\psi) = j^k_0(\varphi \circ \psi)\,. \tag{1.9}$$

Since the origin is fixed, we can identify $j^k_0(\varphi) \in \mathrm{Gl}^k(m)$ with the Taylor polynomial $T^k_0(\varphi) \in T^k_0(m,m)$ of φ at 0. With respect to this identification, $\mathrm{Gl}^k(m)$ consists of the open subset of polynomials $T \in T^k_0(m,m)$ such that $T'(0)$ is invertible. Multiplication in $Gl^k(m)$ corresponds to composition of polynomials in $T^k_0(m,m)$, where terms of order higher than k are cancelled. In this sense, $T^k_0(m,m) = \mathfrak{gl}^k(m)$, the Lie algebra of $\mathrm{Gl}^k(m)$.

By definition, $\mathrm{Gl}^0(m)$ is the trivial group. We view $\mathrm{Gl}^1(m)$ as the general linear group $\mathrm{Gl}(m)$ of invertible $(m \times m)$–matrices with real entries and, more generally, $\mathrm{Gl}^k(m)$ as the space of all k–tuples

$$\mathrm{Gl}^k(m) = \{(a_1, \dots, a_k)\}\,, \tag{1.10}$$

where $a_1 \in \mathrm{Gl}(m)$ and $a_i \in \mathrm{Sym}^i(m,m)$, $2 \le i \le k$. In this notation, multiplication in $\mathrm{Gl}^2(m)$ is given by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot b_1, a_1 \cdot b_2 + a_2 \cdot (b_1, b_1))\,,$$

and there are similar formulas for $k \ge 3$.

Let $p \in M$. A *frame of order k at p* is the k-jet $j^k(f,0)$ of a local diffeomorphism f from $\mathbb{R}^m$ to M with $f(0) = p$. The space of frames of order k at p is denoted $\mathrm{Gl}^k_p(M)$ and we set

$$\mathrm{Gl}^k(M) = \cup_{p \in M} \mathrm{Gl}^k_p(M)\,.$$

With respect to the smooth structure on $J^k(\mathbb{R}^m, M)$ defined above, $\mathrm{Gl}^k(M)$ is an open subset of the submanifold $J^k_0(\mathbb{R}^m, M)$.

The projection $\mathrm{Gl}^k(M) \to M$ with fibres $\mathrm{Gl}^k_p(M)$ is a principal bundle with structure group $\mathrm{Gl}^k(m)$. Here right multiplication by $j^k_0(\varphi)$ is defined by right composition as above,

$$j^k_0(f) \cdot j^k_0(\varphi) = j^k_0(f \circ \varphi)\,. \tag{1.11}$$

Translations. By $\tau(u)$, $u \in \mathbb{R}^m$, we denote the translation by u,

$$\tau_u = \tau(u) : \mathbb{R}^m \to \mathbb{R}^m\,, \quad \tau_u(h) = u + h\,. \tag{1.12}$$

It usually serves as the preferred smooth map sending $0 \in \mathbb{R}^m$ to u.

For a local diffeomorphism φ of $\mathbb{R}^m$ and $u \in \mathbb{R}^m$ in the domain of φ, we let

$$\begin{aligned} g^k(\varphi, u) &= j^k_0(\tau(-\varphi(u)) \circ \varphi \circ \tau(u)) \\ &= j^k_0(h \mapsto \varphi(u+h) - \varphi(u)) \in \mathrm{Gl}^k(m)\,. \end{aligned} \tag{1.13}$$

We may think of $g^k(\varphi, u)$ as the Taylor polynomial of φ at u. Whenever convenient, we also write $g^k_u(\varphi)$ or $g^k_\varphi(u)$ instead of $g^k(\varphi, u)$.

Let ψ be another local diffeomorphism of $\mathbb{R}^m$. Then the chain rule tells us that

$$g^k(\varphi \circ \psi, u) = g^k(\varphi, \psi(u)) \cdot g^k(\psi, u) \tag{1.14}$$

for all u in the domain of $\varphi \circ \psi$. We also have

$$g^k(\varphi, u)^{-1} = g^k(\varphi^{-1}, \varphi(u))\,. \tag{1.15}$$

2. Geometric Structures

Suppose Σ is a smooth manifold and $\lambda : \mathrm{Gl}^k(m) \times \Sigma \to \Sigma$ is a smooth action. Whenever convenient we also write gs or $g \cdot s$ instead of $\lambda(g, s)$.

2.1. Definition. A *geometric structure of type* λ on M is an equivariant map $\sigma : \mathrm{Gl}^k(M) \to \Sigma$, that is, we require $\sigma(x \cdot g) = g^{-1} \cdot \sigma(x)$ for all $x \in \mathrm{Gl}^k(M)$ and $g \in \mathrm{Gl}^k(m)$. The number k is called the *order of* σ.

We may interpret geometric structures of type λ as sections of the associated fiber bundle $\mathrm{Gl}^k(M) \times_\lambda \Sigma$.

Frequently, there is no need to specify the particular λ and then we simply speak of a *geometric structure*.

Let σ be a structure of type λ on M and k be the order of σ. A *(local) automorphism of* σ is a (local) diffeomorphism F of M such that $\sigma \circ F_* = \sigma$, where F_* denotes the map of $Gl^k(M)$ induced by F, $F_*(j_0^k(f)) = j_0^k(F \circ f)$. The (pseudo) group of (local) automorphisms of σ is denoted $\mathrm{Aut}(\sigma)$ (respectively $\mathrm{Aut}^{loc}(\sigma)$).

2.2. Example (Parallelizations). Let $\Sigma = \mathrm{Gl}^1(m) = Gl(m)$ and λ be the action of $\mathrm{Gl}(m)$ on itself by left translations. Then a geometric structure σ on M of type λ corresponds to a parallelization Φ of M: Define $\Phi(p)$ to be the unique frame $j_0^1(\varphi)$ at p with $\sigma(j_0^1(\varphi)) = 1$.

2.3. Example (Connections). Let Σ be the vector space of bilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^m$ and, for $g = (a_1, a_2) \in \mathrm{Gl}^2(m)$ and $\Gamma \in \Sigma$, let

$$\lambda(g, \Gamma)(\xi, \eta) = a_1 \cdot \Gamma(a_1^{-1}\xi, a_1^{-1}\eta) - a_2(a_1^{-1}\xi, a_1^{-1}\eta)\,.$$

This is the transformation rule for the Christoffel symbols of a connection under a change of coordinates. In other words, connections are geometric structures of order 2.

2.4. Example (Riemannian Metrics). Let Σ_+ the space of inner products on $\mathbb{R}^m$, where $\mathrm{Gl}^1(m) = \mathrm{Gl}(m)$ acts by

$$\lambda(a, s)(\xi, \eta) = s(a^{-1}\xi, a^{-1}\eta)\,.$$

In this case, a geometric structure σ of type λ is the same as a Riemannian metric on M: View $\sigma(j_0^1(f))$ as the fundamental matrix of the metric with respect to the frame $j_0^1(f)$.

The same λ also works when we replace Σ_+ by the vector space Σ of all symmetric bilinear forms on $\mathbb{R}^m$. The corresponding geometric structures are symmetric $(2, 0)$–tensor fields.

2.5. Example (G–Structures). Let $G \subset \mathrm{Gl}^k(m)$ be a closed subgroup. By definition, a *G–structure on M* is a principal subbundle $P \subset \mathrm{Gl}^k(M)$

with induced structure group G. We may view G–structures as geometric structures: We let $\Sigma = \mathrm{Gl}^k(m)/G$ with the natural $\mathrm{Gl}^k(m)$–action by left translations and define σ by

$$\sigma(j^k_0(f)) = [G] \in \Sigma \quad \text{iff} \quad j^k_0(f) \in P\,.$$

Riemannian metrics are geometric structures of this kind, connections are not.

2.6. Exercise. Show that distributions and tensor fields are geometric structures of order 1. A distribution of rank d is a $\mathrm{Gl}(d) \times \mathrm{Gl}(m-d)$–structure. Give an example of a tensor field which is not a G–structure.

In what follows, σ denotes a structure of type λ on M and k denotes the order of σ.

Local Form. Suppose $x : U' \to U \subset \mathbb{R}^m$ is a local coordinate chart of M. Then we define the *local form* $\sigma_x : U \to \Sigma$ of σ with respect to x by

$$\begin{aligned} \sigma_x(u) &= \sigma(j^k_0(x^{-1} \circ \tau(u))) \\ &= \sigma(j^k_0(h \mapsto x^{-1}(u+h)))\,. \end{aligned} \tag{2.7}$$

Note that σ is only defined on jets taken at 0, so that $\tau(u)$ enters into the definition. If $y : V' \to V$ is another local coordinate chart of M, $\varphi := y \circ x^{-1}$ and u is in the domain of φ, then

$$\begin{aligned} j^k_0(x^{-1} \circ \tau(u)) &= j^k_0(y^{-1} \circ y \circ x^{-1} \circ \tau(u)) \\ &= j^k_0(y^{-1} \circ \tau(\varphi(u))) \cdot g^k_\varphi(u)\,. \end{aligned}$$

Therefore the *transformation rule* for local forms is

$$g^k_\varphi(u) \cdot \sigma_x(u) = \sigma_y(\varphi(u))\,. \tag{2.8}$$

Vice versa, any system (σ_x) of local forms satisfying this rule gives rise to a geometric structure σ of type λ.

Higher Order. We obtain geometric structures of higher order by taking derivatives. Suppose that $\lambda : \mathrm{Gl}^k(m) \times \Sigma \to \Sigma$ is an action. Let $r \geq k$ and set $l = r - k \geq 0$ and $\Sigma^l = J^l_0(\mathbb{R}^m, \Sigma)$. For $j^r_0(\varphi) \in \mathrm{Gl}^r(m)$ and $j^l_0(s) \in \Sigma^l$, set

$$\begin{aligned} \lambda^l(j^r_0(\varphi), j^l_0(s)) :&= j^l_0((g^k_\varphi \cdot s) \circ \varphi^{-1}) \\ &= j^l_0(u \mapsto g^k_\varphi(\varphi^{-1}(u)) \cdot s(\varphi^{-1}(u)))\,. \end{aligned} \tag{2.9}$$

2.10. Lemma. *λ^l is an action of* $\mathrm{Gl}^r(m)$ *on* Σ^l.

Proof. We compute

$$\begin{aligned}(j_0^r(\varphi)j_0^r(\psi))\cdot j_0^l(s) &= j_0^r(\varphi\psi)\cdot j_0^l(s) = j_0^l((g^k_{\varphi\psi}\cdot s)\circ(\varphi\psi)^{-1})\\ &= j_0^l(u\mapsto g^k_\varphi(\varphi^{-1}(u))\cdot g^k_\psi(\psi^{-1}\varphi^{-1}(u))\cdot s(\psi^{-1}\varphi^{-1}(u)))\\ &= j_0^r(\varphi)\cdot j_0^l(v\mapsto g^k_\psi(\psi^{-1}(v))\cdot s(\psi^{-1}(v)))\\ &= j_0^r(\varphi)\cdot(j_0^r(\psi)\cdot j_0^l(s))\,,\end{aligned}$$

hence the claim. □

Let $g=(a_1,\dots,a_r)\in Gl^r(m)$ and $j_0^l(s)\in\Sigma^l$. Then in terms of local coordinates on Σ about $s(0)$ and $g\cdot s(0)$ respectively, we have

$$\lambda^l((a_1,\dots,a_r),(s_0,\dots,s_l)) = (\tilde s(0),\tilde s'(0),\dots,\tilde s^{(l)}(0))\,, \tag{2.11}$$

where

$$\tilde s(u)=\lambda(g^k_\varphi(\varphi^{-1}(u)),s(\varphi^{-1}(u)))\,.$$

Frequently, Σ is an open subset in a Euclidean space and then we use (2.11) to compute λ^l.

2.12. Example (Riemannian Metrics). Let Σ be the vector space of symmetric bilinear forms on $\mathbb{R}^m$ and λ be as in Example 2.4. We compute the action of $j_0^2(\varphi)=(a_1,a_2)\in \mathrm{Gl}^2(m)$ on $j_0^1(s)=(s_0,s_1)\in\Sigma^1=J_0^1(\mathbb{R}^m,\Sigma)$. We have

$$(a_1,a_2)\cdot(s_0,s_1)=(a_1\cdot s_0,\tilde s_1)\,,$$

where $a_1\cdot s_0=\lambda(a_1,s_0)$ is as before. To compute $\tilde s_1$ we fix $\xi_1,\xi_2\in\mathbb{R}^m$ and consider the map

$$u\mapsto s(\varphi^{-1}(u))(\varphi'(\varphi^{-1}(u))^{-1}(\xi_1),\varphi'(\varphi^{-1}(u))^{-1}(\xi_2))\,.$$

The derivative at $u=0$ in the direction of $\xi_0\in\mathbb{R}^m$ is

$$\begin{aligned}\tilde s_1(\xi_0)(\xi_1,\xi_2) &= s_1(a_1^{-1}\xi_0)(a_1^{-1}\xi_1,a_1^{-1}\xi_2)\\ &\quad - s_0(a_1^{-1}a_2(a_1^{-1}\xi_0,a_1^{-1}\xi_1),a_1^{-1}\xi_2)\\ &\quad - s_0(a_1^{-1}\xi_1,a_1^{-1}a_2(a_1^{-1}\xi_0,a_1^{-1}\xi_2))\,.\end{aligned}$$

In particular, if $a_1=1$ and $a_2=:a$, then $\tilde s_1-s_1$ is given by

$$-s_0(a(\xi_0,\xi_1),\xi_2)-s_0(\xi_1,a(\xi_0,\xi_2))\,.$$

Now suppose that $\sigma:\mathrm{Gl}^k(M)\to\Sigma$ is a geometric structure on M of order k. We obtain a smooth map $\sigma^l:\mathrm{Gl}^r(M)\to\Sigma^l$ by setting

$$\sigma^l(j_0^r(f)) = j_0^l(u\mapsto\sigma(j_0^k(f\circ\tau(u))))\,. \tag{2.13}$$

2.14. Lemma. *σ^l is $\mathrm{Gl}^r(m)$–equivariant, hence a geometric structure of order r.*

Proof. We recall that for $j_0^r(\varphi) \in \mathrm{Gl}^r(m)$

$$g_u^k(\varphi)^{-1} = g_{\varphi(u)}^k(\varphi^{-1}) .$$

We also have

$$j_0^k(f \circ \varphi \circ \tau(u)) = j_0^k(f \circ \tau(\varphi(u))) \cdot g_u^k(\varphi) .$$

Hence since σ in a geometric structure,

$$\begin{aligned}
\sigma^l(j_0^r(f) \cdot j_0^r(\varphi)) &= \sigma^l(j_0^r(f \circ \varphi)) \\
&= j_0^l(u \mapsto \sigma(j_0^k(f \circ \varphi \circ \tau(u)))) \\
&= j_0^l(u \mapsto \sigma(j_0^k(f \circ \tau(\varphi(u)))) \cdot g_u^k(\varphi)) \\
&= j_0^l(u \mapsto g_u^k(\varphi)^{-1} \cdot \sigma(j_0^k(f \circ \tau(\varphi(u))))) \\
&= j_0^l(u \mapsto g_{\varphi(u)}^k(\varphi^{-1}) \cdot \sigma(j_0^k(f \circ \tau(\varphi(u))))) \\
&= j_0^r(\varphi^{-1}) \cdot j_0^l(v \mapsto \sigma(j_0^k(f \circ \tau(v)))) \\
&= j_0^r(\varphi)^{-1} \cdot \sigma^l(j_0^r(f)) .
\end{aligned}$$

We conclude that σ^l is $\mathrm{Gl}^r(m)$–equivariant. □

If x is a coordinate chart on M, then the local form of σ^l with respect to x is

$$\sigma_x^l(u) = j_0^l(h \mapsto \sigma(j_0^k(x^{-1} \circ \tau(u+h)))) . \tag{2.15}$$

That is, in terms of local coordinates on Σ,

$$\sigma_x^l(u) = (\sigma_x(u), \sigma_x'(u), \ldots, \sigma_x^{(l)}(u)) . \tag{2.16}$$

This formula will be used in computations.

3. Rigid Geometric Structures

It is well known that the group of isometries of a Riemannian metric on M is a Lie group with respect to the compact–open topology. On the other hand, the group of automorphisms of a vector field X on M, that is, the group of diffeomorphisms of M commuting with the flow of X, is infinite dimensional in many cases. We are interested in the question when $\mathrm{Aut}(\sigma)$ is finite dimensional in a natural topology. This leads us to the notion of rigidity of geometric structures.

We let $\sigma : \mathrm{Gl}^k(M) \to \Sigma$ be a geometric structure on M. By $D^k(M) \subset J^k(M, M)$ we denote the open subset of k-jets of local diffeomorphisms of M. For points $p, q \in M$, we set

$$\begin{aligned}
\mathrm{Aut}_{pq}^k(\sigma) := \{ j_p^k(F) \in \mathrm{D}^k(M) \mid F(p) = q \text{ and} \\
\sigma(j_0^k(F \circ f)) = \sigma(j_0^k(f)) \text{ for all } f \in \mathrm{Gl}_p^k(M) \} .
\end{aligned} \tag{3.1}$$

By the equivariance of σ, we can replace the term "for all"on the right hand side by "for one". We also set

$$\mathrm{Aut}^k(\sigma) := \cup_{p,q\in M} \mathrm{Aut}^k_{pq}(\sigma)\,. \tag{3.2}$$

For $r \geq k$ and $l = r - k$, we set $\mathrm{Aut}^r_{pq}(\sigma) := \mathrm{Aut}^r_{pq}(\sigma^l)$ and define $\mathrm{Aut}^r(\sigma)$ correspondingly. Similar notation will be used for automorphisms and local automorphisms of σ. For example, $\mathrm{Aut}^{loc}_{pq}(\sigma)$ will denote the set of local automorphisms of σ mapping p to q.

Recall that an isometry of a connected Riemannian manifold is completely determined by its 1–jet at one point. Now the idea behind rigidity is that for some r, the $(r+1)$–jet of a local automorphism F of σ is determined by its r–jet. More precisely, in the case $r \geq k$ we require that the projection $\mathrm{Aut}^{r+1}_{pp}(\sigma) \to \mathrm{Aut}^r_{pp}(\sigma)$ is injective for all points $p \in M$.

A straightforwarded computation shows that $j^k_p(F) \in \mathrm{Aut}^k_{pp}(\sigma)$ if and only if for one (or any) $x = j^k_0(f) \in \mathrm{Gl}^k_p(M)$, $j^k_0(f^{-1}Ff) \in \mathrm{Gl}^k(m)$ is in the stabilizer

$$\mathrm{Stab}(y) = \{g \in \mathrm{Gl}^k(m) \mid gy = y\} \tag{3.3}$$

of $y = \sigma(x) \in \Sigma$. The corresponding statement holds for $j^r_p(F)$, $x = j^r_0(f)$ and $y = \sigma^l(x)$, $l = r - k$.

Let $\Delta^{r+1}(m)$ be the kernel of the projection $\mathrm{Gl}^{r+1}(m) \to \mathrm{Gl}^r(m)$. Then $\Delta^1(m) = \mathrm{Gl}(m)$ and $\Delta^2(m) = \{(1,a) \mid a \in \mathrm{Sym}^2(m,m)\}$. More generally, if 1_r denotes the r–tupel $(1,0,\dots,0)$, then

$$\Delta^{r+1}(m) = \{(1_r, a) \mid a \in \mathrm{Sym}^{r+1}(m,m)\}\,. \tag{3.4}$$

For $r \geq 1$ we have

$$(1_r, a)\cdot(1_r, b) = (1_r, a+b)\,. \tag{3.5}$$

We see that $\Delta^{r+1}(m)$ is abelian if $r \geq 1$.

In terms of stabilizers, our requirement on injectivity of the projection $\mathrm{Aut}^{r+1}_{pp}(\sigma) \to \mathrm{Aut}^r_{pp}(\sigma)$ means that

$$\Delta^{r+1}(m) \cap \mathrm{Stab}^{r+1}(\sigma^{l+1}(x)) = \{1\} \tag{3.6}$$

for all $p \in M$ and $x = j^{r+1}_p(f) \in \mathrm{Gl}^{r+1}_p(M)$. In other words, the requirement on injectivity means that $\Delta^{r+1}(m)$ acts freely on the image of σ^{l+1}, $l = r - k$. This requirement also makes sense in the case $r = k - 1$ and leads to our main definition.

3.7. Definition. Let $r \geq k - 1$ and $l = r - k \geq -1$. We say that a geometric structure $\sigma : \mathrm{Gl}^k(M) \to \Sigma$ is *r–rigid* if there is an open $\mathrm{Gl}^{r+1}(m)$–invariant neighborhood Σ^{l+1}_0 of the image of σ^{l+1} in Σ^{l+1} such that the action of $\Delta^{r+1}(m)$ on Σ^{l+1}_0 is proper and free.

By *proper* we mean that for any compact subset $K \subset \Sigma_0^{l+1}$, the set of $g \in \Delta^{r+1}(m)$ with $gK \cap K \neq \emptyset$ is compact in $\Delta^{r+1}(m)$.

3.8. Example (Parallelizations). The action of $\Delta^1(m) = \mathrm{Gl}(m)$ on itself by left translation is proper and free. Hence parallelizations are 0–rigid.

3.9. Example (Connections). Let Σ be the vector space of bilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^m$ with the action of $\mathrm{Gl}^2(m)$ as in Example 2.3. Now $(1,a) \in \Delta^2(m)$ acts on $\Gamma \in \Sigma$ by

$$(1,a) \cdot \Gamma = \Gamma - a\,.$$

This action is proper and free, hence connections are 1–rigid.

3.10. Example (Riemannian Metrics). Let Σ_+ be the space of inner products on $\mathbb{R}^m$ with the action of $\mathrm{Gl}^1(m)$ as in Example 2.4. In Example 2.12 above we computed the action of $\Delta^2(m)$ on Σ_+^1. We see that $(1,a) \in \Delta^2(m)$ is in the stabilizer of $(s_0, s_1) \in \Sigma_+^1$ if and only if

$$s_0(a(\xi_0,\xi_1),\xi_2) + s_0(\xi_1, a(\xi_0,\xi_2)) = 0$$

for all $\xi_0, \xi_1, \xi_2 \in \mathbb{R}^m$. Now s_0 is an inner product and a is a symmetric bilinear form with values in $\mathbb{R}^m$. It follows easily that $a = 0$. Hence Riemannian metrics are 1–rigid.

The Case $r \geq k$. We assume $r \geq k$ and discuss the action of $\Delta^{r+1}(m)$ on Σ^{l+1}, $l = r-k$, in more detail. We do not assume that our geometric structure is r–rigid.

Let $j_0^{l+1}(s) \in \Sigma^{l+1}$ and $\mathrm{j}_0^{r+1}(\varphi) \in \mathrm{Gl}^{r+1}(m)$. Let $P : \Sigma^{l+1} \to \Sigma^l$ be the projection. Then

$$P(j_0^{r+1}(\varphi) \cdot j_0^{l+1}(s)) = j_0^r(\varphi) \cdot j_0^l(s)\,. \tag{3.11}$$

We conclude that the action of $\Delta^{r+1}(m)$ does not affect $j_0^l(s)$. In other words, in terms of local coordinates on Σ we may consider the action locally about $\Delta^{r+1}(m) \subset \mathrm{Gl}^{r+1}(m)$ as an action on

$$W \times \big(\oplus_{i=1}^l \mathrm{Sym}^i(m,n)\big) \oplus \mathrm{Sym}^{l+1}(m,n)\,, \tag{3.12}$$

where $W \subset \mathbb{R}^n$ is an open subset, $n = \dim \Sigma$. All that matters is the $(l+1)$–st derivative of

$$u \mapsto \lambda\big(g_\varphi^k(\varphi^{-1}(u)), s(\varphi^{-1}(u))\big) \tag{3.13}$$

at $u = 0$, where we assume $\varphi'(0) = 1$ and $\varphi^{(i)}(0) = 0$ for $2 \leq i \leq r$. Now the derivative of φ^{-1} at u is $\varphi'(\varphi^{-1}(u))^{-1}$, hence the first derivative of the above map is

$$\sum_{i=1}^{k}(\partial_i\lambda)\big(g^k_\varphi(\varphi^{-1}(u)),s(\varphi^{-1}(u))\big)\circ(\varphi^{(i)})'(\varphi^{-1}(u))\circ\varphi'(\varphi^{-1}(u))^{-1}$$
$$+(\partial_s\lambda)\big(g^k_\varphi(\varphi^{-1}(u)),s(\varphi^{-1}(u))\big)\circ(s\circ\varphi^{-1})'(u)\,,$$

where $\partial_i\lambda$ and $\partial_s\lambda$ denote the partial derivative of λ in the direction of $\mathrm{Sym}^i(m,m)$ and Σ respectively. We need to differentiate all these terms l more times and evaluate the result at $u=0$. By the condition $j_0^{r+1}(\varphi)\in\Delta^{r+1}(m)$, only the term

$$(\partial_k\lambda)(1_k,s(0))\circ(\varphi^{(k)})^{(l+1)}(0) \tag{3.14}$$

survives among the terms from the sum. Note that with the usual identification $(\varphi^{(k)})^{(l+1)}(0)=\varphi^{(k+l+1)}(0)$.

The second term involving $\partial_s\lambda$ is a complicated expression involving higher derivatives of $\partial_s\lambda$ and $(s\circ\varphi^{-1})'$. However, this term has to simplify to $s^{(l+1)}(0)$ since it does not involve derivatives of φ of order $r+1$ or higher and therefore has to give the same expression as in the case where $\varphi^{(r+1)}(0)=0$. We summarize the result of our computation in the following lemma.

3.15. Lemma. *Let $r\geq k$ and view Σ^{l+1} as the space of holonomic linear maps from $\mathbb{R}^m=T_0\mathbb{R}^m$ to $T\Sigma^l$. Then the action of $\Delta^{r+1}(m)$ on Σ^{l+1} is affine on the fibers of $P:\Sigma^{l+1}\to\Sigma^l$. In fact, in terms of local coordinates we have*

$$\lambda^l((1_r,a),(s_0,\dots,s_l,s_{l+1}))=(s_0,\dots,s_l,\tilde{s}_{l+1})$$

with

$$\tilde{s}_{l+1}-s_{l+1}=(\partial_k\lambda)(1_k,s_0)\circ a\,.$$

Here we note that for $\xi_0,\dots,\xi_l\in\mathbb{R}^m$,

$$a(\xi_0,\dots,\xi_l,\dots)=\xi_0\lrcorner\cdots\xi_l\lrcorner a$$

is a symmetric k–linear map, to which $(\partial_k\lambda)(1_k,s_0)$ can be applied.

The formula for $\tilde{s}_{l+1}-s_{l+1}$ only involves the first derivative of λ and no higher order terms of λ or s. In particular, $(\partial_k\lambda)(1_k,s_0)\circ a$ is well defined as an $(l+1)$–linear symmetric map from $\mathbb{R}^m$ to $T_{s_0}\Sigma$.

3.16. Corollary. *Let $r\geq k$. Then σ is r–rigid if and only if the linear map*

$$\mathrm{Sym}^{r+1}(m,m)\ni a\mapsto(\partial_k\lambda)(1_k,s_0)\circ a\in\mathrm{Sym}^{l+1}(\mathbb{R}^m,T_{s_0}\Sigma)$$

is injective for all s_0 in a $\mathrm{Gl}^k(m)$–invariant open neighborhood Σ_0 of the image of σ in Σ.

Proof. Injectivity of the action of $\Delta^{r+1}(m)$ on Σ_0^{l+1} is immediate from Lemma 3.15. Properness follows since the action is affine. □

3.17. Example (G–structures). Let $G \subset \mathrm{Gl}^k(m)$ be a closed subgroup and denote by $\mathfrak{g}$ the Lie algebra of G. Let $\Sigma = Gl^k(m)/G$. Since the natural action of $Gl^k(m)$ on Σ is transitive, r–rigidity only needs to be checked at the point $s_0 = [G]$ in Σ.

Recall that the $(l+1)$*–st prolongation of* $\mathfrak{g}$ consists precisely of those $a \in \mathrm{Sym}^{k+l+1}(m, m)$ such that

$$(0, \dots, 0, \xi_0 \llcorner \cdots \xi_l \llcorner a) \in \mathfrak{g}$$

for all $\xi_0, \dots, \xi_l \in \mathbb{R}^m$. Now for $b \in \mathrm{Sym}^k(m, m)$,

$$(\partial_k \lambda)(1_k, s_0)(b) = 0 \Longleftrightarrow (0, \dots, 0, b) \in \mathfrak{g}$$

by the definition of λ. Hence G–structures are r–rigid for $r \geq k$ if and only if the $(l+1)$–st prolongation of $\mathfrak{g}$ vanishes, $l = r - k$.

3.18. Proposition. *Let $\sigma : \mathrm{Gl}^k(M) \to \Sigma$ be a geometric structure on M. Suppose that σ is r–rigid for some $r \geq k - 1$. Then σ is r'–rigid for all $r' \geq r$.*

Proof. If σ is $(k-1)$–rigid, then $(\partial_k \lambda)(1, s_0)$ is injective for all s_0 in an open invariant neighborhood Σ_0 of the image of σ. The rest is clear from Corollary 3.16. □

Cross Sections. Suppose now that σ is a geometric structure of order k which is r–rigid. Set $l = r - k \geq -1$.

A *cross section* for the action of $\Delta^{r+1}(m)$ on Σ_0^{l+1} is a submanifold $T \subset \Sigma_0^{l+1}$ which meets each orbit of $\Delta^{r+1}(m)$ exactly once and transversally.

3.19. Example (Parallelizations). We let $r = 0$, $k = 1$ and $\Sigma = \mathrm{Gl}(m)$. The action of $\mathrm{Gl}(m)$ on itself by left translations is simply transitive, hence any point in $\mathrm{Gl}(m)$ is a cross section.

3.20. Example (Connections). We let $r = 1$, $k = 2$ and Σ be the space of bilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^m$. We have

$$(1, a) \cdot \Gamma = \Gamma - a\,.$$

Now a is an arbitrary symmetric bilinear map from $\mathbb{R}^m$ to $\mathbb{R}^m$, hence we may take $T = \{\Gamma \mid \Gamma(v, w) = -\Gamma(w, v)\}$ as a cross section. Interpretation: With respect to exponential coordinates, the Christoffel symbols at the origin are skew symmetric.

3.21. Example (Riemannian Metrics). We let $r = k = 1$ and Σ_+ be the space of inner products s on $\mathbb{R}^m$. We checked that we have 1–rigidity. Hence for reasons of dimension, $T = \{(s_0, 0) \mid s_0 \in \Sigma_+\}$ is a cross section. Interpretation: In Riemannian normal coordinates, the first derivative of the fundamental tensor of the metric vanishes in the origin.

Let $\pi : \mathrm{Gl}^r(M) \to M$ be the projection. Let $j_0^{r+1}(f) \in Gl^{r+1}(M)$. Recall that we can interpret $j_0^{r+1}(f)$ as a holonomic linear map from $\mathbb{R}^m$ to the tangent space of $Gl^r(M)$ in $j_0^r(f)$, namely as the differential of the map

$$u \mapsto j_0^r(f \circ \tau(u)) \in \mathrm{Gl}^r(M)$$

at $u = 0$. This map lifts the map f, that is, $\pi(j_0^r(f \circ \tau(u))) = f(u)$. Since f is a local diffeomorphism, we conclude that $j_0^{r+1}(f)$, viewed as a holonomic linear map, is injective and that its image $\mathrm{Im}(j_0^{r+1}(f))$ is a *horizontal space*, that is, $\pi_* : \mathrm{Im}(j_0^{r+1}(f)) \to T_pM$, $p = f(0)$, is an isomorphism.

3.22. Theorem. *Suppose σ is r–rigid and the action of $\Delta^{r+1}(m)$ on Σ_0^{l+1} is proper and free and admits a cross section. Then there is a parallelization Φ of $\mathrm{Gl}^r(M)$ such that a local diffeomorphism of M is a local automorphisms of σ if and only if the induced map of $\mathrm{Gl}^r(M)$ preserves Φ.*

Proof. Let T be a cross section. Now $\pi : Gl^{r+1}(M) \to Gl^r(M)$ is a principal bundle with structure group $\Delta^{r+1}(m)$. Hence for any $x \in Gl^r(M)$ there is precisely one $\tilde{x} \in Gl^{r+1}(M)$ with $\sigma^{l+1}(\tilde{x}) \in T$. Since $\tilde{x}$ is an isomorphism from $\mathbb{R}^m$ to a horizontal subspace $\mathcal{H}_x \subset T_xGl^r(M)$, we obtain in this way a smooth horizontal distribution $\mathcal{H}$ of $\mathrm{Gl}^r(M)$ together with a trivialization of this distribution,

$$\mathrm{Gl}^r(M) \times \mathbb{R}^m \ni (x, \xi) \mapsto \tilde{x}(\xi) \in \mathcal{H}\,.$$

Now $Gl^r(M)$ is a principal bundle over M and therefore has a natural trivialization of the vertical distribution. Hence $\mathrm{Gl}^r(M)$ has a natural parallelization Φ determined by T. Now the induced map F_* of any local diffeomorphism F of M preserves the vertical trivialization. It is an automorphism of σ^l, and hence of σ, if and only if it preserves the horizontal trivialization. □

3.23. Corollary. *Suppose σ is r–rigid and the action of $\Delta^{r+1}(m)$ on Σ_0^{l+1} is proper and free and admits a cross section. Then for any $j_0^r(f) \in \mathrm{Gl}^r(M)$, the orbit map*

$$\mathrm{Aut}(\sigma) \to \mathrm{Gl}^r(M)\,, \quad F \mapsto F_*(j_0^r(f)) = j_0^r(F \circ f)\,,$$

is an inclusion of $\mathrm{Aut}(\sigma)$ *in* $\mathrm{Gl}^r(M)$ *and the image is a smooth submanifold of* $\mathrm{Gl}^r(M)$*. Furthermore, the corresponding smooth structure on* $\mathrm{Aut}(\sigma)$ *is independent of the choice of* $j_0^r(f) \in \mathrm{Gl}^r(M)$ *and turns* $\mathrm{Aut}(\sigma)$ *into a Lie group such that the action of* $\mathrm{Aut}(\sigma)$ *on* $\mathrm{Gl}^r(M)$ *is smooth.*

Proof. By Theorem 3.22, $\mathrm{Aut}(\sigma)$ consists precisely of those diffeomorphisms of M such that the induced diffeomorphism F_* of $\mathrm{Gl}^r(M)$ preserves the parallelization Φ. This represents $\mathrm{Aut}(\sigma)$ as a closed subgroup of $\mathrm{Aut}(\Phi) \subset \mathrm{Diff}(\mathrm{Gl}^r(M))$. Now the claim follows from the corresponding statements about $\mathrm{Aut}(\Phi)$, see [Ba]. □

Note that Corollary 3.23 also gives the trivial estimate

$$\dim \mathrm{Aut}(\sigma) \leq \dim \mathrm{Gl}^r(M)$$

on the dimension of $\mathrm{Aut}(\sigma)$. Equality occurs under very specific circumstances only. For example, the group of isometries of a Riemannian metric on M has dimension at most $m(m+1)/2 < \dim \mathrm{Gl}(M)$.

In general, we cannot guarantee the existence of cross sections. However, the following lemma ensures that Theorem 3.22 and Corollary 3.23 apply in the case of r–rigid structures of order k whenever $r \geq k$.

3.24. Lemma. *Let* $r \geq k$*. Suppose* $\Delta^{r+1}(m)$ *acts freely on* Σ_0^{l+1}*. Then there is a cross section* $T \subset \Sigma_0^{l+1}$*.*

Proof. By Lemma 3.15, the action of $\Delta^{r+1}(m)$ is affine. By our assumption, the orbits are affine subspaces of dimension equal to $\dim \Delta^{r+1}(m)$.

Now for each point $z = j_0^l(s) \in \Sigma_0^l$, we may choose a linear subspace T_z in $\mathrm{Hom}(\mathbb{R}^m, T_z\Sigma_0^l)$ transverse to the orbits of $\Delta^{r+1}(m)$ and so that T depends smoothly on z. For example, we may choose any Riemannian metric on Σ_0^l and define T_z to be the orthogonal complement through 0 of the orbits of $\Delta^{r+1}(m)$. Then the distribution $T \subset \Sigma_0^{l+1}$ is a cross section. □

4. The Automorphism Relation

We let $Z^k = J^k(M, M)$ and denote by $D^k(M) \subset Z^k$ the open subset of k–jets of local diffeomorphisms of M. We consider $\mathrm{Aut}^k(\sigma) \subset D^k(M)$ as a partial differential relation, compare Section 8. By definition, solutions of $\mathrm{Aut}(\sigma)$ are local automorphisms of σ. Following our discussion in Section 8, we discuss smoothness, consistence and completeness of $\mathrm{Aut}^k(\sigma)$.

Let $p, q \in M$ and F be a local diffeomorphism of M with $F(p) = q$. Then in terms of local coordinates x about p and y about q we have

$$\lambda(g_u^k(\varphi), \sigma_x(u)) = \sigma_y(v) \Longleftrightarrow j_F^k(p) \in \mathrm{Aut}^k(\sigma)\,, \tag{4.1}$$

where $u = x(p)$, $v = y(q)$ and $\varphi = y \circ F \circ x^{-1}$. Hence we can reduce our discussion to a discussion of this equation.

We say that *the equation* $\lambda(g, \sigma_x(u)) = \sigma_y(v)$ *has constant rank about* $(u_0, v_0, g_0) \in \mathrm{Aut}^k(\sigma)$ if, in terms of appropriate local coordinates of Σ about $\sigma_y(v_0)$, the map

$$(u, v, g) \mapsto \lambda(g, \sigma_x(u)) - \sigma_y(v) \tag{4.2}$$

has constant rank about (u_0, v_0, g_0). This condition implies, by the implicit function theorem, that locally about (u_0, v_0, g_0) the above map is a submersion followed by an embedding. We conclude the following.

4.3. Lemma. *Suppose that the equation* $\lambda(g, \sigma_x(u)) = \sigma_y(v)$ *has constant rank about the point* $(u_0, v_0, g_0) \in \mathrm{Aut}^k(\sigma)$. *Then there are open neighborhoods* U_0 *of* u_0, V_0 *of* v_0 *and* W_0 *of* g_0 *such that in* $U_0 \times V_0 \times W_0$, $\mathrm{Aut}^k(\sigma)$ *is smooth and the tangent space of* $\mathrm{Aut}^k(\sigma)$ *in* (u_0, v_0, g_0) *is given by the equation*

$$(\partial_g \lambda)(g_0, \sigma_x(u_0)) \cdot \zeta + (\partial_s \lambda)(g_0, \sigma_x(u_0)) \cdot \sigma_x'(u_0) \cdot \xi = \sigma_y'(v_0) \cdot \eta\,,$$

where $\xi, \eta \in \mathbb{R}^m$ *and* $\zeta \in T_{g_0} \mathrm{Gl}^k(m)$. □

An easy case where this applies is contained in the following result.

4.4. Corollary. *Suppose* $\mathrm{Aut}_{pq}^k(\sigma) \neq \emptyset$ *for all* $p, q \in M$. *Then the equation* $\lambda(g, \sigma_x(u)) = \sigma_y(v)$ *has constant rank.*

Proof. The assumption $\mathrm{Aut}_{pq}^k \neq \emptyset$ for all $p, q \in M$ is equivalent to requiring that the image of σ is contained in an orbit Σ_0 of $\mathrm{Gl}^k(m)$ in Σ. Hence by the equivariance of σ, the equation $\lambda(g, \sigma_x(u)) = \sigma_y(v)$ has rank equal to $\dim \Sigma_0$ everywhere. □

Assume now that $\Sigma_0 \subset \Sigma$ is a submanifold which is invariant under $\mathrm{Gl}^k(m)$ such that the orbit space $\bar{\Sigma}_0 = \Sigma_0 / \mathrm{Gl}^k(m)$ is a smooth manifold and the projection $\bar{\pi} : \Sigma_0 \to \bar{\Sigma}_0$ a smooth submersion. Let

$\pi : \mathrm{Gl}^k(M) \to M$ be the projection and suppose that $M_0 \subset M$ is an open subset such that σ maps $\pi^{-1}(M_0)$ into Σ_0. By the equivariance of σ, there is a smooth map $\bar\sigma : M_0 \to \bar\Sigma_0$ such that $\bar\sigma \circ \pi = \bar\pi \circ \sigma$.

4.5. Lemma. *The rank of σ and $\bar\sigma$ differ by a constant,*

$$\operatorname{rank}\sigma = \operatorname{rank}\bar\sigma + \dim\Sigma_0 - \dim\bar\Sigma_0 \,.$$

Proof. Since σ is equivariant, the image of σ' contains the tangent spaces to the fibers of $\bar\pi$. □

4.6. Lemma. *Suppose $\bar\sigma$ has constant rank in M_0. Then the preimages of points under $\bar\sigma$ foliate M_0 into closed submanifolds of dimension* $\dim M - \operatorname{rank}\bar\sigma$.

Proof. By the implicit function theorem, a map of constant rank locally is a submersion followed by an embedding. □

4.7. Proposition. *Suppose σ has constant rank in M_0. Then in an open neighborhood of $\{j^k_p(F) \in \mathrm{Aut}^k(\sigma) \mid p \in M_0$ and $F(p) = p\}$ and in terms of appropriate coordinates x in M_0, the equation $\lambda(g, \sigma_x(u)) = \sigma_x(v)$ has constant rank equal to* $\operatorname{rank}\sigma$.

Proof. With respect to appropriate local coordinates x about a given point $p_0 \in M_0$ and $\bar z$ about its image $\bar\sigma(p_0) \in \bar\Sigma_0$,

$$\bar\sigma(u_1, \dots, u_m) = (u_1, \dots, u_l, 0, \dots, 0) \,,$$

where $l = \operatorname{rank}\bar\sigma$. Now let $u_0 = x(p_0)$ and suppose $(u_0, u_0, g_0) \in \mathrm{Aut}^k(\sigma)$, that is

$$\lambda(g_0, \sigma_x(u_0)) = \sigma_x(u_0) \,.$$

There are local coordinates z about $\sigma_x(u_0)$ in Σ_0, such that with respect to z and the above coordinates $\bar z$, the projection $\bar\pi$ is given by

$$\bar\pi(w_1, \dots, w_{n_0}) = (w_1, \dots, w_{n_1}) \,,$$

where $n_0 = \dim\Sigma_0$ and $n_1 = \dim\bar\Sigma_0$. In terms of such coordinates,

$$\sigma_x(u) = (u_1, \dots, u_l, 0, \dots, 0, \psi(u))$$

and

$$\lambda(g, \sigma_x(u)) = (u_1, \dots, u_l, 0, \dots, 0, \Psi(g, u)) \,,$$

where the number of zeroes in the middle is equal to $n_1 - l$ in both cases. Since u and v are independent variables and λ is equivariant in g, the image of the derivative of $\lambda(g, \sigma_x(u)) - \sigma_x(v)$ spans the first l and last $n_0 - n_1$ coordinates. □

4.8. PROPOSITION (Consistence). *In terms of local coordinates on M, suppose that the equation $\lambda(g,\sigma_x(u))=\sigma_y(v)$ has constant rank about $z=(u_0,v_0,g_0)\in \operatorname{Aut}^k(\sigma)$. Let $\tilde z=(u_0,v_0,\tilde g_0)\in \operatorname{Aut}^{k+1}(\sigma)$ be a point over z. Then $\tilde z$, viewed as a holonomic linear map $T_{u_0}M\to T_zZ^k$, is tangent to $\operatorname{Aut}^k(\sigma)$.*

Proof. Let $\tilde g_0=j_0^{k+1}(\varphi)$. Let $u=u(t)$ be a curve through $u_0=u(0)$ and set $v=\varphi\circ u$. Then $\varphi^{-1}\circ v=u$ and

$$\partial_t\big(\lambda(g^k_\varphi(u),\sigma_x(u))\big)(0)= \\ (\partial_g\lambda)(g_0,\sigma_x(u_0))\cdot(g^k_\varphi\circ u)'(0)+(\partial_s\lambda)(g_0,\sigma_x(u_0))\cdot\sigma_x'(u_0)\cdot u'(0)\,.$$

Now $\tilde z\in \operatorname{Aut}^{k+1}(\sigma)$, hence the right hand side is equal to $\sigma_y'(v_0)\cdot v'(0)$. Hence by Lemma 4.3,

$$(u'(0),\varphi'(u_0)\cdot u'(0),(g^k_\varphi)'(u_0)\cdot u'(0))$$

is tangent to $\operatorname{Aut}^k(\sigma)$ at z. On the other hand, this is just $\tilde z$ applied to $u'(0)$, where $\tilde z$ is viewed as a holonomic linear map. □

4.9. PROPOSITION (Completeness). *Suppose $\sigma:\operatorname{Gl}^k(M)\to\Sigma$ is a $(k-1)$–rigid geometric structure. Then the partial differential relation $\operatorname{Aut}^k(\sigma)$ is complete and $P:\operatorname{Aut}^k(\sigma)\to Z^{k-1}$ is a proper embedding.*

Proof. Without loss of generality we can assume that the action of $\Delta^k(m)$ on all of Σ is proper and free. Now the graph

$$\Gamma=\{(s,t,g)\mid \lambda(g,s)=t\}\subset\Sigma\times\Sigma\times\operatorname{Gl}^k(m)$$

of the action of $\operatorname{Gl}^k(m)$ on Σ is a smooth submanifold. We consider the natural projection

$$\bar P:\Sigma\times\Sigma\times\operatorname{Gl}^k(m)\to\Sigma\times\Sigma\times\operatorname{Gl}^{k-1}(m)$$

which sends (s,t,g) to $(s,t,\bar g)$, where $\bar g$ is the element of $\operatorname{Gl}^{k-1}(m)$ under g. Now $\bar P$ is proper and injective on Γ since the action of $\Delta^k(m)$ on Σ is proper and free. Hence $\bar P:\Gamma\to\Sigma\times\Sigma\times\operatorname{Gl}^{k-1}(m)$ is a proper topological embedding.

Suppose now that $c=c(\tau)=(g\exp(\tau X),s(\tau),t(\tau))$, $-\varepsilon<\tau<\varepsilon$, is a smooth curve in Γ such that $\bar P\circ c$ has derivative 0 in $\tau=0$. Then $s'(0)=t'(0)=0$ and X is in the Lie algebra of $\Delta^k(m)$. Now the image of c is in Γ, hence

$$\lambda(g\exp(\tau X),s(\tau))=t(\tau)\,.$$

Hence for the derivative at $\tau=0$, we have

$$(\partial_g\lambda)(g,s(0))\cdot L_g'(X)+(\partial_s\lambda)(g,s(0))\cdot s'(0)=t'(0)\,,$$

where L_g denotes left translation by g in $\mathrm{Gl}^k(m)$. Since $s'(0) = t'(0) = 0$ and the action of $\Delta^k(m)$ is free, we conclude that $X = 0$. Hence $\bar{P} : \Gamma \to \Sigma \times \Sigma \times \mathrm{Gl}^{k-1}(m)$ is a proper smooth embedding. Therefore there is an open neighborhood W of $\bar{\Gamma} = \bar{P}(\Gamma)$ in $\Sigma \times \Sigma \times \mathrm{Gl}^{k-1}(m)$ and a smooth section $\bar{Q} : W \to \Sigma \times \Sigma \times \mathrm{Gl}^k(m)$ of $\bar{P}$ such that $\bar{Q} \circ \bar{P}$ is the identity on Γ. By the definition of $\bar{P}$,

$$\bar{Q}(s,t,\bar{g}) = (s,t,q(s,t,\bar{g}))$$

for some appropriate smooth map q.

In terms of local coordinates, $D^k(M) = U \times V \times \mathrm{Gl}^k(m)$ and $D^{k-1}(M) = U \times V \times \mathrm{Gl}^{k-1}(m)$. There is a commutative diagram

$$\begin{array}{ccc} U \times V \times \mathrm{Gl}^k(m) & \longrightarrow & \Sigma \times \Sigma \times \mathrm{Gl}^k(m) \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle \bar{P}} \\ U \times V \times \mathrm{Gl}^{k-1}(m) & \longrightarrow & \Sigma \times \Sigma \times \mathrm{Gl}^{k-1}(m) \end{array}$$

where the horizontal maps are given by

$$(u,v,g) \mapsto (\sigma_x(u), \sigma_y(v), g) \quad \text{and} \quad (u,v,\bar{g}) \mapsto (\sigma_x(u), \sigma_y(v), \bar{g})$$

respectively. Over $U \times V$, $\mathrm{Aut}^k(\sigma)$ is the preimage of Γ. It follows that $P : \mathrm{Aut}^k(\sigma) \to D^{k-1}(M)$ is a proper smooth embedding with lift

$$Q(u,v,\bar{g}) = (u,v,q(\sigma_x(u), \sigma_y(v), \bar{g}))$$

over $U \times V$. Hence $\mathrm{Aut}^k(\sigma)$ is complete. $\square$

Recall that for $r \geq k$, $\mathrm{Aut}^r(\sigma) = \mathrm{Aut}^r(\sigma^l)$, $l = r - k$. Hence the results of this section also apply to $\mathrm{Aut}^r(\sigma)$.

5. Singer's Homogeneity Theorem

As a first application of the general theory developed above we discuss a theorem of Singer in which he establishes a criterion for the homogeneity of Riemannian manifolds.

5.1. Theorem (Singer [Si]). *Let M be a Riemannian manifold of dimension m. Suppose that for all points $p, q \in M$, there is an orthogonal linear transformation preserving the curvature tensor and its first $m(m-1)/2$ covariant derivatives. Then M is locally homogeneous.*

Proof. With respect to Riemannian normal coordinates, the k–jet of the fundamental matrix of the metric at the origin is a universal polynomial in the curvature tensor and its covariant derivatives of order $\leq k-2$. Hence the assumption in the theorem means that $\mathrm{Aut}_{pq}^{3+m(m-1)/2} \neq \emptyset$ for all $p, q \in M$. Hence Theorem 5.3 applies. Note that we need to take $k = 2$ in Theorem 5.3 since we assume there that the structure is $(k-1)$–rigid. ☐

5.2. Remark. There are curvature homogeneous Riemannian manifolds which are not homogeneous, see [FKM]. As far as this author knows, there are no examples of Riemannian manifolds in the literature which are locally homogeneous with respect to the curvature and its first covariant derivative, but which are not locally homogeneous as Riemannian manifolds.

5.3. Theorem (Gromov [Gr]). *Let σ be a geometric structure of order k, rigid of order $k-1$. Let $N = \dim \mathrm{Stab}(\sigma(x))$ for some $x \in \mathrm{Gl}^k(M)$. Suppose that $\mathrm{Aut}_{pq}^r(\sigma) \neq \emptyset$ for all $p, q \in M$, where $r = k + N + 1$. Then σ is locally homogeneous, that is, $\mathrm{Aut}^{loc}(\sigma)$ is transitive on M. More precisely, the partial differential relation $\mathrm{Aut}^r(\sigma)$ is smooth and integrable.*

Proof. By our assumption, $\mathrm{Aut}_{pq}^{k+l}(\sigma) \neq \emptyset$ for all $p, q \in M$ and $l \in \{0, \ldots, N+1\}$. Hence $\mathrm{Aut}^{k+l}(\sigma)$ is smooth for all such l, see Corollary 4.4. Hence we have a sequence of smooth manifolds and embeddings

$$\mathrm{Aut}^{k+N+1}(\sigma) \to \mathrm{Aut}^{k+N}(\sigma) \to \ldots \to \mathrm{Aut}^{k+1}(\sigma) \to \mathrm{Aut}^k(\sigma)\,.$$

By assumption, the dimension of $\mathrm{Aut}^{k+l}(\sigma)$ is $m^2 + N_l$, where N_l is the dimension of a stabilizer of an element in the image of σ^l and $N_0 = N$. By rigidity,

$$N_0 \geq N_1 \geq \ldots \geq N_N \geq N_{N+1} \geq 0\,.$$

Hence there is a first $l \in \{0, \dots, N\}$ such that $N_l = N_{l+1}$. But then the image $A \subset \mathrm{Aut}^{k+l}(\sigma)$ of the embedding

$$\mathrm{Aut}^{k+l+1}(\sigma) \to \mathrm{Aut}^{k+l}(\sigma)$$

is an open submanifold in $\mathrm{Aut}^{k+l}(\sigma)$.

We want to apply the general theory in Section 8 and show that A is an integrable PDR. Now σ is $(k-1)$–rigid, hence also $(k+l-1)$–rigid by Proposition 3.18. Hence A is complete by Proposition 4.9. By definition, A is precisely the image of $\mathrm{Aut}^{k+l+1}(\sigma)$ and open in $\mathrm{Aut}^{k+l}(\sigma)$, hence A is consistent by Proposition 4.8. Hence Theorem 8.14 applies and shows that A is integrable.

Now solutions of A are solutions of $\mathrm{Aut}^{k+l}(\sigma)$ and hence local automorphisms of σ. By definition, A is the image of $\mathrm{Aut}^{k+l+1}(\sigma)$, hence $\mathrm{Aut}^{k+j}(\sigma)$ is integrable for all $j \geq l+1$. □

5.4. Remark. Following [Si] we can improve on the magnitude of the number N as follows: For a Lie algebra $\mathfrak{g}$, let $N = N(\mathfrak{g})$ be the maximal length of a chain of Lie subalgebras

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \ldots \supset \mathfrak{g}_N$$

with

$$\dim \mathfrak{g} = \dim \mathfrak{g}_1 > \dim \mathfrak{g}_2 > \ldots > \dim \mathfrak{g}_N > 0\,.$$

Clearly $N(\mathfrak{g}) \leq \dim \mathfrak{g}$. It is also clear from the above proof, that we can replace the number $N = \dim \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of the stabilizer of $\sigma(x)$ in Theorem 5.3 by the number $N = N(\mathfrak{g})$.

6. Gromov's Open Orbit Theorem

The problem in applying the results of the general theory on the integrability of PDRs from Section 8 to the automorphism relation lies in the smoothness question. In the case where the action λ of $\mathrm{Gl}^k(m)$ on Σ is not too wild this problem can be overcome on an open and dense subset of M.

6.1. Definition. We say that the action λ of $\mathrm{Gl}^k(m)$ on Σ is *tame* if there is a stratification of Σ by G–invariant submanifolds,

$$\Sigma = \Sigma_0 \cup \ldots \cup \Sigma_s\,,$$

such that for all $l \in \{0, \dots, s\}$

1. $\Sigma_l \cup \ldots \cup \Sigma_s$ is closed in Σ;
2. $\bar\Sigma_l = \Sigma_l / G$ is a smooth manifold such that $\Sigma_l \to \bar\Sigma_l$ is a smooth submersion.

If λ is tame, then the orbits of G are embedded submanifolds of Σ.

The following result will be used without proof. It is an easy consequence of Bezout's theorem.

6.2. THEOREM. *Suppose that Σ is a smooth real algebraic variety and that λ is algebraic. Then λ is tame. In fact, there is a stratification of Σ by G–invariant submanifolds as above such that for all $l \in \{0, \dots, s\}$*

(1) *$\Sigma_l \cup \dots \cup \Sigma_s$ is Zariski closed in Σ;*
(2) *$\bar{\Sigma}_l = \Sigma_l / G$ is a smooth real algebraic variety such that $\Sigma_l \to \bar{\Sigma}_l$ is a submersion.*

If Σ is a smooth real algebraic variety and λ is algebraic, then Σ^l is a smooth real algebraic variety and the induced action of $\mathrm{Gl}^r(m)$ on Σ^l is algebraic, where $l \geq 0$ and $r = l + k$. Hence if λ is algebraic, then λ^l is tame for all $r \geq k$. Note that in all our examples λ is algebraic.

From now on we assume that $\sigma : \mathrm{Gl}^k(M) \to \Sigma$ is a $(k-1)$–rigid structure. We let N be the maximal dimension of a stabilizer in the image of σ and assume that λ^l is tame for all $l \in \{0, \dots, N+1\}$.

For all $l \in \{0, \dots, N+1\}$, we fix stratifications $\Sigma^l = \Sigma^l_0 \cup \dots \cup \Sigma^l_s$, $s = s(l)$, of Σ^l as in Definition 6.1. We let M^l_i be the open subset of points $p \in M$ such that $\sigma(x) \in \Sigma^l_i$ for all $x \in \mathrm{Gl}^{k+l}(M)$ with foot point $\pi(x)$ sufficiently close to p.

6.3. LEMMA. *For all $l \in \{0, \dots, N+1\}$, $M^l_0 \cup \dots \cup M^l_s$ is open and dense in M.*

Proof. We prove recursively that $\pi^{-1}(M^l_0 \cup \dots \cup M^l_i)$ is dense in the preimage of $\Sigma^l_0 \cup \dots \cup \Sigma^l_i$, $0 \leq i \leq s$. Now Σ^l_0 is open, hence $\pi^{-1}(M^l_0)$ is actually equal to the preimage of Σ^l_0. Let $i \geq 1$ and assume that $\pi^{-1}(M^l_0 \cup \dots \cup M^l_{i-1})$ is dense in the preimage of $\Sigma^l_0 \cup \dots \cup \Sigma^l_{i-1}$. Now $\Sigma^l_0 \cup \dots \cup \Sigma^l_i$ is open. Hence if $\pi^{-1}(p)$ is not in the closure of the preimage of $\Sigma^l_0 \cup \dots \cup \Sigma^l_{i-1}$, but is in the preimage of Σ^l_i, then $p \in M^l_i$ by the definition of M^l_i. Hence recursion applies. □

We are now ready to apply the counting argument from Section 5 in the present situation.

6.4. THEOREM (Gromov [Gr]). *Assume that $\sigma : \mathrm{Gl}^k(M) \to \Sigma$ is a $(k-1)$–rigid geometric structure and let N be the maximal dimension of a stabilizer in the image of σ. Assume that λ^l is tame for all $l \in \{0, \dots, N+1\}$. Then there is an open and dense set U in M such that $\mathrm{Aut}^{k+N+1}(\sigma)$ is integrable over a neighborhood of the diagonal in $U \times U$ and such that the orbits of $\mathrm{Aut}^{loc}(\sigma)$ foliate U into closed submanifolds of constant dimension locally.*

Proof. We let U be the open subset of points such that any $p \in U$ has an open neighborhood V in M such that, for all $l \in \{0, \dots, N+1\}$,

(1) $V \subset U_i^l$ for some $i = i(V, l)$ and
(2) $\operatorname{rank} \sigma^l$ is constant in the preimage of V.

By the semi–continuity of the rank and Lemma 6.3, U is open and dense in M.

Let U_0 be a component of U and $l \in \{0, \dots, N+1\}$. Then $U_0 \subset U_i^l$ for some fixed $i = i(l)$. We let N_i^l be the dimension of the stabilizers in Σ_i^l. By Proposition 4.7, in an open neighborhood of

$$\{j_F^{k+l}(p) \mid p \in U_0 \text{ and } F(p) = p\} ,$$

$\operatorname{Aut}^{k+l}(\sigma)$ is a smooth submanifold of dimension

$$\begin{aligned} &\dim D^{k+l}(M) - \operatorname{rank} \sigma^l \\ &= m^2 + \dim \operatorname{Gl}^{k+l}(m) - \dim \Sigma_i^l + \dim \bar{\Sigma}_i^l - \operatorname{rank} \bar{\sigma}^l \\ &= m^2 + N_i^l - \operatorname{rank} \bar{\sigma}^l , \end{aligned}$$

where $\bar{\sigma}^l : U_0 \to \bar{\Sigma}_i^l$ denotes the induced map as in Lemma 4.5. Hence by the argument in the proof of Theorem 5.3, $\operatorname{Aut}^{k+N+1}(\sigma)$ is integrable in a neighborhood of $\{j_F^{k+N+1}(p) \mid p \in U_0, F(p) = p\}$.

By Lemma 4.6, the preimages of points of $\bar{\sigma}^{k+N+1}$ foliate U_0 into submanifolds of dimension $\dim M - \operatorname{rank} \bar{\sigma}^{k+N+1}$. By the definition of $\bar{\sigma}^{k+N+1}$, $\operatorname{Aut}_{pq}^{k+N+1}(\sigma) \neq \emptyset$ if $p, q \in U_0$ are contained in the same such submanifold. By the integrability of $\operatorname{Aut}^{k+N+1}(\sigma)$ established above, the orbits of $\operatorname{Aut}^{loc}(\sigma)$ in U_0 are open subsets of these submanifolds. □

A special case of the above main theorem is the following result, Gromov's Open Orbit Theorem.

6.5. Theorem. *Assume that $\sigma : \operatorname{Gl}^k(M) \to \Sigma$ is a $(k-1)$–rigid geometric structure and let N be the maximal dimension of a stabilizer in the image of σ. Assume that λ^l is tame for all $l \in \{0, \dots, N+1\}$. Then an orbit of $\operatorname{Aut}^{loc}(\sigma)$ is open if it is dense.*

7. Example: Conformal Structures

Conformal structures are a bit more involved than the examples discussed in the text. Therefore we discuss them in this extra section.

Let Σ, Σ_+ and λ be as in Example 2.4. Note that λ is linear in $s \in \Sigma$. Hence λ descends to an action $\bar{\lambda}$ of $\mathrm{Gl}(m)$ on the projective space $\bar{\Sigma} := P(\Sigma)$. The subset $\bar{\Sigma}_+ = P(\Sigma_+)$ is open and invariant under $\mathrm{Gl}(m)$. Conformal structures are geometric structures of type $\bar{\lambda}$ with image in $\bar{\Sigma}_+$.

Now $\mathrm{Gl}(m)$ acts transitively on $\bar{\Sigma}_+$ and $\mathrm{CO}(m)$, the group of conformal transformations of $\mathbb{R}^m$ with respect to the standard inner product, is the stabilizer of the standard conformal structure. This identifies $\bar{\Sigma}_+ \cong \mathrm{Gl}(m)/\mathrm{CO}(m)$. Hence we may view conformal structures as G–structures in the sense of Example 2.5, where $G = \mathrm{CO}(m)$ and $P \subset \mathrm{Gl}^1(M)$ is the preimage of $[\mathrm{CO}(m)] \in \bar{\Sigma}_+$ under $\bar{\sigma}$.

Let $\xi \in \mathbb{R}^m$, $\xi \neq 0$. The image of a conformal structure $\bar{\sigma}$ is contained in the open subset of $[s] \in \bar{\Sigma}$ with $s(\xi,\xi) \neq 0$, the domain of the coordinate chart $[s] \mapsto s/s(\xi,\xi)$ of $\bar{\Sigma}$. Hence a conformal structure $\bar{\sigma}$ corresponds to an equivariant map σ_ξ into the affine subspace $\Sigma_\xi = \{s(\xi,\xi) = 1\} \subset \Sigma$, where the action λ_ξ of $\mathrm{Gl}(m)$ on Σ_ξ is given by

$$\lambda_\xi(a,s)(\xi_0,\xi_1) = \frac{s(a^{-1}\xi_0, a^{-1}\xi_1)}{s(a^{-1}\xi, a^{-1}\xi)} \,. \tag{7.1}$$

This representation of conformal structures will be used below.

We now discuss the induced geometric structure of order two. Let $(a_1,a_2) = j_0^2(\varphi) \in \mathrm{Gl}^2(m)$, $(s_0,s_1) = j_0^1(s) \in \Sigma_\xi^1$ and set

$$(a_1,a_2)\cdot(s_0,s_1) = (a_1\cdot s_0, \tilde{s}_1)\,, \tag{7.2}$$

where $a_1 \cdot s_0 = \lambda_\xi(a_1,s_0)$ is as before. To determine $\tilde{s}_1$ we fix $\xi_1,\xi_2 \in \mathbb{R}^m$ and consider the map

$$u \mapsto \frac{s(\varphi^{-1}(u))(\varphi'(\varphi^{-1}(u))^{-1}\xi_1, \varphi'(\varphi^{-1}(u))^{-1}\xi_2)}{s(\varphi^{-1}(u))(\varphi'(\varphi^{-1}(u))^{-1}\xi, \varphi'(\varphi^{-1}(u))^{-1}\xi)} \,.$$

The differential at $u = 0$ in the direction of $\xi_0 \in \mathbb{R}^m$ is

$$\begin{aligned}
\tilde{s}_1(\xi_0)(\xi_1,\xi_2) = {} & s_1(a_1^{-1}\xi_0)(a_1^{-1}\xi_1, a_1^{-1}\xi_2)s_0(a_1^{-1}\xi, a_1^{-1}\xi) \\
& - s_0(a_1^{-1}a_2(a_1^{-1}\xi_0, a_1^{-1}\xi_1), a_1^{-1}\xi_2)s_0(a_1^{-1}\xi, a_1^{-1}\xi) \\
& - s_0(a_1^{-1}\xi_1, a_1^{-1}a_2(a_1^{-1}\xi_0, a_1^{-1}\xi_2))s_0(a_1^{-1}\xi, a_1^{-1}\xi) \\
& - s_0(a_1^{-1}\xi_1, a_1^{-1}\xi_2)s_1(a_1^{-1}\xi_0)(a_1^{-1}\xi, a_1^{-1}\xi) \\
& + 2s_0(a_1^{-1}\xi_1, a_1^{-1}\xi_2)s_0(a_1^{-1}a_2(a_1^{-1}\xi_0, a_1^{-1}\xi), a_1^{-1}\xi)\,,
\end{aligned}$$

divided by $s_0(a_1^{-1}\xi, a_1^{-1}\xi)^2$.

We are mainly interested in the case $(a_1, a_2) \in \Delta^2(m)$, that is $a_1 = 1$ and $a_2 =: a \in \mathrm{Sym}^2(m,m)$. Then $\tilde{s}_1 - s_1 \in \mathrm{Hom}(\mathbb{R}^m, T_{s_0}\Sigma_\xi)$ is given by

$$-s_0(a(\xi_0,\xi_1),\xi_2) - s_0(\xi_1, a(\xi_0,\xi_2)) + 2s_0(\xi_1,\xi_2)s_0(a(\xi_0,\xi),\xi)\,. \tag{7.3}$$

Here we use $s_0(\xi,\xi) = 1$ and $s_1(\xi_0)(\xi,\xi) = 0$, which follows from $s(u)(\xi,\xi) \equiv 1$.

We conclude that $(1,a) \in \Delta^2(m)$ is in the stabilizer of $(s_0,s_1) \in \Sigma^1_\xi$ if and only if a is in the first prolongation of $\mathfrak{co}(s_0)$, the Lie algebra of the conformal group $CO(s_0)$, compare Example 3.17.

7.4. Lemma. *The first prolongation of* $\mathfrak{co}(s_0)$ *has dimension* m *and consists precisely of the maps* a_ζ, $\zeta \in \mathbb{R}^m$, *given by*

$$a_\zeta(\xi_0,\xi_1) = s_0(\xi_0,\xi_1)\zeta - s_0(\zeta,\xi_1)\xi_0 - s_0(\xi_0,\zeta)\xi_1$$

Proof. Consider the linear map which associates to a in the first prolongation of $\mathfrak{co}(s_0)$ the linear map

$$\mathbb{R}^m \ni \xi_0 \mapsto s_0(a(\xi_0,\xi),\xi) \in \mathbb{R}\,.$$

If a is in the kernel, then a belongs to the first prolongation of $\mathfrak{so}(m)$. This implies $a = 0$, see Example 3.10. Hence the first prolongation of $\mathfrak{co}(s_0)$ has dimension at most m. On the other hand, the maps a_ζ, $\zeta \in \mathbb{R}^m$, belong to the first prolongation of $\mathfrak{co}(s_0)$. $\square$

It follows that conformal structures are not 1–rigid. Now we also have

$$\dim \Delta^2(m) - \dim \mathrm{Hom}(\mathbb{R}^m, T_{s_0}\Sigma_\xi) = m\,. \tag{7.5}$$

We conclude that the linear map which associates to $a \in \mathrm{Sym}^2(m,m)$ the 3–linear map $\tilde{s}_1 - s_1 \in \mathrm{Hom}(\mathbb{R}^m, T_{s_0}\Sigma_\xi)$ in (7.3) is surjective. It follows that $\mathrm{Gl}^2(m)$ acts transitively on $\bar{\Sigma}^1_+$. Let s_0 be the standard conformal structure on $\mathbb{R}^m$ and $\mathrm{CO}^1(m)$ be the stabilizer of $(s_0,0)$ in $\mathrm{Gl}^2(m)$. Then we may view $\bar{\sigma}^1$ as a G–structure of order 2, where $G = \mathrm{CO}^1(m)$ and the principal bundle $P^1 \subset \mathrm{Gl}^2(M)$ is the preimage of $(s_0,0)$ under $\bar{\sigma}^1$.

Now conformal structures are 2–rigid for $m \ge 3$. As explained in Example 3.17, we need to check that $a \in \mathrm{Sym}^3(m,m)$ vanishes if

$$\begin{aligned} &- s_0(a(\xi_0,\xi_1,\xi_2),\xi_3) - s_0(\xi_2, a(\xi_0,\xi_1,\xi_3)) \\ &\qquad\qquad + 2s_0(\xi_2,\xi_3)s_0(a(\xi_0,\xi_1,\xi),\xi) = 0 \end{aligned}$$

for all $\xi_0,\xi_1,\xi_2,\xi_3 \in \mathbb{R}^m$. This is an elementary exercise.

8. Appendix: Partial Differential Relations

Let M and N be smooth manifolds of dimension m and n respectively. Let $k \geq 0$, $Z = J^k(M,N)$ and $\pi : Z \to M$ be the projection to the foot point.

A *partial differential relation* (PDR) of order k (for local maps from M to N) is a subset $\mathcal{R} \subset Z$. A *solution* of such a relation $\mathcal{R}$ is a local map f from M to N such that $j^k_f(p)$ is in $\mathcal{R}$ for all p in the domain of f, a *formal solution* is a local section s of π with $s(p) \in \mathcal{R}$ for all p in the domain of s. In this appendix we discuss solvability of PDRs in the sense of the Frobenius Integrability Theorem.

Completeness. Let $\tilde{Z} := J^{k+1}(M,N)$ and $P : \tilde{Z} \to Z$, $\tilde{\pi} : \tilde{Z} \to M$ be the canonical projections.

The first idea we discuss in connection with PDRs is that of completeness. Let $\tilde{\mathcal{R}} \subset \tilde{Z}$ be a PDR of order $k+1$ and set $\mathcal{R} = P(\tilde{\mathcal{R}})$. On a more informal level, completeness means that for $j^{k+1}_f(p) \in \tilde{\mathcal{R}}$, the derivative of order $k+1$ of f at p is a function of the k–jet $j^k_f(p) \in \mathcal{R}$.

8.1. Definition. We say that $\tilde{\mathcal{R}}$ is *complete* if $P : \tilde{\mathcal{R}} \to \mathcal{R}$ is injective and if, for each point $z \in \mathcal{R}$, there is an open neighborhood W of z in Z and a smooth section $Q : W \to P^{-1}(W)$ of P such that $Q(\mathcal{R} \cap W) \subset \tilde{\mathcal{R}}$.

It follows that $P : \tilde{\mathcal{R}} \cap P^{-1}(W) \to \mathcal{R} \cap W$ is a homeomorphism with inverse $Q|\mathcal{R} \cap W$.

From now on we assume that $\tilde{\mathcal{R}}$ is complete. We will see that completeness corresponds to uniqueness of solutions of $\tilde{\mathcal{R}}$.

In terms of local coordinates, a section Q as in Definition 8.1 is a map of the form

$$Q(u, v, \varphi_1, \dots, \varphi_k) = (u, v, \varphi_1, \dots, \varphi_k, q(u, v, \varphi_1, \dots, \varphi_k)) \,, \tag{8.2}$$

where q is a smooth map which is defined in a neighborhood in Z of a point $z \in \mathcal{R}$. Locally about z, solutions of $\tilde{\mathcal{R}}$ are maps φ from $\mathbb{R}^m$ to $\mathbb{R}^n$ which satisfy the PDE

$$\varphi^{(k+1)}(u) = q(u, \varphi(u), \varphi'(u), \dots, \varphi^{(k)}(u)) = q(j^k_\varphi(u)) \tag{8.3}$$

for all u in the domain of φ. If φ is a solution of $\tilde{\mathcal{R}}$ and c is a smooth curve in M, then along c we have

$$(\varphi^{(i)} \circ c)' = c' \llcorner (\varphi^{(i+1)} \circ c) \,, \quad 0 \leq i \leq k \,,$$

with $\varphi^{(0)} = \varphi$ and $\varphi^{(k+1)} = q \circ j^k_\varphi$, an ODE for $j^k_\varphi \circ c$. In our interpretation of $(k+1)$–jets as linear holonomic maps, we can write this ODE as follows,

$$(j^k_\varphi \circ c)' = c' \llcorner Q(j^k_\varphi \circ c)\,. \tag{8.4}$$

This discussion leads us to the first result concerning solutions of $\tilde{\mathcal{R}}$.

8.5. Lemma. *Let $z \in \mathcal{R}$. Then there is at most one solution of $\tilde{\mathcal{R}}$ through z.*

Proof. Locally, the k–jet j^k_φ of a solution satisfies an ODE as in (8.4). Now the assertion is immediate from the uniqueness of solutions of ODEs with smooth coefficients. □

8.6. Remark. The above ODE also implies that solutions of $\tilde{\mathcal{R}}$, if they exist, are smooth and depend smoothly on z.

Consistence. The second idea we discuss in connection with PDRs is that of consistence. We saw that completeness of $\tilde{\mathcal{R}}$ leads to an ODE for the k-jet j^k_φ of solutions φ of $\tilde{\mathcal{R}}$ along smooth curves $c : [0,1] \to M$. Consistence requires that the terminal values of the solutions of the ODE only depend on their initial values and the homotopy class of c. Thus in connection with completeness, consistence corresponds to the existence of solutions of $\tilde{\mathcal{R}}$.

8.7. Definition. We say that $\tilde{\mathcal{R}}$ is *consistent* if $\tilde{\mathcal{R}}$ is a smooth submanifold of $\tilde{Z}$ and if, for any $\tilde{z} \in \tilde{\mathcal{R}}$, there is a holonomic linear map $L : T_pM \to T_{\tilde{z}}\tilde{Z}$ tangent to $\tilde{\mathcal{R}}$, that is, with $\operatorname{Im} L \subset T_{\tilde{z}}\tilde{\mathcal{R}}$.

8.8. Lemma. *Suppose $\tilde{\mathcal{R}}$ is consistent and let $\tilde{z} \in \tilde{\mathcal{R}}$. Let $L : T_pM \to T_{\tilde{z}}\tilde{\mathcal{R}}$ be a linear holonomic map. Then there is a formal solution $\tilde{s}$ of $\tilde{\mathcal{R}}$ through $\tilde{z}$ such that $\tilde{s}'(p) = L$.*

Proof. Suppose that $\tilde{\mathcal{R}}$ is consistent and let $\tilde{z} \in \tilde{\mathcal{R}}$. Let $L : T_pM \to T_{\tilde{z}}\tilde{\mathcal{R}}$ be a holonomic linear map. Let $\tilde{s}_h$ be a local holonomic section of $\tilde{\pi}$ with $\tilde{s}_h'(p) = L$. Then $\tilde{\pi} \circ \tilde{s}_h = \mathrm{id}$ and hence $\tilde{\pi}'(\tilde{z}) : \operatorname{Im} L \to T_pM$ is an isomorphism. Now by assumption $\operatorname{Im} L \subset T_{\tilde{z}}\tilde{\mathcal{R}}$. In particular, $\tilde{\pi} : \tilde{\mathcal{R}} \to M$ has rank m at $\tilde{z}$. It follows that there is a formal solution $\tilde{s}$ of $\tilde{\mathcal{R}}$ with $\tilde{s}'(p) = L$. □

From now on we assume that $\tilde{\mathcal{R}}$ is complete and consistent and set $\mathcal{R} = P(\tilde{\mathcal{R}})$ as above.

8.9. Lemma. *$\mathcal{R}$ is a smooth submanifold of Z, $P : \tilde{\mathcal{R}} \to \mathcal{R}$ is a diffeomorphism and $\tilde{z} \in \tilde{\mathcal{R}}$, viewed as a holonomic linear map $T_pM \to T_z\mathcal{R}$, where $z = P(\tilde{z})$ and $p = \pi(z)$, is tangent to $\mathcal{R}$. In particular, solutions of the ODE (8.4) with initial value in $\mathcal{R}$ stay in $\mathcal{R}$.*

Proof. The first claim is clear since the local sections Q of P are smooth and satisfy $Q \circ P = \mathrm{id}$ on $\tilde{\mathcal{R}}$. As for the second, let $\tilde{s}$ be a formal solution of $\tilde{\mathcal{R}}$ through $\tilde{z}$ as in Lemma 8.8. Let $s = P \circ \tilde{s}$. Then s is a formal solution of $\mathcal{R}$ through z and $s'(p)$ is the holonomic linear map $T_pM \to T_zZ$ associated to $\tilde{z}$. □

In terms of local coordinates, let

$$\tilde{z} = (u, v, \varphi_1, \ldots, \varphi_k, \varphi_{k+1}) = (z, q(z))$$

with

$$z = (u, v, \varphi_1, \ldots, \varphi_k) \quad \text{and} \quad \varphi_{k+1} = q(z)\,.$$

Let $L : T_pM \to T_{\tilde{z}}\tilde{\mathcal{R}}$ be a linear holonomic map and $\tilde{s}$ be a formal solution as in Lemma 8.8. Then

$$L = (\mathrm{id}, \varphi_1, \ldots, \varphi_{k+1}, \varphi_{k+2}) \tag{8.10}$$

for some $\varphi_{k+2} \in \mathrm{Sym}^{k+2}(m, n)$. Taking the differential of $\tilde{s}$ at u, we see that

$$\varphi_{k+2} = q'(z) \cdot (\mathrm{id}, \varphi_1, \ldots, \varphi_{k+1})\,. \tag{8.11}$$

In other words, φ_{k+2} is uniquely determined by q and $\tilde{z}$. Hence L is uniquely determined by Q and $\tilde{z}$. We obtain a smooth local section $\tilde{Q} : \tilde{\mathcal{R}} \to \hat{\mathcal{R}}$ with $\hat{\mathcal{R}} \subset \hat{Z} := J^{k+2}(M, N)$, where $L = \tilde{Q}(\tilde{z})$ is the holonomic linear map as in Definition 8.7.

Again in terms of local coordinates and in the above notation, since φ_{k+2} is symmetric,

$$\begin{aligned} \xi \llcorner (q'(z) \cdot (\mu, \mu \llcorner \varphi_1, \ldots, \mu \llcorner \varphi_{k+1})) \\ = \mu \llcorner (q'(z) \cdot (\xi, \xi \llcorner \varphi_1, \ldots, \xi \llcorner \varphi_{k+1})) \end{aligned} \tag{8.12}$$

for all $\xi, \mu \in \mathbb{R}^m$. This is the formula which we need in the proof of the next lemma.

8.13. Lemma. *In terms of local coordinates, suppose that*

$$(c, \varphi_0, \ldots, \varphi_k) = z = z(t, \tau) : [0, 1]^2 \to \mathcal{R}$$

is a solution of

$$\partial_t \varphi_i = (\partial_t c) \llcorner \varphi_{i+1}\,, \quad 0 \le i \le k\,,$$

with $\varphi_{k+1} = q \circ z$ such that

$$\partial_\tau \varphi_i = (\partial_\tau c) \llcorner \varphi_{i+1}\,, \quad 0 \le i \le k\,,$$

holds along the curve $t = 0$. Then the latter equation holds for all (t, τ).

Proof. We compute the differential of the difference of the sides with respect to t. For $0 \le i \le k-1$, we have

$$\begin{aligned}\partial_t\partial_\tau\varphi_i - \partial_t(\partial_\tau c\llcorner\varphi_{i+1}) &= \partial_\tau\partial_t\varphi_i - \partial_t(\partial_\tau c\llcorner\varphi_{i+1}) \\ &= \partial_\tau(\partial_t c\llcorner\varphi_{i+1}) - \partial_t(\partial_\tau c\llcorner\varphi_{i+1}) \\ &= \partial_t c\llcorner(\partial_\tau\varphi_{i+1}) - \partial_\tau c\llcorner(\partial_t c\llcorner\varphi_{i+2})\,.\end{aligned}$$

For $i = k$, we have

$$\begin{aligned}\partial_t\partial_\tau\varphi_k &- \partial_t(\partial_\tau c\llcorner\varphi_{k+1}) = \partial_\tau\partial_t\varphi_k - \partial_t(\partial_\tau c\llcorner(q\circ z)) \\ &= \partial_\tau(\partial_t c\llcorner\varphi_{k+1}) - \partial_t(\partial_\tau c\llcorner(q\circ z)) \\ &= \partial_t c\llcorner(\partial_\tau\varphi_{k+1}) - \partial_\tau c\llcorner(q'(z)\cdot(\partial_t c, \partial_t c\llcorner\varphi_1, \dots, \partial_t c\llcorner\varphi_k)) \\ &= \partial_t c\llcorner(\partial_\tau\varphi_{k+1}) - \partial_t c\llcorner(q'(z)\cdot(\partial_\tau c, \partial_\tau c\llcorner\varphi_1, \dots, \partial_\tau c\llcorner\varphi_k)) \\ &= \partial_t c\llcorner(\partial_\tau\varphi_{k+1}) - \partial_t c\llcorner\partial_\tau(q\circ z) = 0\,.\end{aligned}$$

Hence $\partial_\tau\varphi_k = \partial_\tau c\llcorner\varphi_{k+1}$ for all (t,τ). But then also

$$\partial_\tau c\llcorner\partial_t c\llcorner\varphi_{k+1} = \partial_t c\llcorner\partial_\tau c\llcorner\varphi_{k+1} = \partial_t c\llcorner\partial_\tau\varphi_k$$

and therefore $\partial_\tau\varphi_{k-1} = \partial_\tau c\llcorner\varphi_k$ for all (t,τ). Recursively we get that $\partial_\tau\varphi_i = \partial_\tau c\llcorner\varphi_{i+1}$ for all i and (t,τ). □

We say that $\tilde{\mathcal{R}}$ is *integrable* if through any $\tilde{z}$ in $\tilde{\mathcal{R}}$, there is a solution of $\tilde{\mathcal{R}}$.

8.14. Theorem. *Suppose $\tilde{\mathcal{R}}$ is complete. Then $\tilde{\mathcal{R}}$ is integrable if and only if $\tilde{\mathcal{R}}$ is consistent.*

Proof. Consistence implies that $\tilde\pi : \tilde{\mathcal{R}} \to M$ is of maximal rank $m = \dim M$. Hence the image U of $\tilde\pi : \tilde{\mathcal{R}} \to M$ is open in M. By Lemma 8.9, for any smooth curve $c : [0,1] \to U$ and point $\tilde z \in \mathcal{R}$ over p, the solution of (8.4) with $j^k_\varphi(0) = z = P(\tilde z)$ is contained in $\mathcal{R}$. By Lemma 8.13, the terminal value does not depend on the (proper) homotopy class of c (since then $(\partial_\tau c)(\tau, 1) = 0$). □

References

[Ba] W. Ballmann. *Automorphism Groups.* Lecture Notes, U Bonn 1999.

[Be] Y. Benoist. Orbites des structures rigides (d'après M. Gromov). *Feuilletages et systèmes intégrables* (Montpellier, 1995), 1–17, Progr. Math. 145, Birkhäuser, 1997.

[BFL] Y. Benoist, P. Foulon & F. Labourie. Flots d'Anosov à distributions stable et instable différentiables. *JAMS* 5 (1992), 33–74.

[Ch] S. S. Chern. The Geometry of G–Structures. *BAMS* 72 (1966), 167–219.

[DG] G. D'Ambra & M. Gromov. Lectures on Transformation Groups: Geometry and Dynamics. *Surveys in differential geometry*, 19–111, Lehigh U 1991.

[FKM] D. Ferus, H. Karcher & H. Münzner. Cliffordalgebren und neue isoparametrische Hyperflächen. *MZ* 177 (1981), 479–502.

[Gr] M. Gromov. Rigid Transformation Groups. In: *Géométrie différentielle*, edited by D. Bernard and Y. Choquet–Bruhat. Hermann, Travaux en cours 33 (1988), 65–139.

[Ko] S. Kobayashi. *Transformation Groups in Differential Geometry.* Springer–Verlag 1972.

[Si] I. M. Singer. Infinitesimally Homogeneous Spaces. *CPAM* 13 (1960), 685–697.

[We] G. Weingart. Holonomic and Semiholonomic Geometries. Proceedings of the 1999 CIRM-Luminy conference *'Analyse Harmonique et Analyse sur les Variétés'.* Preprint, U Bonn 2000.

[Ze] A. Zeghib. On Gromov's Theory of Rigid Transformation Groups: A Dual Approach. Preprint, ENS Lyon 1998.