BASIC DIFFERENTIAL GEOMETRY: VARIATIONAL THEORY OF GEODESICS

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Introduction

We assume throughout that M is a Riemannian manifold. For points p, q in M, we ask for the critical points of the length functional L on the space Ω_{pq}^{ps} of piecewise smooth curves from p to q. Since Ω_{pq}^{ps} is not a manifold and L is not smooth, the notion of critical point and derivative of L require an explanation. Following one of the possible definitions of a tangent vector at a point in a smooth manifold, we consider smooth curves $s \mapsto c_s \in \Omega_{pq}^{ps}$, $-\varepsilon < s < \varepsilon$, of piecewise smooth curves in Ω_{pq}^{ps} so that the length $L(s) := L(c_s)$ depends smoothly on s. Then $\partial_s L(0)$ represents the derivative of L at $c = c_0$ in the direction represented by the curve $s \mapsto c_s$ in Ω_{pq}^{ps}.

The notion of smooth curves in Ω_{pq}^{ps} is made precise by the concept of *proper piecewise smooth variation* and the corresponding derivative $\partial_s L(0)$ is obtained in the First Variation Formula. As an application of the First Variation Formula we obtain that the set of critical points of the length functional L on Ω_{pq}^{ps} consists precisely of the geodesic segments from p tp q.

For a smooth function f on a smooth manifold, the second derivative of f is well-defined in the critical points of f.[1] Correspondingly we can define the second derivative of the length functional for a geodesic segment c in Ω_{pq}^{ps}. The Second Variation Formula determines this second derivative in terms of the geometry of M along c.

Last update: 2003-05-25.

[1]In general, there is no reasonable way of defining the second derivative of f at a point, except the point is critical.

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Conventions

The covariant derivative of a vector field X along a piecewise smooth curve c is denoted by X'. Correspondingly, the covariant derivative of c' is denoted c''.

1. First and Second Variation of Arc Length

Let $c : [a, b] \to M$ be a piecewise smooth curve. By definition, the *length* $L(c)$ of c is

$$L(c) = \int_a^b \|c'(t)\| dt. \tag{1.1}$$

The length of c is invariant under reparameterization. More generally, if $\varphi : [\alpha, \beta] \to [a, b]$ is a piecewise smooth, (weakly) monotone and surjective map, then $L(c \circ \varphi) = L(c)$.

A piecewise smooth n-parameter *variation* of c is a map $H : U \times [a, b] \to M$, where U is an open neighborhood of 0 in $\mathbb{R}^n$, such that $H(0, \cdot) = c$ and such that there is a subdivision $a = t_0 < \ldots < t_k = b$ of $[a, b]$ such that H is smooth on the sets $U \times [t_{i-1}, t_i]$, $1 \le i \le k$. We say that a variation H of c is *proper* if $H(s, a) = c(a)$ and $H(s, b) = c(b)$ for all $s \in U$.

We consider variations as a formalization of the idea of smooth families of curves. For a given variation H of c, we set $c_s = H(s, \cdot)$. If H is a 1-parameter variation, then U is an open interval and the family (c_s) is a smooth curve — in the above sense — in the space of piecewise smooth curves. We may then consider the *variation field* $V = \partial_s H(0, \cdot)$ as the tangent vector to this curve at time $s = 0$. If H is proper, $V(a) = 0$ and $V(b) = 0$. Note that V is piecewise smooth.

Lemma 1.1. *Let $c : [a, b] \to M$ be piecewise smooth and V be a piecewise smooth vector field along c. Then there is a piecewise smooth 1-parameter variation H of c with variation field V and with $H(s, t) = c(t)$ for all (s, t) with $V(t) = 0$. In particular, H is proper if $V(a) = 0$ and $V(b) = 0$.*

Remark 1.2. The variation H in the Lemma is not unique. This is similar to the fact that a tangent vector in a manifold is not represented by a unique smooth curve through the foot point of the vector but rather by an equivalence class of such curves.

Proof of Lemma 1.1. Set $H(s, t) := \exp(s \cdot V(t))$. □

1.1. First Variation of Arc Length.

Since the length of a curve is invariant under reparameterization, we let $c : [a, b] \to M$ be a piecewise smooth curve with constant speed $v \neq 0$, that is, $\|c'(t)\| = v$ for all $t \in [a, b]$.

First Variation Formula 1.3. *Let $H : U \times [a, b] \to M$ be a piecewise smooth 1-parameter variation of c. Let V be the variation field of H and set $c_s := H(s, \cdot)$, $L(s) := L(c_s)$. Then $L : U \to \mathbb{R}$ is smooth about 0 and*

$$\partial_s L(0) = \frac{1}{v} \cdot \Big\{ \langle V, c' \rangle \Big|_a^b + \sum_{i=1}^{k-1} \langle V(t_i), \Delta c'(t_i) \rangle - \int_a^b \langle V, c'' \rangle \, dt \Big\},$$

where the subdivision $a = t_0 < \ldots < t_k = b$ *of* $[a, b]$ *is chosen such that* c *is smooth on the sets* $[t_{i-1}, t_i]$, *and* $\Delta c'(t_i) := c'(t_i - 0) - c'(t_i + 0)$.

Proof. By assumption we have $c'(t) \neq 0$ for all $t \in [a, b]$. Hence by diminishing the size of U if necessary, we can assume $c_s'(t) \neq 0$ for all $s \in U$ and $t \in [a, b]$. Then $\|c_s'(t)\|$ is smooth on the sets $U \times [t_{i-1}, t_i]$ and hence L is smooth in s. Moreover,

$$\begin{aligned}
\partial_s L(0) &= \partial_s \Big\{ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|c_s'(t)\| \, dt \Big\} \Big|_{s=0} \\
&= \Big\{ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \partial_s \Big(\|c_s'(t)\| \Big) \, dt \Big\} \Big|_{s=0} \\
&= \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{\langle D_s \partial_t H(0,t), \partial_t H(0,t) \rangle}{\|c_0'(t)\|} \, dt \\
&= \frac{1}{v} \cdot \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle D_t \partial_s H(0,t), c'(t) \rangle \, dt \\
&= \frac{1}{v} \cdot \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \Big\{ \partial_t \langle V, c' \rangle - \langle V, D_t c' \rangle \Big\} \, dt \\
&= \frac{1}{v} \cdot \Big\{ \sum_{i=1}^{k} \langle V, c' \rangle \Big|_{t_{i-1}}^{t_i} - \int_a^b \langle V, D_t c' \rangle \, dt \Big\}. \qquad \square
\end{aligned}$$

We call $\partial_s L(0)$ the *first variation of arc length* with respect to the given variation. The first variation of arc length corresponds to the derivative of the length functional in the direction of the "tangent vector" V of the variation. In fact, $\partial_s L(0)$ only depends on V, not on the particular choice of variation H defining V. This is in accordance with derivatives of functions on smooth manifolds.

In many applications, $c = c_0$ is a geodesic segment. Then c is smooth and hence the term involving the (possible) breaks $\Delta c'(t_i)$ vanishes. Furthermore, c'' vanishes by definition, and therefore the integral on the right hand side vanishes as well.

Corollary 1.4. *Let* $c : [a, b] \to M$ *be a geodesic segment and* $H : U \times [a, b] \to M$ *a piecewise smooth 1-parameter variation of* c. *Let* V *be the variation field of* H *and set* $c_s := H(s, \cdot)$ *and* $L(s) := L(c_s)$. *Then*

$$\partial_s L(0) = -\cos\alpha \cdot \|V(a)\| - \cos\beta \cdot \|V(b)\|,$$

where $\alpha = \angle(c'(a), V(a))$ *and* $\beta = \angle(-c'(b), V(b))$. $\square$

1.2. **Geodesic Segments as Critical Points.** In the First Variation Formula 1.3 it is not required that the variation be proper. If we restrict to proper variations, then it turns out that the curve c is a critical point of the length functional if and only if c is a geodesic segment.

THEOREM 1.5. *A regular piecewise smooth curve $c : [a, b] \to M$ is a geodesic if and only if c has constant (nonzero) speed and the first variation of arc length vanishes for every proper variation of c.*

Proof. If c is a geodesic, then c is smooth and of constant speed v. Now $v \neq 0$ since c is regular. By definition $c'' = 0$, and hence the first variation of arc length vanishes for every proper variation of c.

We assume now that c has constant speed $v \neq 0$ and that the first variation of arc length vanishes for every proper variation of c. Let $a = t_0 < t_1 < \ldots < t_k = b$ be a subdivision of $[a, b]$ such that c is smooth on $[t_{i-1}, t_i]$, $1 \leq i \leq k$. Fix $i \in \{1, \ldots, k\}$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function which is positive on (t_{i-1}, t_i) and 0 otherwise. Let $V = \varphi \cdot c''$. Then V is a piecewise smooth vector field along c with $V(t) = 0$ for $t \in [a, t_{i-1} \cup [t_i, b]$. In particular, there is a proper variation (c_s) of c with variation field V. By the first variation formula and our assumption on c we have

$$0 = \partial_s L(0) = \int_{t_{i-1}}^{t_i} \varphi \cdot \|D_t c'\|^2 \, dt.$$

Hence $c''(t) = 0$ for all $t \in [t_{i-1}, t_i]$. Since i was arbitrary, we conclude that $c|[t_{i-1}, t_i]$ is a geodesic, $1 \leq i \leq k$.

It remains to show $c'(t_i - 0) = c'(t_i + 0)$ for each $i \in \{1, \ldots, k-1\}$. To that end, fix such an i and let E be the parallel field along c with $E(t_i) = c'(t_i - 0) - c'(t_i + 0)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function which is positive on (t_{i-1}, t_{i+1}) and 0 otherwise and set $V = \varphi \cdot E$. Let (c_s) be a proper variation of c with variation field V. Since the first variation of arc length of any proper variation of c vanishes and c is a geodesic on $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$, we get

$$0 = \partial_s L(0) = \langle V(t_i), c'(t_i - 0) - c'(t_i + 0)\rangle = \varphi(t_i) \cdot \|c'(t_i - 0) - c'(t_i + 0)\|^2.$$

Hence $c'(t_i - 0) = c'(t_i + 0)$, $1 \leq i \leq k - 1$, and hence c is a geodesic. ☐

A geodesic segment $c : [a, b] \to M$ with $\|c'\| \neq 0$ is not a critical point of the length functional on the space $\Omega^{ps}(M)$ of all piecewise smooth curves on M. For example, for the smooth variation $H(s, t) = c(a + (1 - s) \cdot (t - a))$ we have $L(c_s) = (1 - s) \cdot L(c)$ and hence $\partial_s L(0) = -L(c) \neq 0$. The point is that boundary conditions have to be imposed, in the case of Theorem 1.5 the boundary condition is that the end points $c(a) = p$ and $c(b) = q$ be fixed by the variation. Other boundary conditions are discussed below.

1.3. **Second Variation of Arc Length.** From now on we assume that $c : [a,b] \to M$ is a geodesic with speed $v \neq 0$. We have shown that c is a critical point of the length functional: The first variation of arc length is zero for any proper variation of c. We now discuss the second derivative of L.

SECOND VARIATION FORMULA 1.6. *Let $H : U \times [a,b] \to M$ be a piecewise smooth 2-parameter variation of c. Set*

$$c_{r,s} := H(r,s,\cdot), \quad V = \partial_r H(0,0,\cdot), \quad W = \partial_s H(0,0,\cdot),$$

and let $\hat{V}$ and $\hat{W}$ be the part of V and W perpendicular to c'. Then $L(r,s) = L(c_{r,s})$ is smooth in (r,s) and

$$\begin{aligned}\partial^2_{r,s} L(0,0) &= \frac{1}{v} \cdot \Big\{ \langle D_r \partial_s H, c' \rangle \Big|_a^b + \int_a^b \Big(\langle \hat{V}', \hat{W}' \rangle - \langle R(\hat{V}, c')c', \hat{W} \rangle \Big)\, dt \Big\} \\ &= \frac{1}{v} \cdot \Big\{ \langle D_r \partial_s H, c' \rangle \Big|_a^b + \langle \hat{V}', \hat{W} \rangle \Big|_a^b \\ &\qquad + \sum_{i=1}^{k-1} \langle \Delta \hat{V}'(t_i), \hat{W}(t_i) \rangle - \int_a^b \langle \hat{V}'' + R(\hat{V}, c')c', \hat{W} \rangle \, dt \Big\},\end{aligned}$$

where the subdivision $a = t_0 < \ldots < t_k = b$ of $[a,b]$ is chosen such that H is smooth on the sets $[t_{i-1}, t_i]$ and $\Delta \hat{V}'(t_i) := \hat{V}'(t_i - 0) - \hat{V}'(t_i + 0)$.

Proof. By diminishing the size of U if necessary, we can assume $c'_{r,s}(t) \neq 0$ for all (r,s,t) in $U \times [t_{i-1}, t_i]$, $1 \leq i \leq k$. Then $\|c'_{r,s}(t)\|$ is smooth in (r,s,t) and we get

$$\begin{aligned}\partial^2_{r,s} L(0,0) &= \partial_r \Big\{ \int_a^b \frac{\langle D_t \partial_s H, \partial_t H \rangle}{\|\partial_t H\|}\, dt \Big\}\Big|_{r=s=0} \\ &= \Big\{ \int_a^b \frac{\langle D_r D_t \partial_s H, \partial_t H \rangle}{\|\partial_t H\|} + \int_a^b \frac{\langle D_t \partial_s H, D_t \partial_r H \rangle}{\|\partial_t H\|} \\ &\qquad - \int_a^b \frac{\langle D_t \partial_s H, \partial_t H \rangle \langle D_t \partial_r H, \partial_t H \rangle}{\|\partial_t H\|^3}\, dt \Big\}\Big|_{r=s=0}.\end{aligned}$$

Now for the second term II and third term III on the right hand side we have

$$II - III = \frac{1}{v} \cdot \int_a^b \langle \hat{V}', \hat{W}' \rangle \, dt.$$

In the first term I, the denominator is v and hence

$$I = \frac{1}{v} \cdot \Big\{ \int_a^b \langle D_t D_r \partial_s H, \partial_t H \rangle \, dt + \int_a^b \langle R(\partial_r H, \partial_t H) \partial_s H, \partial_t H \rangle \, dt \Big\}\Big|_{r=s=0}.$$

For the second term I_2 on the right hand side we have

$$I_2 = -\frac{1}{v} \cdot \int_a^b \langle R(V, c')c', W \rangle \, dt = -\frac{1}{v} \cdot \int_a^b \langle R(\hat{V}, c')c', \hat{W} \rangle \, dt.$$

Since c is a geodesic, $D_t\partial_t H(0,0,\cdot) = 0$, hence the first term I_1 on the right hand side above is

$$I_1 = \frac{1}{v}\cdot\int_a^b \partial_t\langle D_r\partial_s H, \partial_t H\rangle\,dt = \frac{1}{v}\cdot\sum_{i=1}^{k}\langle D_r\partial_s H, c'\rangle\Big|_{t_{i-1}}^{t_i} = \frac{1}{v}\cdot\langle D_r\partial_s H, c'\rangle\Big|_a^b.$$

This is the first of the asserted formulas. As for the second, we note that

$$\partial_t\langle\hat{V}', \hat{W}\rangle = \langle\hat{V}'', \hat{W}\rangle + \langle\hat{V}', \hat{W}'\rangle. \qquad \square$$

Note that the term $\langle D_r\partial_s H, c'\rangle\Big|_a^b$ depends on the chosen variation, not only on the "tangent vectors" V and W to the variation. This is due to the fact that the geodesic segment $c : [a,b] \to M$ is not a critical point of the length functional on the space of all piecewise smooth curves on M. The *index form* of c is defined as

$$\begin{aligned} I(V,W) :&= \frac{1}{v}\cdot\Big\{\int_a^b \Big(\langle\hat{V}', \hat{W}'\rangle - \langle R(\hat{V}, c')c', \hat{W}\rangle\Big)\,dt\Big\} \\ &= \frac{1}{v}\cdot\Big\{\sum_{i=1}^{k-1}\langle\Delta\hat{V}'(t_i), \hat{W}(t_i)\rangle - \int_a^b\langle\hat{V}'' + R(\hat{V}, c')c', \hat{W}\rangle\,dt\Big\}, \end{aligned}$$

where the notation is as in the Second Variation Formula 1.6. The index form depends only on V and W, not on the variation defining V and W and is a symmetric bilinear form on the space of piecewise smooth vector fields along c.

THEOREM 1.7. *Suppose $c : [a,b] \to M$ is a geodesic segment with speed $\|c'\| = v \neq 0$. Then for any* proper *2-parameter variation H of c*

$$\partial^2_{r,s}L(0,0) = I(V,W).$$

That is, the index form is the second derivative of the length functional L at c in the space of piecewise smooth curves joining $p = c(a)$ and $q = c(b)$.

2. Boundary Conditions

So far we discussed only one boundary condition, the fixed end point condition. There are other interesting boundary conditions. For example, for given submanifolds M_a and M_b of M, we may consider the condition $c(a) \in M_a$ and $c(b) \in M_b$. The fixed end point case is the special case where M_a and M_b are points. Another important example is the periodicity condition $c(b) = c(a)$. In general, a boundary condition is defined by a submanifold N of $M \times M$ and the boundary condition is $(c(a), c(b)) \in N$. In the first example mentioned above, $N = M_a \times M_b$; in the second, N is the diagonal in $M \times M$.

We fix a boundary condition $N \subset M \times M$. We ask for critical points of the length functional L on the space Ω_N^{ps} of piecewise smooth curves $c : [a,b] \to M$ with $(c(a), c(b)) \in N$. To that end, we say that a variation (c_s) of such a curve c is an *N-variation* if $c_s \in \Omega_N^{ps}$ for all s. We say that $c \in \Omega_N^{ps}$ is a *critical point* of the length functional L on Ω_N^{ps} if the first variation of arc length vanishes for each

N-variation of c. Note that proper variations are N-variations for any boundary condition N. Hence by Theorem 1.5, critical points of the length functional L on Ω_N^{ps} are geodesic segments. Now the following characterization is immediate from the First Variation Formula.

THEOREM 2.1. *Suppose $c \in \Omega_N^{ps}$ is a geodesic segment with $\|c'\| \neq 0$. Then c is a critical point of the length functional L on Ω_N^{ps} if and only if $(c'(a), -c'(b))$ is perpendicular to N in $(c(a), c(b))$.* □

In the first example $N = M_a \times M_b$, this just means that $c'(a)$ is perpendicular to M_a in $c(a)$ and that $c'(b)$ is perpendicular to M_b in $c(b)$. In the special case of fixed end points the latter conditions are empty. In the second example, the periodic boundary condition, where N is the diagonal in $M \times M$, this means that c is a *closed geodesic*; that is $c'(b) = c'(a)$.

Now we assume that c is a geodesic segment with $\|c'\| \neq 0$ and $(c'(a), -c'(b))$ perpendicular to N in $(c(a), c(b))$. By Theorem 2.1 this means that C is a critical point of the length functional L on Ω_N^{ps}. We observe that for an N-variation of c, the first term in the Second Variation Formula is precisely the second fundamental form S of N in the point $(p, q) := (c(a), c(b))$ in the direction of the normal vector $(-c'(a), c'(b))$. Using the notation in the Second Variation Formula 1.6, we get the following result.

THEOREM 2.2. *Suppose c is a geodesic segment with speed $\|c'\| = v \neq 0$ and $(c'(a), -c'(b))$ perpendicular to N in $(c(a), c(b))$. Then for any 2-parameter variation H of c we have*

$$\begin{aligned}\partial^2_{r,s} L(0) &= \frac{1}{v} \cdot \Big\{ \big\langle S\big((V(a), V(b)), (-c'(a), c'(b))\big)\big\rangle \\ &\qquad + \int_a^b \Big(\langle \hat{V}', \hat{W}' \rangle - \langle R(\hat{V}, c')c', \hat{W} \rangle \Big)\, dt \Big\} \\ &= \frac{1}{v} \cdot \Big\{ \big\langle S\big((V(a), V(b)), (-c'(a), c'(b))\big)\big\rangle + \langle \hat{V}', \hat{W} \rangle \Big|_a^b \\ &\qquad + \sum_{i=1}^{k-1} \langle \Delta \hat{V}'(t_i), \hat{W}(t_i) \rangle - \int_a^b \langle \hat{V}'' + R(\hat{V}, c')c', \hat{W} \rangle\, dt \Big\}.\end{aligned}$$

In the first example $N = M_a \times M_b$, the second fundamental form S of N in $(c(a), c(b))$ is the sum of the second fundamental forms S_a of M_a in $c(a)$ and S_b of M_b in $c(b)$. Hence

$$\big\langle S\big((V(a), V(b)), (-c'(a), c'(b))\big)\big\rangle = \big\langle S_b\big(V(b), c'(b)\big)\big\rangle - \big\langle S_a\big(V(a), c'(a)\big)\big\rangle.$$

In the second example, the periodic boundary condition, we observe that the diagonal is totally geodesic in $M \times M$. Hence in this case, the first term in the Second Variation Formula vanishes for all N-variations.

3. Energy Versus Arc Length

The length of a piecewise smooth curve $c : [a, b] \to M$ is invariant under reparameterization. In considerations of a more analytic nature this is a disadvantage and it is better to use the *energy* of c,

$$E(c) = \frac{1}{2} \int_a^b \|c'(t)\|^2 \, dt \,. \tag{3.1}$$

One advantage is clear right from the definition: Whereas the norm$\| \cdot \|$ as a function on the vector space T_pM, $p \in M$, is not smooth in the zero-vector $0_p \in T_pM$, the square $\|\cdot\|^2$ of the norm is. A disadvantage of the energy functional is its less geometric nature.

The energy is not invariant under reparameterization, this is the heart of the matter here. In fact, by the Cauchy-Schwarz Inequality we have

$$L^2(c) \leq 2(b - a) \cdot E(c) \tag{3.2}$$

with equality if and only if c has constant speed, $\|c'(t)\| = v = \text{const.}$

As in the case of arc length, there are formulas for the first and second variation of energy. These formulas are very similar to, but somewhat simpler than the ones for the first and second variation of arc length.

First Variation of Energy 3.1. *Let $c : [a, b] \to M$ be piecewise smooth and $H : U \times [a, b] \to M$ a piecewise smooth 1-parameter variation of c. Let V be the variation field of H and set $c_s := H(s, \cdot)$, $E(s) := E(c_s)$. Then $E : U \to \mathbb{R}$ is smooth and*

$$\partial_s E(0) = \langle V, c' \rangle \Big|_a^b + \sum_{i=1}^{k-1} \langle V(t_i), \Delta c'(t_i) \rangle - \int_a^b \langle V, D_t c' \rangle \, dt,$$

where the subdivision $a = t_0 < \ldots < t_k = b$ of $[a, b]$ is such that c is smooth on the sets $[t_{i-1}, t_i]$ and $\Delta c'(t_i) := c'(t_i - 0) - c'(t_i + 0)$. □

We leave the proof of this formula and of the next assertions as an exercise. They consist of simplifications of the proofs of the corresponding statements for the length functional. Note that it is not necessary to assume that c has constant and nonzero speed. As a result, we have the following more elegant characterization of geodesics.

Theorem 3.2. *A piecewise smooth curve $c : [a, b] \to M$ is a geodesic if and only if the first variation of energy vanishes for any proper variation of c.* □

Second Variation of Energy 3.3. *Let $H : U \times [a,b] \to M$ be a piecewise smooth 2-parameter variation of c. Set*

$$c_{r,s} = H(r,s,\cdot), \quad V = \partial_r H(0,0,\cdot), \quad W = \partial_s H(0,0,\cdot).$$

Then $E(r,s) = E(c_{r,s})$ is smooth in (r,s) and

$$\begin{aligned}\partial^2_{r,s} E(0,0) &= \langle D_r \partial_s H, c' \rangle \Big|_a^b + \int_a^b \Big(\langle V', W' \rangle - \langle R(V,c')c', W \rangle \Big)\, dt \\ &= \langle D_r \partial_s H, c' \rangle \Big|_a^b + \langle V', W \rangle \Big|_a^b \\ &\qquad + \sum_{i=1}^{k-1} \langle \Delta V'(t_i), W(t_i) \rangle - \int_a^b \langle V'' + R(V,c')c', W \rangle\, dt,\end{aligned}$$

where the subdivision $a = t_0 < \ldots < t_k = b$ of $[a,b]$ is such that H is smooth on the sets $[t_{i-1}, t_i]$ and $\Delta V'(t_i) := V'(t_i - 0) - V'(t_i + 0)$. □

3.1. **Semi-Riemannian metrics.** In the case of Semi-Riemannian manifolds, the energy functional can be considered on the space of all piecewise smooth curves, the length functional can be considered on the space of piecewise smooth curves with constant positive speed, $\langle c', c' \rangle = \text{const} > 0$. The formulas for the first and second variation of energy and length remain the same.

Acknowledgment

I would like to thank Alexander Lytchak for proof-reading.

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