# BASIC DIFFERENTIAL GEOMETRY: GLOBAL RIEMANNIAN GEOMETRY

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## Introduction

For a Riemannian manifold $M$, we are interested in relations between the geometry and the topology of $M$. Now the local topology of $M$ is the same as that of Euclidean space $\mathbb{R}^m$, $m = \dim M$. Therefore the topological properties we consider are of a global nature. For that reason, we always need global assumptions on the geometry as well. The most important one is completeness, discussed in the Theorem of Hopf-Rinow below. This theorem is at the basis of global Riemannian Geometry.

I discuss two rather elementary geometric and topological results which involve the Hopf-Rinow Theorem, the Theorems of Bonnet-Myers and Hadamard-Cartan, respectively.

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Last update: 2003-05-25.

## 1. The Theorem of Hopf-Rinow

The length of curves gives a natural way of defining the distance of points in $M$,

$$d(p,q) := \inf\{L(c) \mid c \in \Omega_{pq}^{ps}(M)\}, \tag{1.1}$$

where $\Omega_{pq}^{ps}(M)$ denotes the space of piecewise smooth curves $c : [0,1] \to M$ with $c(0) = p$ and $c(q) = 1$. Note that we allow the value $d(p,q) = \infty$.

Exercise 1.1. The distance $d(p,q) = \infty$ if and only if $p$ and $q$ lie in different components of $M$.

For any curve $c \in \Omega_{pq}^{ps}(M)$, the curve $c^{-1} \in \Omega_{qp}^{ps}(M)$, $c^{-1}(t) = c(1-t)$, has the same length as $c$ and hence $d(q,p) = d(p,q)$.

For any three points $p, q, r$ and curves $c \in \Omega_{pq}^{ps}(M)$, $\tilde{c} \in \Omega_{qr}^{ps}(M)$, the curve $c * \tilde{c} \in \Omega_{pr}(M)$,

$$(c * \tilde{c})(t) := \begin{cases} c(2t) & 0 \le t \le 1/2, \\ \tilde{c}(2t-1) & 1/2 \le t \le 1, \end{cases}$$

has length $L(c) + L(\tilde{c})$ and hence $d$ satisfies the triangle inequality.

Since a point curve has length 0, we have $d(p,p) = 0$ for all $p \in M$. For $d$ to be a true distance, we also need the converse: $d(p,q) = 0$ implies $p = q$. In addition we want that $d$ induces the given topology on $M$. This and more is contained in the lemma below.

Note first that for any $p \in M$ and any $v \in T_pM$ in the domain of definition of $\exp_p$, the curve

$$c_v(t) = \exp_p(tv), \quad 0 \le t \le 1, \tag{1.2}$$

is a smooth curve from $p$ to $q = \exp_p(v)$ of length $\|v\|$. In particular, $d(p,q) \le \|v\|$.

For $r > 0$ let $B_r(0_p) := \{v \in T_pM \mid \|v\| < r\}$ and, if $B_r(0_p)$ is in the domain of definition of $\exp_p$, set $B_r(p) := \exp_p(B_r(0_p))$ . The notation suggests that $B_r(p)$ is the ball of radius $r$ about $p$ with respect to $d$. So far this is not proved, so far we only know that all points $q$ in $B_r(p)$ satisfy $d(p,q) \le r$. We will show below that $B_r(p)$ is indeed the ball of radius $r$ about $p$ whenever $M$ is complete with respect to the metric $d$.

Let $(p_n)$ be a sequence in $M$ converging to $p$. Since $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism for $\varepsilon > 0$ sufficiently small, there is a sequence $v_n \to 0$ in $B_\varepsilon(0_p)$ with $\exp_p(v_n) = p_n$ for all $n$ sufficiently large. Hence $d(p,p_n) \to 0$. If $(q_n)$ is a second convergent sequence in $M$ and $q = \lim q_n$, then by the triangle inequality

$$|d(p_n,q_n) - d(p,q)| \le d(p_n,p) + d(q_n,q) \to 0,$$

and hence $d$ is continuous on $M \times M$.

Lemma 1.2. *Let $p \in M$ and $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism. Then*
*1) for any $v \in B_\varepsilon(0_p)$, $d(p, \exp_p(v)) = \|v\|$, and the curve $\exp_p(tv)$, $0 \le t \le 1$, is up to (weakly monotone) reparameterization the unique minimal connection between $p$ and $\exp_p(v)$.*
*2) for any $q \notin B_\varepsilon(p)$ we have $d(p,q) > \varepsilon$. For any $\delta \in (0,\varepsilon)$ there is a $v \in B_\varepsilon(0_p)$ with $\|v\| = \delta$ such that*

$$d(p,q) = d(p, \exp_p(v)) + d(\exp(v), q) = \delta + d(\exp(v), q).$$

*Proof.* Consider the continuous function $r : B_\varepsilon(p) \to \mathbb{R}$ defined by $r(\exp(v)) = \|v\|$. Recall that $r$ is smooth on $B_\varepsilon(p)\backslash\{p\}$, and that in $q = \exp_p(v)$

$$\operatorname{grad} r(q) = (\exp_p)_{*v}(v/\|v\|),$$

the radial field of norm 1 about $p$. Let $c : [a,b] \to B_\varepsilon(p)$ be piecewise smooth with $c(a) = p$. We show that $L(c) \ge r(c(b))$. To that end, we can assume that

$$a = \max\{t \in [a,b] \mid c(t) = p\} \quad \text{and} \quad b = \min\{t \in [a,b] \mid r(c(t)) = r(c(b))\}.$$

Then since $\|\operatorname{grad} r\| = 1$,

$$L(c) \ge \int_a^b \|c'(t)\|\, dt \ge \int_a^b \langle c'(t), \operatorname{grad} r(c(t))\rangle\, dt = r(c(b) - r(c(a)) = r(c(b)).$$

Note that equality holds iff $c'(t) = \alpha(t) \operatorname{grad} r(c(t))$ for all $t \in (a,b)$, where $\alpha = \alpha(t) \ge 0$. Hence 1) and the first assertion in 2).

Now let $\delta \in (0,\varepsilon)$ and $q \in M\backslash B_\varepsilon(p)$. Let $c_n : [0,1] \to M$ be a sequence of piecewise smooth curves from $p$ to $q$ with $L(c_n) \to d(p,q)$. Since $\exp_p$ is a diffeomorphism on $B_\varepsilon(0_p)$ and $M$ is Hausdorff, there are $t_n \in (0,1)$ and $v_n \in T_pM$ with $\|v_n\| = \delta$ such that $c_n(t_n) = \exp_p(v_n)$. Since the sphere of radius $\delta$ in $T_pM$ is compact, we can assume that the sequence $(v_n)$ converges to a vector $v \in T_pM$ of norm $\delta$. Since $d$ and $\exp_p$ are continuous and

$$d(p,q) \le d(p, \exp_p(v_n)) + d(\exp_p(v_n), q) \le L(c_n) \to d(p,q),$$

we conclude that

$$d(p,q) = d(p, \exp_p(v)) + d(\exp_p(v), q) = \delta + d(\exp_p(v), q). \qquad \square$$

Note that Lemma 1.2 justifies the notation $B_\varepsilon(p)$ for $\exp_p(B_\varepsilon(0_p))$ — $B_\varepsilon(p)$ is the ball of radius $\varepsilon$ about $p$ with respect to the metric $d$.

Corollary 1.3. *The Riemannian distance $d$ is a metric on $M$ and $d$ induces the given topology on $M$.*

*Proof.* By Assertion 1) of Lemma 1.2, $d(p,q) = 0$ implies $p = q$. Hence $d$ is a metric. Since $d$ is continuous, the balls $B_r(p)$, $r > 0$, are open in $M$. Vice versa, if $U$ is open and $p$ is a point in $U$, then $\exp_p^{-1}(U)$ contains an open neighborhood of $0_p$ in $T_pM$. Therefore there is an $\varepsilon > 0$ with $B_\varepsilon(0_p) \subset \exp_p^{-1}(U)$. Then $B_\varepsilon(p) \subset U$. $\square$

Lemma 1.4 (Main Argument). *Let $p, q \in M$ and set $r := d(p,q)$. Suppose that the closed ball $\overline{B}_r(0_p)$ is contained in the domain of definition of* $\exp_p$*. Then there is a unit vector $v \in T_v M$ with $\exp_p(rv) = q$. In particular, $\exp_p(tv)$, $0 \le t \le r$, is a minimal curve from $p$ to $q$.*

*Proof.* By Lemma 1.2, there are a unit vector $v$ and a number $\delta > 0$ such that

$$r = d(p,q) = d(\exp_p(\delta v) + d(\exp_p(\delta v), q) = \delta + d(\exp_p(\delta v), q)\,.$$

The idea is that this particular $v$ shows in the right direction: we show that $\exp_p(rv) = q$. To that end, we let $A \subset [0,r]$ be the subset of $t$ with

$$r = d(p, \exp_p(tv)) + d(\exp_p(tv), q) = t + d(\exp_p(tv), q)\,.$$

Suppose $t_0 \in A$ and let $t \in (0, t_0)$. Then by the triangle inequality

$$\begin{aligned} d(p,q) &\le t + d(\exp_p(tv), q) \\ &\le t + (t_0 - t) + d(\exp_p(t_0 v), q) \\ &= t_0 + d(\exp_p(t_0 v), q) = d(p,q). \end{aligned}$$

Hence $t \in A$ and therefore $A$ is a closed interval, $A = [0, t_0]$ for some $t_0 \le r$. By the choice of $v$ we have $t_0 \ge \delta > 0$. It remains to show that $t_0 = r$.

To arrive at a contradiction we assume $t_0 < r$ and set $p_0 = \exp_r(t_0 v)$. By Lemma 1.2, there are a unit vector $v_0$ at $p_0$ and a number $\delta_0 > 0$ such that

$$d(p_0, q) = d(p, \exp_{p_0}(\delta_0 v_0)) = \delta_0 + d(\exp_{p_0}(\delta_0 v_0), q)\,.$$

We conclude that for $p_1 = \exp_{p_0}(\delta_0 v_0)$,

$$d(p,q) = t_0 + d(p_0, q) = d(p, p_0) + d(p_0, p_1) + d(p_1, q) \ge d(p,q)\,.$$

Hence the unit speed curve $c : [0, t_0 + \delta_0] \to M$ defined by

$$c(t) = \begin{cases} \exp_p(tv), & 0 \le t \le t_0, \\ \exp_{p_0}((t - t_0)v_0), & t_0 \le t \le t_0 + \delta_0, \end{cases}$$

is a minimal connection from $p$ to $p_1$. It follows that $c$ is a geodesic and hence that

$$c(t) = \exp_p(tv), \quad 0 \le t \le t_0 + \delta_0\,.$$

We conclude that $t_0 + \delta_0 \in A$, a contradiction. □

Theorem of Hopf-Rinow 1.5. *Let $M$ be a connected Riemannian manifold. Then the following four assertions are equivalent:*
*1) $(M,d)$ is a complete metric space.*
*2) $M$ is geodesically complete; that is, the real line $\mathbb{R}$ is the maximal domain of definition of geodesics in $M$.*
*3) There is a $p \in M$ such that $\exp_p$ is defined on all of $T_pM$.*
*4) Bounded (with respect to $d$) subsets of $M$ are relatively compact.*
*Any of these four assertions implies that for any two points $p, q \in M$, there is a minimal geodesic from $p$ to $q$.*

*Proof.* The implications 2) $\Longrightarrow$ 3) and 4) $\Longrightarrow$ 1) are trivial. As for 3) $\Longrightarrow$ 4), suppose that $A \subset M$ is bounded with respect to $d$. That is, there are a point $q \in M$ and a number $r > 0$ with $A \subset B_r(q)$. Since $M$ is connected, $d(p,q) < \infty$. By the triangle inequality, $A \subset B_R(p)$ with $R = r + d(p,q)$. But then $A$ is contained in the compact set $\exp_p(\overline{B}_R(0_p))$ by the previous main argument.

We now prove 1) $\Longrightarrow$ 2). Suppose that the maximal interval $I$ of definition of a unit speed geodesic $c$ has an upper bound $t_0 < \infty$. Let $(t_n)$ be an increasing sequence in $I$ converging to $t_0$. Then

$$d(c(t_n), c(t_m)) \le |t_n - t_m|.$$

Hence $(c(t_n))$ is a Cauchy sequence. Since $M$ is complete, there is a (unique) limit point $p$. Now there is an $\varepsilon > 0$ such that $B_\varepsilon(0_q)$ is in the domain of definition of $\exp_q$ for all $q \in B_\varepsilon(p)$. But then

$$\exp_{c(t_n)}((t - t_n) \cdot c'(t_n)), \quad |t - t_n| < \varepsilon,$$

is an honest geodesic extension of $c$ as soon as $t_0 - t_n < \varepsilon$, a contradiction. In a similar way we conclude that $I$ has no lower bound. Hence the maximal interval of definition of a unit speed geodesic is the real line. Hence 2). □

In [CV], S. S. Cohn-Vossen proves a version of the Hopf-Rinow Theorem for locally compact geodesic spaces, see also Chapter I in [NP].

Exercise 1.6. Show by example that the last assertion in Theorem 1.5 is not equivalent to (any of) the first four assertions. (Hint: Start in dimension 1.)

## 2. The Theorem of Bonnet-Myers

In what follows, let $M$ be a connected Riemannian manifold of dimension $m$.

Lemma 2.1. *Let $\kappa > 0$ and suppose that $c : [0,l] \to M$ is a unit speed geodesic such that $\mathrm{ric}(c'(t), c'(t)) \ge (m-1)\kappa$ for all $t \in [0,l]$. Then $l > \pi/\sqrt{\kappa}$ implies that there is a proper smooth variation $(c_s)$ of $c = c_0$ with $L(c_s) < L(c)$ for all $s \ne 0$. In particular, $c$ is not a minimal connection of its end points $p = c(0)$ and $q = c(l)$.*

*Proof.* Let $(E_1 = c', E_2, \dots, E_m)$ be a parallel $ON$-frame along $c$. For $2 \le i \le m$ set

$$V_i(t) := \sin(\pi t/l) \cdot E_i(t), \quad 0 \le t \le l.$$

Then $V_i(0) = 0$ and $V_i(l) = 0$. F or the index form of $c$ we obtain

$$\begin{aligned}\sum_{i=2}^m I(V_i, V_i) &= \sum_{i=2}^m \int_0^l \sin^2(\pi t/l)\Big(\frac{\pi^2}{l^2} - \langle R(E_i(t), c'(t))c'(t), E_i(t)\rangle\Big)\,dt \\ &= \int_0^l \sin^2(\pi t/l)\Big((m-1)\frac{\pi^2}{l^2} - \mathrm{ric}(c'(t), c'(t))\Big)\,dt < 0.\end{aligned}$$

Hence there is an $i$ with $I(V_i, V_i) < 0$. Now there is a proper smooth variation $(c_s)$ of $c$ with variation field $V_i$. Then $L(s) := L(c_s)$ is smooth in $s$ with $L'(0) = 0$ and $L''(0) = I(V_i, V_i) < 0$. □

THEOREM OF BONNET-MYERS (GEOMETRIC VERSION) 2.2. *Let $M$ be a complete and connected Riemannian manifold of dimension $m \geq 2$. Suppose that there is a constant $\kappa > 0$ such that*

$$ric(v, v) \geq (m-1) \cdot \kappa \cdot \|v\|^2$$

*for all tangent vectors $v$ of $M$. Then the diameter* $\operatorname{diam}(M)$ *of $M$ satisfies*

$$\operatorname{diam}(M) \leq \pi/\sqrt{\kappa}.$$

A round sphere of radius $r$ in Euclidean space has Ricci curvature $(m-1)/r^2$ and (inner) diameter $\pi r$. Hence the diameter estimate in Theorem 2.2 is optimal.

EXERCISE 2.3. Show that the completeness assumption on $M$ cannot be deleted.

*Proof ofTheorem 2.2.* Suppose there are points $p, q \in M$ with $d(p, q) > \pi/\sqrt{\kappa}$. By the Theorem of Hopf-Rinow there is a minimal geodesic $c$ from $p$ to $q$. The length of $c$ is $l = d(p, q) > \pi/\sqrt{\kappa}$. By the previous lemma this contradicts the minimality of $c$. □

LEMMA 2.4. *Let $M$, $\tilde{M}$ be connected Riemannian manifolds and $\pi : \tilde{M} \to M$ be a Riemannian covering. Then $\tilde{M}$ is complete iff $M$ is complete.*

*Proof.* Suppose $M$ is complete. Let $\tilde{p} \in \tilde{M}$ and $\tilde{v}$ be a tangent vector of $\tilde{M}$ at $\tilde{p}$. Let $c : \mathbb{R} \to M$ be the (maximal) geodesic with $c'(0) = \pi_*(\tilde{v})$. Now $\pi : \tilde{M} \to M$ is a covering, and hence there is exactly one lift $\tilde{c} : \mathbb{R} \to \tilde{M}$ of $c$ with $\tilde{c}(0) = \tilde{p}$. Since $\pi$ is a local diffeomorphism, $c$ is smooth and has initial velocity $\tilde{v}$. Since $\pi$ is a local isometry and $c$ is a geodesic in $M$, $\tilde{c}$ is a geodesic in $\tilde{M}$. It follows that $\tilde{M}$ is complete. The other direction is clear since $\pi$ is a surjective local isometry. □

THEOREM OF BONNET-MYERS (TOPOLOGICAL VERSION) 2.5. *Let $M$ be a complete and connected Riemannian manifold. Suppose that there is a constant $\kappa > 0$ such that*

$$ric(v, v) \geq (m-1) \cdot \kappa \cdot \|v\|^2$$

*for all tangent vectors $v$ of $M$. Then $M$ is compact and the fundamental group of $M$ is finite.*

*Proof.* Compactness of $M$ follows from the Theorem of Hopf-Rinow and finiteness of the diameter, see Theorem 2.2.

Let $\pi : \tilde{M} \to M$ be the universal covering of $M$. Endow $\tilde{M}$ with the pull back $\tilde{g} = \pi^* g$ of the metric $g$ of $M$. Then $\pi$ is a Riemannian covering and hence, by the previous lemma, $\tilde{M}$ is complete with respect to $\tilde{g}$. Since $\pi$ is a local isometry, local geometric invariants of $\tilde{M}$ and $M$ are the same (in corresponding points). In particular, we have the same estimate for the Ricci curvature,

$$\widetilde{\operatorname{ric}}(\tilde{v}, \tilde{v}) \geq (m-1)\kappa \cdot \|\tilde{v}\|^2$$

for all tangent vectors $\tilde{v}$ of $\tilde{M}$. Now $\tilde{M}$ is complete and connected. Hence $\operatorname{diam}(\tilde{M}) \le \pi/\sqrt{\kappa}$. By the Hopf-Rinow Theorem we conclude that $\tilde{M}$ is compact. Therefore the number of sheets of the covering $\pi$ is finite, and hence $\pi_1(M)$ is finite. □

## 3. The Theorem of Hadamard-Cartan

Lemma 3.1. *Let $M$, $\tilde{M}$ be connected Riemannian manifolds and $\pi : \tilde{M} \to M$ be a local isometry. Suppose that $\tilde{M}$ is complete. Then $M$ is complete and $\pi$ is a covering map.*

*Proof.* Fix a point $p \in M$ in the image of $\pi$. Let $v \in T_pM$. Choose a point $\tilde{p}$ in the preimage of $p$. Then there is a unique tangent vector $\tilde{v}$ at $\tilde{p}$ with $\pi_*(\tilde{v}) = v$. Let $\tilde{c} : \mathbb{R} \to \tilde{M}$ be the (maximal) geodesic with initial velocity $\tilde{v}$. Then $c = \pi \circ \tilde{c}$ is a geodesic in $M$ since $\pi$ is a local isometry. The initial velocity of $c$ is $v$ since $\pi_*(\tilde{v}) = v$. Hence for every $v \in T_pM$ the maximal geodesic with initial velocity $v$ is defined on all of $\mathbb{R}$. Therefore $M$ is complete.

Let $q$ be another point in $M$. Then by the Theorem of Hopf-Rinow, there is a geodesic segment $c : [0,1] \to M$ from $p$ to $q$. Let $\tilde{w}$ be a tangent vector at $\tilde{p}$ with $\pi_*(\tilde{w}) = c'(0)$. Let $\tilde{c} : [0,1] \to \tilde{M}$ be the geodesic segment in $\tilde{M}$ with initial velocity $\tilde{w}$. Then $c = \pi \circ \tilde{c}$ and hence $\pi(\tilde{c}(1)) = c(1) = q$. Hence $\pi$ is surjective.

Choose $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism. In our last step of the proof we show that $B_\varepsilon(p)$ is evenly covered by the balls $B_\varepsilon(\tilde{p})$, where $\tilde{p}$ runs over the points in the preimage $\pi^{-1}(p)$. Since $\pi$ is a local isometry, for any such point $\tilde{p}$,

$$(\pi\big|B_\varepsilon(\tilde{p})) \circ (\exp_{\tilde{p}}\big|B_\varepsilon(0_{\tilde{p}})) = (\exp_p\big|B_\varepsilon(0_p)) \circ (\pi_{*\tilde{p}}\big|B_\varepsilon(0_{\tilde{p}})).$$

Each of the two maps on the right hand side is a diffeomorphism. It follows that $\exp_{\tilde{p}}$ is injective and has maximal rank on $B_\varepsilon(0_{\tilde{p}})$. Hence $\exp_{\tilde{p}}|B_\varepsilon(0_{\tilde{p}})$ is a diffeomorphism, hence also $\pi|B_\varepsilon(\tilde{p})$.

Let $\hat{p}$ be another point in the preimage of $p$ and $\tilde{c} : [0,l] \to \tilde{M}$ be a minimal unit speed geodesic from $\tilde{p}$ to $\hat{p}$. Then $c = \pi \circ \tilde{c}$ is a unit speed geodesic in $M$ with $c(0) = c(l) = p$. Hence $l \ge 2\varepsilon$, and hence $B_\varepsilon(\tilde{p}) \cap B_\varepsilon(\hat{p}) = \emptyset$. It follows that $B_\varepsilon(p)$ is evenly covered by the balls $B_\varepsilon(\tilde{p})$. □

Theorem of Hadamard-Cartan 3.2. *Let $M$ be a complete and connected Riemannian manifold. Suppose that the sectional curvature $K_M$ of $M$ is non-positive. Then for any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is the universal covering.*

Exercise 3.3. Show that the completeness assumption on $M$ cannot be deleted.

*Proof of Theorem 3.2.* Let $c : \mathbb{R} \to M$ be a geodesic and $V$ be a Jacobi field along $c$. Then

$$\langle V, V\rangle'' = 2(\langle V', V'\rangle - \langle R(V, c')c', V\rangle) \ge 0,$$

hence $\langle V, V\rangle$ is convex. Therefore $V(t) \neq 0$ for all $t \neq 0$ if $V(0) = 0$ and $V'(0) \neq 0$. In particular, $\exp_p : T_pM \to M$ is a local diffeomorphism. Since $M$ is complete, $\exp_p$ is surjective.

Now let $\tilde{g} = \exp_p^*(g)$ be the pullback of the metric $g$ on $M$. Then $\tilde{g}$ is a Riemannian metric on the manifold $T_pM$ and by definition, $\exp_p : T_pM \to M$ is a local isometry with respect to $\tilde{g}$ and $g$. For any $v \in T_pM$, the image $\exp(tv)$ of the line $tv$, $t \in \mathbb{R}$, through $0_p$ in $T_pM$ is a geodesic in $M$. Hence this line is a geodesic in $T_pM$ (with respect to $\tilde{g}$). By the Theorem of Hopf-Rinow, $T_pM$ is a complete Riemannian manifold with respect to $\tilde{g}$. By Lemma 3.1, $\exp_p$ is a covering. Since $T_pM$ is simply connected, $\exp_p$ is the universal covering. □

COROLLARY 3.4. *Let $M$ be a complete and simply connected Riemannian manifold. Suppose that the sectional curvature $K_M$ of $M$ is nonpositive. Then $\exp_p : T_pM \to M$ is a diffeomorphism.* □

In [AB], S. Alexander and R. Bishop prove a version of the Hadamard-Cartan Theorem for convex geodesic spaces, see also Chapter I in [NP].


## ACKNOWLEDGMENTS

I would like to thank Alexander Lytchak for proof-reading.


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