BASIC DIFFERENTIAL GEOMETRY: RIEMANNIAN IMMERSIONS AND SUBMERSIONS

WERNER BALLMANN

Introduction

Immersions and submersions between SR-manifolds which respect the SR-structures are called *Riemannian immersions* respectively *Riemannian submersions*. A typical example of the first kind of map are immersions $f : M \to \mathbb{R}^n$ considered in [SE], where $\mathbb{R}^n$ is endowed with the Euclidean metric and M with the corresponding first fundamental form. An important example of the second kind is the Hopf map $f : S^{2n+1} \to \mathbb{C}P^n$ which we discuss below.

For both kinds of maps a natural splitting of the tangent bundle plays a crucial role. For Riemannian immersions, it is the splitting of the tangent bundle of the target manifold into tangential and normal part. For Riemannian submersions, it is the splitting of the tangent bundle of the source manifold into horizontal and vertical part.

A general reference is [Be]. My discussion is influenced by the work of Hermann Karcher in [Ka]. I start with some general remarks on vector bundles with a given decomposition into a direct sum.

Contents

Date: Last update: 10.02.03.

1. Some General Remarks

Let $E \to M$ be a vector bundle and D be a connection on E. Assume that $E = E' + E''$ is a decomposition of E into a direct sum. Let

$$\mathcal{P}' : E \to E \quad \text{and} \quad \mathcal{P}'' : E \to E \tag{1.1}$$

be the projection of E onto E' along E'' and of E onto E'' along E', respectively. For a section S of E, denote by $S' = \mathcal{P}' \cdot S$ and $S'' = \mathcal{P}'' \cdot S''$ the part of S in E' and E'', respectively. More generally, S' will denote a section of E' and S'' a section of E''. Define connections D' on E' and D'' on E'' by

$$D'_X S' = \mathcal{P}' \cdot D_X S' \quad \text{and} \quad D''_X S'' = \mathcal{P}'' \cdot D_X S'', \tag{1.2}$$

respectively. Here and below, X is an (arbitrary) vector field on M.

Exercise 1.1. Show that D' and D'' are connections.

Now we have $\mathcal{P}' + \mathcal{P}'' = \mathrm{id}$ and $D_X \, \mathrm{id} = 0$. Hence

$$D_X \mathcal{P}' = -D_X \mathcal{P}''. \tag{1.3}$$

Furthermore $\mathcal{P}' \cdot S'' = 0$ and $\mathcal{P}'' \cdot S' = 0$, therefore

$$D_X \mathcal{P}' \cdot S'' = -\mathcal{P}' \cdot D_X S'' \quad \text{and} \quad D_X \mathcal{P}'' \cdot S' = -\mathcal{P}'' \cdot D_X S', \tag{1.4}$$

respectively. In particular,

$$D_X \mathcal{P}', D_X \mathcal{P}'' \text{ maps } E' \text{ to } E'' \text{ and } E'' \text{ to } E'. \tag{1.5}$$

We also have $\mathcal{P}' \cdot S' = S'$, hence

$$D_X S' = D'_X S' + D_X \mathcal{P}' \cdot S' \quad \text{and} \quad D_X S'' = D''_X S'' + D_X \mathcal{P}'' \cdot S'', \tag{1.6}$$

respectively. Since the roles of E' and E'' are exchangeable, we prefer E' in the statements below.

Remark 1.2. If D is a symmetric connection on $E = TM$ and X and Y are vector fields tangent to the distribution E', then

$$[X, Y] = D_X Y - D_Y X = D'_X Y - D'_Y X + D_X \mathcal{P}' \cdot Y - D_Y \mathcal{P}' \cdot X.$$

The part $D'_X Y - D'_Y X$ is tangent to E', the rest is tangent to E''. Hence E' is integrable iff $D\mathcal{P}'$ is symmetric on E', that is, iff $D_X \mathcal{P}' \cdot Y = D_Y \mathcal{P}' \cdot X$ for all vector fields X, Y tangent to E'.

Fundamental Equations 1.3. *In the above setup we have*

$$\mathcal{P}' \cdot R(X,Y)S' = \big(R'(X,Y) - [D_X \mathcal{P}', D_Y \mathcal{P}']\big) \cdot S';$$
$$\mathcal{P}'' \cdot R(X,Y)S' = \big(D_X D_Y \mathcal{P}' - D_Y D_X \mathcal{P}' - D_{[X,Y]} \mathcal{P}'\big) \cdot S'.$$

Proof. We compute

$$\begin{aligned} D_X D_Y S' &= D_X(D'_Y S' + D_Y\mathcal{P}' \cdot S') \\ &= D'_X D'_Y S' + D_X\mathcal{P}' \cdot D'_Y S' + D''_X(D_Y\mathcal{P}' \cdot S') + D_X\mathcal{P}'' \cdot D_Y\mathcal{P}' \cdot S'. \end{aligned}$$

Here we recall that $D_Y\mathcal{P}' \cdot S'$ is a section of E'', so that D''_X and $D_X\mathcal{P}''$ are responsible for the second term on the right hand side of the first line. We also have

$$D_{[X,Y]}S' = D'_{[X,Y]}S' + D_{[X,Y]}\mathcal{P}' \cdot S',$$

where the first term on the right hand side belongs to E', the second to E''. Collecting terms which belong to E' in $R(X,Y)S'$, and using (1.3) we get the first equation. As for the second equation, we differentiate $\mathcal{P}'' \cdot S' = 0$ and get

$$\begin{aligned} 0 &= D_X D_Y(\mathcal{P}'' \cdot S') \\ &= (D_X D_Y\mathcal{P}'') \cdot S' + (D_Y\mathcal{P}'') \cdot D_X S' + (D_X\mathcal{P}'') \cdot D_Y S' + \mathcal{P}'' \cdot D_X D_Y S'. \end{aligned}$$

Now $D\mathcal{P}'' = -D\mathcal{P}'$, hence the claim. □

Remark 1.4. If $\langle .,. \rangle$ is an SR-metric on E with E' and E'' non-degenerate and perpendicular and if D is metric, then the projections $\mathcal{P}'$ and $\mathcal{P}''$ are symmetric,

$$\langle \mathcal{P}' \cdot S_1, S_2 \rangle = \langle S_1, \mathcal{P}' \cdot S_2 \rangle \quad \text{and} \quad \langle \mathcal{P}'' \cdot S_1, S_2 \rangle = \langle S_1, \mathcal{P}'' \cdot S_2 \rangle$$

for all sections S_1 and S_2 of E. Furthermore, the connections D' and D'' are metric and the covariant derivatives $D_X\mathcal{P}'$ and $D_X\mathcal{P}''$ are symmetric,

$$\langle D_X\mathcal{P}' \cdot S_1, S_2 \rangle = \langle S_1, D_X\mathcal{P}' \cdot S_2 \rangle \quad \text{and} \quad \langle D_X\mathcal{P}'' \cdot S_1, S_2 \rangle = \langle S_1, D_X\mathcal{P}'' \cdot S_2 \rangle.$$

Similarly, all higher covariant derivatives of $\mathcal{P}'$ and $\mathcal{P}''$ are symmetric.

2. Riemannian Immersions

Let M and $\bar{M}$ be SR-manifolds. We say that a smooth map $f : M \to \bar{M}$ is a *Riemannian immersion* if

$$\langle f_{*p}v, f_{*p}w \rangle_{f(p)} = \langle v, w \rangle_p \tag{2.1}$$

for all $p \in M$ and $v, w \in T_pM$. A local isometry is a Riemannian immersion, but in the definition of Riemannian immersion it is not required that $\dim M = \dim \bar{M}$.

Typical examples are the immersions $f : M \to \mathbb{R}^n = \bar{M}$ considered in [SE], where $\mathbb{R}^n$ is endowed with the Euclidean metric and M with the corresponding first fundamental form. A similar construction works in our present context, but we have to be a bit more careful in the indefinite case: Let $\bar{M}$ be an SR-manifold with SR-metric $\bar{g}$. Assume that $f : M \to \bar{M}$ is an immersion such that $\operatorname{Im} f_{*p}$ is a non-degenerate subspace of $T_{f(p)}\bar{M}$ for all $p \in M$[1]. For each point $p \in M$, define a non-degenerate bilinear form $g_p = \langle .,. \rangle_p$ on T_pM by

$$\langle v, w \rangle_p = \langle f_{*p}v, f_{*p}w \rangle_{f(p)}. \tag{2.2}$$

[1]This condition is empty if M is Riemannian.

With respect to local coordinates (x, U) on M and (y, V) on $\bar{M}$ we have

$$g_{ij}(p) = \bar{g}_{kl}(f(p))\, f_i^k(p)\, f_j^l(p) \tag{2.3}$$

for all $p \in U \cap f^{-1}(V)$. Here $f^k = y^k \circ f$, $f^l = y^l \circ f$ for $1 \leq k, l \leq m$. It follows that the family of non-degenerate symmetric bilinear forms g_p is smooth in the sense of the definition of SR-metrics. The SR-metric g is called the *first fundamental form* of f. By definition, $f : M \to \bar{M}$ is a Riemannian immersion if M is endowed with the first fundamental form of f. The first fundamental form is also called the *pull-back* of $\bar{g}$, denoted $g = f^*(\bar{g})$.

Let $f : M \to \bar{M}$ be a Riemannian immersion. Let $\pi : TM \to M$ and $\bar{\pi} : T\bar{M} \to \bar{M}$ be the natural projections. Recall that $f^*T\bar{M} = \{(p, v) \in M \times T\bar{M} \mid \bar{\pi}(v) = f(p)\}$ is a bundle over M, the pull-back of $T\bar{M}$. A section of $f^*T\bar{M}$ is a map of the form (id, Z), where Z is a vector field along f. The first component is tautological — there is no information in it concerning the section. The second component Z is called the *principal part* of the section. It determines the section completely. For that reason, we do not distinguish between sections of $f^*T\bar{M}$ and vector fields along f and switch freely from one language to the other. Note however that in the definition of $f^*T\bar{M}$, it is necessary to keep the first component: it serves to remind us which point in the preimage of the foot of the second component we consider. If f is an embedding, then the first component is redundant.

The bundle $f^*T\bar{M}$ carries a natural SR-metric and metric connection,

$$\langle (p, v), (p, w) \rangle := \langle v, w \rangle \quad \text{and} \quad \bar{D}_X(\mathrm{id}, Z) := (\mathrm{id}, \bar{D}_X Z),$$

where $\bar{D}_X Z$ is the covariant derivative along f and $\bar{D}$ is the Levi-Civita connection of $\bar{M}$. It would be more precise to reserve a special symbol for the connection on $f^*T\bar{M}$ and not to use the same symbol $\bar{D}$ as for the connection on $\bar{M}$, but there will be no danger of confusion.

There are two natural perpendicular subbundles of $f^*T\bar{M}$, the tangent and normal bundle of f. We define the fibres of these bundles for each point in M, Lemma 2.1 below then shows that the corresponding subbundles are smooth. For $p \in M$, let

$$\begin{aligned} \mathcal{T}_p(M, f) &:= \{(p, v) \in (f^*T\bar{M})_p \mid v \in \operatorname{Im} f_{*p}\}, \\ \mathcal{N}_p(M, f) &:= \{(p, v) \in (f^*T\bar{M})_p \mid v \perp \operatorname{Im} f_{*p}\}. \end{aligned} \tag{2.4}$$

We call $\mathcal{T}_p(M, f)$ the *tangent space* and $\mathcal{N}_p(M, f)$ the *normal space* to f at p. By assumption, $\mathcal{T}_p(M, f)$ is non-degenerate, hence $\mathcal{N}_p(M, f)$ is non-degenerate as well and

$$(f^*T\bar{M})_p = \mathcal{T}_p(M, f) + \mathcal{N}_p(M, f). \tag{2.5}$$

as a direct sum. The unions $\mathcal{T}(M, f) := \cup \mathcal{T}_p(M, f)$ and $\mathcal{N}(M, f) := \cup \mathcal{N}_p(M, f)$ are called the *tangent* and *normal bundle* of f, respectively. Sections of $\mathcal{T}(M, f)$ are called *tangential fields* and sections of $\mathcal{N}(M, f)$ *normal fields* (along f).

LEMMA 2.1. *Let $p \in M$. Then there are an open neighborhood U of p in M, smooth vector fields $E_1, \dots, E_m$ of M on U and smooth vector fields $N_1, \dots, N_k$ along f, where $k = n - m$, such that*

$$(f_{*q}E_q(q), \dots, f_{*q}E_m(q), N_1(q), \dots, N_k(q))$$

is an ON-basis of $T_{f(q)}\bar{M}$ for each $q \in U$.

Proof. In a neighborhood of p choose smooth vector fields $X_1, \dots, X_m$ of M and $Y_1, \dots, Y_k$ along f such that

$$f_{*p} \cdot X_1(p), \dots, f_{*p} \cdot X_m(p), Y_1(p), \dots, Y_k(p)$$

is a basis of $T_{f(p)}\bar{M}$ and such that the linear hulls of

$$(f_{*p} \cdot X_1(p), \dots, f_{*p} \cdot X_i(p)) \quad \text{and} \quad (f_{*q}E_q(q), \dots, f_{*q}E_m(q), N_1(q), \dots, N_j(q))$$

are non-degenerate subspaces for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n-m\}$. These properties are open, therefore they also hold for all q in an open neighborhood U of $p \in M$. Now apply the Gram-Schmidt orthonormalization procedure to get vector fields $E_1, \dots, E_m$ of M and $N_1, \dots, N_k$ along f as claimed. $\square$

COROLLARY 2.2. *$\mathcal{T}(M, f)$ and $\mathcal{N}(M, f)$ are perpendicular and non-degenerate subbundles of $f^*T\bar{M}$ and*

$$f^*T\bar{M} = \mathcal{T}(M, f) + \mathcal{N}(M, f)$$

is a decomposition of $f^*T\bar{M}$ into a direct sum. The bundle morphism

$$TM \to \mathcal{T}(M, f), \quad v \mapsto (p, f_{*p} \cdot v),$$

where $p = \pi(v)$, is an isomorphism. $\square$

If f is an embedding, then the above morphism $TM \to \mathcal{T}(M, f)$ corresponds to the usual identification of the tangent spaces T_pM with the subspaces $\operatorname{Im} f_{*p}$ of $T_{f(p)}\bar{M}$, $p \in M$.

We are now in the situation of the previous section: We have a bundle, $f^*T\bar{M}$, together with a decomposition into perpendicular and non-degenerate subbundles, $\mathcal{T}(M, f)$ and $\mathcal{N}(M, f)$. We denote by $\mathcal{T}$ and $\mathcal{N}$ the (orthogonal) projections.

For a vector field Z along f (viewed as a section of $f^*T\bar{M}$), we denote by $Z^{\mathcal{T}}$ respectively $Z^{\mathcal{N}}$ the components of Z in $\mathcal{T}(M, f)$ and $\mathcal{N}(M, f)$, respectively,

$$Z^{\mathcal{T}} = \mathcal{T} \cdot Z, \quad Z^{\mathcal{N}} = \mathcal{N} \cdot Z. \tag{2.6}$$

We call $Z^{\mathcal{T}}$ and $Z^{\mathcal{N}}$ the *tangential part* and *normal part* of Z.

The connection $\bar{D}$ on $f^*T\bar{M}$ and the projections $\mathcal{T}$ and $\mathcal{N}$ give rise to connections $\bar{D}^{\mathcal{T}}$ on $\mathcal{T}(M, f)$ and $\bar{D}^{\mathcal{N}}$ on $\mathcal{N}(M, f)$. Since $\bar{D}$ is metric, $\bar{D}^{\mathcal{T}}$ and $\bar{D}^{\mathcal{N}}$ are metric, see Remark 1.4.

We are now ready to apply the results of the previous section. Before we do this, we discuss the most important part of the story, namely the relation with the intrinsic geometry of M — this is achieved in the crucial Fundamental Lemma below. It is the key observation for Riemannian immersions.

For a vector field X of M, define a vector field f_*X along f by

$$f_*X(p) = f_{*p} \cdot X(p). \tag{2.7}$$

If X is smooth, then f_*X is smooth. In fact, with respect to local coordinates (x, U) on M and (y, V) on $\bar{M}$ we have

$$f_*X(p) = \xi^i(p)(\partial_i f^j)(p) \cdot Y_j(f(p)), \tag{2.8}$$

on $U \cap f^{-1}(V)$, where ξ denotes the principal part of X with respect to the coordinates x. By our standard identification, we view f_*X as a section of $\mathcal{T}(M, f)$.

FUNDAMENTAL LEMMA 2.3. *Let $f : M \to \bar{M}$ be an isometric immersion. Let D be the Levi-Civita connection of M, $\bar{D}$ the Levi-Civita connection on $\bar{M}$. Then*

$$(\bar{D}_X f_* Y)^{\mathcal{T}} = \bar{D}^{\mathcal{T}}_X f_* Y = f_* D_X Y$$

for all $X, Y \in \mathcal{V}(M)$. Furthermore,

$$(\bar{D}_X f_* Y)^{\mathcal{N}} = \bar{D}_X \mathcal{T} \cdot f_* Y =: II(X, Y)$$

is tensorial and symmetric in X and Y.

In other words, via the identification $TM = \mathcal{T}(M, f)$ in Corollary 2.2, the tangential connection $\bar{D}^{\mathcal{T}}$ corresponds to the Levi-Civita connection on M. The normal part $II(X, Y) = \bar{D}_X \mathcal{T} \cdot f_* Y$ is called the *second fundamental form* of f.

Proof of Fundamental Lemma. It remains to show that $\bar{D}^{\mathcal{T}}$ corresponds to a symmetric connection on TM and that $\bar{D}_X \mathcal{T} \cdot Y$ is symmetric in X and Y. (It is obviously tensorial in X and Y.) Now $\bar{D}$ is symmetric, hence

$$\begin{aligned} 0 &= \bar{D}_X f_* Y - \bar{D}_Y f_* X - f_*[X, Y] \\ &= (\bar{D}_X f_* Y)^{\mathcal{T}} - (\bar{D}_Y f_* X)^{\mathcal{T}} f_*[X, Y] + II(X, Y) - II(Y, X). \end{aligned}$$

The first three terms on the right hand side are tangential, the last two terms are normal. Hence

$$(\bar{D}_X f_* Y)^{\mathcal{T}} - (\bar{D}_Y f_* X)^{\mathcal{T}} = f_*([X, Y]) \quad \text{and} \quad II(X, Y) = II(Y, X). \qquad \square$$

FUNDAMENTAL EQUATIONS 2.4. *Let $f : M \to \bar{M}$ be an isometric immersion. For vector fields X, Y, Z on M and normal field N along f we have*

$$\big(\bar{R}(X, Y) f_* Z\big)^{\mathcal{T}} = f_* R(X, Y) Z - [\bar{D}_X \mathcal{T}, \bar{D}_Y \mathcal{T}] \cdot f_* Z \qquad \text{(Gauß Equation)}$$

$$\big(\bar{R}(X, Y) f_* Z\big)^{\mathcal{N}} = \big(\bar{D}^2_{X,Y} \mathcal{T} - \bar{D}^2_{Y,X} \mathcal{T}\big) \cdot f_* Z \qquad \text{(Codazzi Equation)}$$

$$\big(\bar{R}(X, Y) N\big)^{\mathcal{N}} = \bar{R}^{\mathcal{N}}(X, Y) N - [\bar{D}_X \mathcal{T}, \bar{D}_Y \mathcal{T}] \cdot N \qquad \text{(Ricci Equation)}$$

There is also an equation for the tangential part of $\bar{R}(X, Y)N$. However, by the skew symmetry of the curvature tensor, $\langle \bar{R}(X, Y)N, f_*Z \rangle = -\langle \bar{R}(X, Y) f_* Z, N \rangle$, it is equivalent to the Codazzi Equation.

Proof of Fundamental Equations. The Gauß Equation follows immediately from the Fundamental Lemma 2.3 and the first of the Fundamental Equations 1.3. As for the Codazzi Equation, note that by definition

$$\bar{D}^2_{X,Y}\mathcal{T} = \bar{D}_X\bar{D}_Y\mathcal{T} - \bar{D}_{D_XY}\mathcal{T}$$

and that $[X,Y] = D_XY - D_YX$. Now the Codazzi Equation follows from the second of the Fundamental Equations 1.3. The Ricci Equation is immediate from the first of the Fundamental Equations 1.3. □

COROLLARY 2.5. *Let $f : M \to \bar{M}$ be an isometric immersion. Let $p \in M$ and $u, v, x, y \in T_pM$. Then*

$$\langle R(x,y)u, v\rangle = \langle \bar{R}(x,y)u, v\rangle + \langle II(x,v), II(y,u)\rangle - \langle II(x,u), II(y,v)\rangle.$$

In particular, if (u,v) span a nondegenerate 2-plane $\sigma \subset T_pM$, then

$$K(\sigma) = \bar{K}(f_{*p}(\sigma)) + \frac{\langle II(u,u), II(v,v)\rangle - \langle II(u,v), II(u,v)\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$

Proof. Let X, Y, U, V be smooth vector fields extending x, y, u, v. Since f is a Riemannian immersion, the Gauß Equation implies

$$\begin{aligned}\langle R(X,Y)U, V\rangle &= \langle f_*R(X,Y)U, f_*V\rangle \\ &= \langle \bar{R}(X,Y)f_*U, f_*V\rangle + \langle[\bar{D}_X\mathcal{T}, \bar{D}_Y\mathcal{T}]\cdot f_*U, f_*V\rangle.\end{aligned}$$

Now $\bar{D}_X\mathcal{T}$ and $\bar{D}_Y\mathcal{T}$ are symmetric, see Remark 1.4. Therefore, by the definition of II in Lemma 2.3,

$$\langle[\bar{D}_X\mathcal{T}, \bar{D}_Y\mathcal{T}]\cdot f_*U, f_*V\rangle = \langle II(Y,U), II(X,V)\rangle - \langle II(X,U), II(Y,V)\rangle. \quad \square$$

EXAMPLE 2.6. Compare Example 1.1.3 in [SRM]: Let V be a vector space of dimension $m+1$ and $\langle\cdot,\cdot\rangle$ be a non-degenerate symmetric bilinear form on V. For $\alpha \neq 0$, set

$$Q_\alpha = \{p \in V \mid \langle p, p\rangle = \alpha\}$$

and assume $Q_\alpha \neq \emptyset$. Then Q_α is a smooth submanifold of V of dimension m, a level set of the function $f(p) = \langle p, p\rangle$ on V.

Let $p \in Q_\alpha$. Then tangent and normal space to Q_α are given by

$$\mathcal{T}_pQ_\alpha \cong p^\perp \text{ and } \mathcal{N}_pQ_\alpha \cong \mathbb{R}\cdot p.$$

We identify vectorfields on Q_α with smooth maps $X : Q_\alpha \to V$ such that $\langle p, X(p)\rangle = 0$ for all $p \in Q_\alpha$. Furthermore, the restriction $N : Q_\alpha \to V$ of the identity map of V, $N(p) = p$, is normal to Q_α. For vector fields X, Y of Q_α, we have $\langle Y, N\rangle = 0$. Since $d = \bar{D}$ is the Levi-Civita connection of $V = \bar{M}$ and since $d_XN = X$, we compute

$$0 = X\langle Y, N\rangle = \langle d_XY, N\rangle + \langle Y, d_XN\rangle = \langle II(X,Y), N\rangle + \langle Y, X\rangle.$$

Now $\dim N_pQ_\alpha = 1$ for all p. Hence

$$II(X,Y) = \frac{1}{\alpha}\langle II(X,Y), N\rangle N.$$

This implies

$$II(X,Y) = -\frac{1}{\alpha}\langle X, Y\rangle N.$$

Since $\bar{R} = 0$, we get

$$R(X,Y)Z = \frac{1}{\alpha}(\langle Y,Z\rangle X - \langle X,Z\rangle Y) = R_\kappa(X,Y)Z$$

with $\kappa = 1/\alpha$. In particular, hyperbolic space has constant sectional curvature -1 and the sectional curvature of Riemann's metric is constant $= \kappa$, see Examples 1.1.2 and 1.3.2 in [SRM].

Definition 2.7. We say that a Riemannian immersion $f : M \to \bar{M}$ is *totally geodesic* in $U \subset M$ if $II_p = 0$ for all $p \in U$.

Proposition 2.8. *Let $f : M \to \bar{M}$ be a Riemannian immersion and let $U \subset M$ be open. Then f is totally geodesic in U if and only if $\bar{c} = f \circ c$ is a geodesic in $\bar{M}$ for each geodesic c in U.*

Proof. By Lemma 2.3 we have $\bar{D}_t c' = f_*(D_t c') + S(c', c')$. Hence if $\bar{c}$ is a geodesic for each geodesic c in U, then $S_p(v,v) = 0$ for all $p \in U$ and $v \in T_pM$. Since S_p is symmetric this implies $S_p = 0$. The other direction is obvious. □

3. Riemannian Submersions

Let M, $\bar{M}$ be SR-manifolds with Levi-Civita connections D and $\bar{D}$, respectively. A smooth map $f : M \to \bar{M}$ is called a *Riemannian submersion* if $\mathcal{V}_p(M,f) = \ker f_{*p}$ is non-degenerate and $f_{*p} : \mathcal{V}_p^\perp(M,f) \to T_{f(p)}\bar{M}$ is an orthogonal isomorphism for all $p \in M$. In that case, $\mathcal{H}_p(M,f) := \mathcal{V}_p^\perp(M,f)$ is called the *horizontal* and $\mathcal{V}_p(\bar{M},f)$ the *vertical subspace* of T_pM. The corresponding distributions $\mathcal{H}(M,f)$ and $\mathcal{V}(M,f)$ are smooth and

$$TM = \mathcal{H}(M,f) + \mathcal{V}(M,f) \tag{3.1}$$

is a decomposition into a direct sum.

In what follows, we fix a Riemannian submersion $f : M \to \bar{M}$ and denote by $\mathcal{H}$ and $\mathcal{V}$ the orthogonal projections onto the horizontal distribution $\mathcal{H}(M,f)$ and vertical distribution $\mathcal{V}(M,f)$, respectively. The horizontal and vertical part of vector field X on M are denoted $X^{\mathcal{H}}$ and $X^{\mathcal{V}}$, respectively.

For every vector field $\bar{H}$ on $\bar{M}$, there is precisely one horizontal vector field H on M that is f-related to H, that is, $f_{*p}H(p) = \bar{H}(f(p))$ for all $p \in \bar{M}$. We say that H is the *horizontal lift* of $\bar{H}$, denoted $H = \mathcal{L}(\bar{H})$.

Lemma 3.1. *For a vertical vector field V and a horizontal lift H, the Lie bracket $[V,H]$ is vertical.*

Proof. Since V is f-related to the vector field 0 and H to a vector field $\bar{H}$ on $\bar{M}$, the Lie bracket $[V, H]$ is f-related to $[0, \bar{H}] = 0$. □

FUNDAMENTAL LEMMA 3.2. *For horizontal lifts $H = \mathcal{L}(\bar{H})$ and $K = \mathcal{L}(\bar{K})$,*

$$D_H K = \mathcal{L}(\bar{D}_{\bar{H}} \bar{K}) + \frac{1}{2} \cdot [H, K]^{\mathcal{V}}.$$

In other words,

$$D^{\mathcal{H}}_H K = (D_H K)^{\mathcal{H}} = \mathcal{L}(\bar{D}_{\bar{H}} \bar{K}) \quad \text{and} \quad D_H \mathcal{H} \cdot K = (D_H K)^{\mathcal{V}} = \frac{1}{2} \cdot [H, K]^{\mathcal{V}}.$$

Proof. Using the Koszul formula, we determine the horizontal and vertical part of $D_H K$ separately. As for the vertical part, we use that the horizontal and vertical distributions are perpendicular and that the Lie bracket of a vertical field and a horizontal lift is vertical. Hence for a vertical field V we have

$$2 \cdot \langle D_H K, V \rangle = -V \langle H, K \rangle + \langle V, [H, K] \rangle.$$

Now $\langle H, K \rangle$ is constant along the fibres of f since H and K are horizontal lifts, hence $V \langle H, K \rangle = 0$. Thus the vertical part of $D_H K$ is as claimed.

Let L be the horizontal lift of a further vector field $\bar{L}$ on $\bar{M}$. Then H, K and L are f-related to $\bar{H}, \bar{K}$ and $\bar{L}$, respectively. Hence, for any point $p \in M$,

$$H \langle K, L \rangle (p) = \bar{H} \langle \bar{K}, \bar{L} \rangle (f(p)), \quad \langle H, [K, L] \rangle (p) = \langle \bar{H}, [\bar{K}, \bar{L}] \rangle (f(p)),$$

and similarly for permutations of H, K and L. Therefore

$$\langle D_H K, L \rangle (p) = \langle \bar{D}_{\bar{H}} \bar{K}, \bar{L} \rangle (f(p))$$

and the claim about the horizontal part of $D_H K$ follows. □

REMARK 3.3. Since the Lie bracket is skew symmetric, $D\mathcal{H}$ is skew symmetric on the horizontal distribution. Compare with Remark 1.4.

COROLLARY 3.4. *The horizontal lift of a geodesic in $\bar{M}$ is a geodesic in M: more precisely, if $c : I \to M$ is a smooth horizontal curve, then c is a geodesic in M iff $f \circ c$ is a geodesic in $\bar{M}$.*

Proof. By the Fundamental Lemma 3.2,

$$(D_t c')^{\mathcal{V}} = 0 \quad \text{and} \quad f_*(D_t c') = \bar{D}_t (f \circ c)'.$$ □

The Fundamental Equation 1.3 can be used to relate the curvature of D with that of the connections $D^{\mathcal{H}}$ and $D^{\mathcal{V}}$ on the horizontal and vertical distributions, respectively. However, our main interest is the relation with the curvature of the base manifold $\bar{M}$. The key for this relation is the Fundamental Lemma 3.2.

O'Neill's Formula 3.5. *Let H, K, X, Y be horizontal vector fields on M. Then the curvatures R of M and $\bar{R}$ of $\bar{M}$ in corresponding points p and $f(p)$ satisfy*

$$\langle \bar{R}(f_*H, f_*K)f_*X, f_*Y\rangle = \langle R(H,K)X,Y\rangle - \frac{1}{2}\cdot\langle [H,K]^{\mathcal{V}}, [X,Y]^{\mathcal{V}}\rangle - \frac{1}{4}\cdot\big\{\langle [H,X]^{\mathcal{V}}, [K,Y]^{\mathcal{V}}\rangle - \langle [K,X]^{\mathcal{V}}, [H,Y]^{\mathcal{V}}\rangle\big\}.$$

Proof. Since all the terms are tensorial in all the variables, we can assume that H, K, X, Y are horizontal lifts of vector fields $\bar{H}, \bar{K}, \bar{X}, \bar{Y}$ on $\bar{M}$. By the Fundamental Lemma 3.2, we have

$$\langle D^{\mathcal{H}}_H D^{\mathcal{H}}_K X, Y\rangle = \langle \bar{D}_{\bar{H}} D_{\bar{K}} \bar{X}, \bar{Y}\rangle.$$

Similarly for the horizontal lift $L = [H,K]^{\mathcal{H}}$ of $\bar{L} = [\bar{H}, \bar{K}]$,

$$\langle D^{\mathcal{H}}_L X, Y\rangle = \langle \bar{D}_{\bar{L}} \bar{X}, \bar{Y}\rangle.$$

By Lemma 3.1, the vertical field $V = [H,K]^{\mathcal{V}}$ satisfies

$$\langle D^{\mathcal{H}}_V X, Y\rangle = \langle D_V X, Y\rangle = \langle D_X V, Y\rangle = -\langle V, D_X Y\rangle = -\frac{1}{2}\cdot\langle [H,K]^{\mathcal{V}}, [X,Y]^{\mathcal{V}}\rangle.$$

Hence in corresponding points,

$$\langle \bar{R}(f_*H, f_*K)f_*X, f_*Y\rangle = \langle R^{\mathcal{H}}(H,K)X,Y\rangle + \frac{1}{2}\cdot\langle [H,K]^{\mathcal{V}}, [X,Y]^{\mathcal{V}}\rangle.$$

Now the claim is immediate from the Fundamental Equation 1.3, the symmetry of $D_H\mathcal{H}$ and $D_K\mathcal{H}$, see Remark 1.4, and the formula for $D\mathcal{H}$, see the Fundamental Lemma 3.2. ☐

Corollary 3.6. *For horizontal vector fields H, K on M,*

$$\langle \bar{R}(f_*H, f_*K)f_*K, f_*H\rangle = \langle R(H,K)K,H\rangle + \frac{3}{4}\cdot\langle [H,K]^{\mathcal{V}}, [H,K]^{\mathcal{V}}\rangle.$$

In the Riemannian case, the sectional curvature in the base manifold increases by the amount $\frac{3}{4}\cdot\|[H,K]^{\mathcal{V}}\|^2$. In particular, the sectional curvature of $\bar{M}$ is nonnegative respectively positive if the sectional curvature of M is nonnegative respectively positive.

4. Projective Spaces, Hopf Map and Standard Metric

In what follows, we denote by $\mathbb{K}$ one of the fields $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers or $\mathbb{H}$ of quaternionic numbers[2]. Then $\mathbb{K}$ is a vector space over $\mathbb{R}$, where the dimension $k = \dim_{\mathbb{R}} \mathbb{K}$ is 1, 2 or 4, respectively.

Any complex number α can be written in the form $\alpha = \alpha_0 + \alpha_1 i$ with $\alpha_0, \alpha_1 \in \mathbb{R}$. Similarly, any quaternion α can be written in the form $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ with $\alpha_0, \ldots, \alpha_3 \in \mathbb{R}$. In both cases, we call α_0 the *real part* and $\alpha - \alpha_0$ the

[2]Readers not familiar with $\mathbb{H}$ should disregard this case.

imaginary part of α, notation $\operatorname{Re}\alpha = \alpha_0$, $\operatorname{Im}\alpha = \alpha - \alpha_0$. For convenience we set $\operatorname{Re}\alpha = \alpha$ and $\operatorname{Im}\alpha = 0$ for $\alpha \in \mathbb{R}$. We say that $\alpha \in \mathbb{K}$ is *real* respectively *imaginary* if $\alpha = \operatorname{Re}\alpha$ respectively $\alpha = \operatorname{Im}\alpha$. The space of imaginary $\alpha \in \mathbb{K}$ is denoted $\operatorname{Im}\mathbb{K}$. It is an $\mathbb{R}$-linear subspace of $\mathbb{K}$ of dimension $k-1$.

For $\alpha \in \mathbb{K}$ we call $\bar\alpha = \operatorname{Re}\alpha - \operatorname{Im}\alpha$ the *conjugate number* of α. We have $\bar\alpha\alpha = |\alpha|^2$ and $\overline{\alpha\beta} = \bar\beta\bar\alpha$. Recall that $\mathbb{H}$ is not commutative, so that we have to be careful with the order of factors in products.

We denote by $\mathbb{K}^{n\times n}$ the space of $(n\times n)$-matrices with entries in $\mathbb{K}$. Since $\mathbb{H}$ is not commutative, we need to be careful when representing $\mathbb{K}$-linear maps of $\mathbb{K}^n$ by multiplication with a matrix from $\mathbb{K}^{n\times n}$ (at least when $\mathbb{K} = \mathbb{H}$). We stick to the following conventions: Scalar multiplication from the right: for scalar $\alpha \in \mathbb{K}$ and vector $x = (x^1, \dots, x^n) \in \mathbb{K}^n$,

$$x \cdot \alpha = (x^1\alpha, \dots, x^n\alpha). \tag{4.1}$$

Matrix multiplication from the left: for matrix $A = (\alpha^i_j) \in \mathbb{K}^{n\times n}$ and vector $x = (x^1, \dots, x^n) \in \mathbb{K}^n$,

$$A \cdot x = \big(\sum \alpha^1_j x^j, \dots, \alpha^n_j x^j\big). \tag{4.2}$$

Under these conventions, multiplication by scalars and matrices do not interfere with each other, and matrix multiplication is $\mathbb{K}$-linear, $A\cdot(x\cdot\alpha) = (A\cdot x)\cdot\alpha$. On $\mathbb{K}^n$ we consider the form

$$(x, y) = \sum_i \bar x^i y^i \tag{4.3}$$

which is conjugate linear in x and linear in y. The real part

$$\langle x, y\rangle = \operatorname{Re}(x, y) \tag{4.4}$$

is the standard Euclidean inner product on $\mathbb{K}^n$, when considered as real vector space $\mathbb{R}^{kn}$ as above. For a matrix $A \in \mathbb{K}^{n\times n}$ we have

$$(A\cdot x, y) = (x, A^*\cdot y), \tag{4.5}$$

where A^* is the conjugate transposed of A, $A^* = \bar A^t$.

A $\mathbb{K}$–*line* in $\mathbb{K}^{n+1}$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}^{n+1}$ of dimension 1 over $\mathbb{K}$. A $\mathbb{K}$-line is a real vector space of dimension $k := \dim_{\mathbb{R}} K \in \{1, 2, 4\}$. Two different $\mathbb{K}$-lines intersect in $\{0\}$ only, and any non-zero vector $x \in \mathbb{K}^{n+1}$ is contained in precisely one $\mathbb{K}$–line, namely $x\cdot\mathbb{K}$. The set $P\mathbb{K}^n$ of all $\mathbb{K}$-lines in $\mathbb{K}^{n+1}$ is called n-dimensional *projective space* over $\mathbb{K}$. For $n = 2$, $P\mathbb{K}^2$ is also called *projective plane* over $\mathbb{K}$.

For non-zero $x = (x_0, \dots, x_n) \in \mathbb{K}^{n+1}$, the line $L = x\cdot\mathbb{K}$ is also denoted $[x^0, \dots, x^n]$. We call $x^0, \dots, x^n$ the *homogeneous coordinates* of L. We obtain a surjective map

$$H : S^{kn+k-1} \to P\mathbb{K}^n, \ \ x \mapsto [x], \tag{4.6}$$

the *Hopf map*. The fibres $H_L := H^{-1}(L) = L \cap S^{kn+k-1}$, $L \in \mathbb{K}P^n$, of the Hopf map are called *Hopf spheres*. They are great spheres in S^{kn+k-1}. For $\mathbb{K} = \mathbb{R}$, Hopf spheres consist of pairs of antipodal points; for $\mathbb{K} = \mathbb{C}$, they are great circles; for $\mathbb{K} = \mathbb{H}$, they are great 3–spheres.

There is a natural distance function on projective space $\mathbb{K}P^n$ derived from angle measurement on the sphere S^{kn+k-1}: For $L, L' \in \mathbb{K}P^N$, set

$$d(L, L') = \min\{\angle(x, x') \mid x \in H_L, x' \in H_{L'}\}. \tag{4.7}$$

Then $0 \leq d(L, L') \leq \pi/2$ and $d(L, L') = 0$ iff $L = L'$. By definition, d is symmetric in L and L'. The triangle inequality for d is immediate from the first assertion of the following proposition.

Proposition 4.1. *Hopf spheres are parallel: If L, L' are Hopf spheres in S^{kn+k-1} and $\delta := d(L, L')$, then for any $x \in L$ there is a point $x' \in L'$ with $\angle(x, x') = \delta$. Moreover, x' is unique if $\delta < \pi/2$ and arbitrary (in L') if $\delta = \pi/2$.*

Proof. Since H_L and H'_L are compact, there are points $x_0 \in L, x'_0 \in L'$ with $\angle(x_0, x'_0) = \delta$. Now there is an $\alpha \in \mathbb{K}$ with $|\alpha| = 1$ such that $x = x_0 \cdot \alpha$. Then $x' = x'_0 \cdot \alpha \in H'_L$ and $\angle(x, x') = \delta$. Here we use that right multiplication by α is an $\mathbb{R}$-linear angle and norm preserving transformation of $\mathbb{K}^{n+1}$.

The case $\mathbb{K} = \mathbb{R}$ is trivial in the discussion of the uniqueness of x'. Therefore we assume $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$ in the rest of the argument. Then Hopf spheres are great spheres of dimension 1 or 3. In particular, for any two points $x', x'' \in L'$ there is a minimal great circle arc from x' to x'' contained in $H_{L'}$.

Since a great circle arc of length δ from x to x' is a curve of minimal length from x to L', it is perpendicular to L' in x'. If there is a second point $x'' \in L'$ with $\angle(x, x'') = \delta$, then the spherical triangle consisting of the minimal great circle arcs from x to x' and x'' respectively and of a minimal great circle arc in L' from x' and x'' has right angles in x' and x'' and hence $\delta = \pi/2$. Vice versa, if $\delta = \pi/2$, we have $\angle(x, x') = \pi/2$ for any $x' \in L'$. □

We will see below that the above distance function on $P\mathbb{K}^n$ arises as the distance associated to a natural Riemannian metric on $P\mathbb{K}^n$. As a first step in that direction we now introduce a manifold structure on $\mathbb{K}P^n$.

Consider the topology associated to the distance function d. From Proposition 4.1 it is immediate that $U \subset \mathbb{K}P^n$ is open iff its preimage $H^{-1}(U)$ under the Hopf map is open in S^{kn+k-1}. In particular, the sets

$$U_i = \{[x] \in P\mathbb{K}^n \mid x^i \neq 0\}, \quad 0 \leq i \leq n,$$

are open and the bijections

$$\kappa_i : U_i \to \mathbb{K}^n, \quad \kappa_i([x^0, \dots, x^n]) = (x^0, \dots, \hat{x}^i, \dots, x^n) \cdot \frac{1}{x^i}, \tag{4.8}$$

where the hat indicates that x^i is to be deleted, are homeomorphisms. Geometrically speaking, $\kappa_i([x])$ corresponds to taking the intersection $[x] \cap A_i$, where

A_i is the affine space of vectors in $\mathbb{K}^{n+1}$ with i-th coordinate equal to 1. The set $\{(\kappa_i, U,)\}$ is a smooth (acutally analytic) atlas of $\mathbb{K}P^n$. Thus $P\mathbb{K}^n$ carries a canonical smooth structure.

The Hopf map is smooth with respect to this structure. We will see below that H is a submersion. For $x \in S^{kn+k-1}$, we define the horizontal and vertical subspace of T_xS^{kn+k-1} by

$$\begin{aligned} \mathcal{H}_x &= \{v \in \mathbb{K}^{n+1} \mid (x, v) = 0\}, \\ \mathcal{V}_x &= \{v \in x \cdot \mathbb{K} \mid \langle x, v\rangle = 0\}. \end{aligned} \tag{4.9}$$

Note that $\mathcal{V}_x = T_xH$, where H is the Hopf sphere through x. In particular, $\mathcal{V}_x \subset \ker H_{*x}$. By definition, $\mathcal{H}_x$ is the orthogonal complement of $\mathcal{V}_x$ in T_xS^{kn+k-1}.

For $x \in S^{kn+k-1}$ and $v \in \mathcal{H}_x$ with $\|v\| = 1$, we consider the parameterized great circle

$$c(t) = x \cdot \cos t + v \cdot \sin t, \quad t \in \mathbb{R}. \tag{4.10}$$

Note that $(c(t), c'(t)) = 0$ for all t, that is, c intersects Hopf spheres perpendicularly.

Lemma 4.2. *For $0 \le t \le \pi/2$, c is the minimal connection between x and the Hopf sphere H_t through $c(t)$.*

Proof. The assertion is trivial in the case $\mathbb{K} = \mathbb{R}$. Suppose now that $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = H$. Then Hopf spheres are great spheres of dimension at least one. In particular, for any two points $x', x'' \in H_t$, there is a minimal great circle arc from x' to x'' contained in H_t.

Suppose that c is not the minimal connection between x and H_t, and let b be a minimal great circle arc from x to H_t. Then the spherical triangle consisting of $c([0,t])$, b and the minimal great circle arc from $c(t)$ to the endpoint x' of b and contained in H_t has right angles in $c(t)$ and x'. Since b is shorter than t, $t > \pi/2$. □

The *standard map* $S : P\mathbb{K}^n \to \mathcal{H}(n+1)$ associates to $L \in \mathbb{K}P^n$ the orthogonal projection of $\mathbb{K}^{n+1}$ onto L. Here we recall that L is a linear subspace of $\mathbb{K}^{n+1}$. If $L = H(x) = [x]$, then

$$S([x])(y) = x \cdot (x, y). \tag{4.11}$$

We study the differential of S through curves c as in (4.10). Note that for such a curve c,

$$S_{*[x]}(H_{*x}(v)) = \partial_t(S([x(t)]))|_{t=0} = v \cdot (x, .) + x \cdot (v, .). \tag{4.12}$$

We conclude that the tangent space to S at x is

$$\mathcal{T}_{[x]}S = \{v \cdot (x, .) + x \cdot (v, .) \mid v \in \mathcal{H}_x\}. \tag{4.13}$$

To determine the first fundamental form of S, we compute the norm of the matrix $A := S_{*[x]}(H_{*x}(v))$: Let $e_0 = x, e_1 = v, e_2, \ldots, e_n$ be an orthonormal basis of $\mathbb{K}^{n+1}$

with respect to $(.,.)$. Then $Ae_0 = e_1$, $Ae_1 = e_0$ and $Ae_j = 0$ for $j \geq 2$. Hence if $v \in \mathcal{H}_x$ has norm 1, then

$$\|S_{*[x]}(H_{*x}(v))\|^2 = 2. \tag{4.14}$$

In other words, up to the factor 2, the Hopf map H is a Riemannian submersion if we endow $P\mathbb{K}^n$ with the first fundamental form g_S of S. To get rid of that factor, we renormalize g_S and call $g = \frac{1}{2}g_S$ the *standard metric*. In the case of complex projective space, the standard metric is also refered to as the *Fubini–Study metric*.

Let c be a curve as in (4.10). Then $H \circ c$ is parameterized by arc length and realizes distances on intervals of length $\leq \pi/2$, see Proposition 4.2. It follows that $H \circ c$ is a unit speed geodesic in $P\mathbb{K}^n$. Clearly, any unit speed geodesic in $P\mathbb{K}^n$ is of this form.

Second fundamental form: As for the second fundamental form, consider a geodesic $H \circ c$ as above. Its second derivative

$$2(v \cdot (v,.) - x \cdot (x,.)) \tag{4.15}$$

is perpendicular to $\mathcal{T}S$ since it is a geodesic. By polarization, we get the second fundamental form of S,

$$II_{[x]}(H_{*x}v, H_{*x}w) = v \cdot (w,.) + w \cdot (v,.) - 2x \cdot \langle v, w\rangle \cdot (x,.). \tag{4.16}$$

It is now possible to compute the curvature tensor of the standard metric by using the Gauß Equation. However, we will follow a different path.

Isometries: Let G be the group of all $\mathbb{K}$-linear maps of $\mathbb{K}^{n+1}$ preserving $(.,.)$. By our representation of $\mathbb{K}$-linear maps by matrices,

$$G = \{A \in \mathbb{K}^{(n+1)\times(n+1)} \mid A^* = A^{-1}\}. \tag{4.17}$$

For $\mathbb{K} = \mathbb{R}$, G is the orthogonal group $O(n+1)$; for $\mathbb{K} = \mathbb{C}$, G is the unitary group $U(n+1)$; for $\mathbb{K} = \mathbb{H}$, G is the symplectic group $Sp(n+1)$. Since G preserves the length of vectors in $\mathbb{K}^{n+1}$, G operates isometrically on S^{kn+k-1}. Since the action of G maps Hopf spheres to Hopf spheres, the induced action on $P\mathbb{K}^n$, $A \cdot [x] = [Ax]$, is isometric. Since the action on $P\mathbb{K}^n$ is transitive, the standard metric is homogeneous. In fact, more is true:

Proposition 4.3. *The standard metric on $\mathbb{K}P^n$ is symmetric. More precisely, if $x \in S^{kn+k-1}$ and if $A \in G$ is the map with $Ax = x$ and $Ay = -y$ for $y \perp x \cdot \mathbb{K}$, then the above map on $P\mathbb{K}^n$ corresponding to A is the geodesic symmetry in $[x]$. In particular, the covariant derivative of the curvature tensor vanishes, $DR = 0$.* □

Totally geodesic submanifolds: For any $\mathbb{K}$-linear subspace $V \subset \mathbb{K}^{n+1}$ of dimension $m+1$ over $\mathbb{K}$, there is an orthonormal basis $e_0, e_1, \dots, e_n$ of $\mathbb{K}^{n+1}$ such that V is the $\mathbb{K}$-linear hull of $e_0, \dots, e_m$. Hence $P_{\mathbb{K}}V := H(V \cap S^{kn+k-1})$ is a $\mathbb{K}P^m$. A curve c as in (4.10) with $x, v \in V$ is contained in V, hence $P_{\mathbb{K}}V$ is a totally geodesic submanifold, isometric to $\mathbb{K}P^m$ with the standard metric.

Suppose now that $V \subset \mathbb{K}^{n+1}$ is *totally real* of dimension $m+1$ over $\mathbb{R}$; that is, there is an orthonormal basis $e_0, \dots, e_n$ of $\mathbb{K}^{n+1}$ such that V is the $\mathbb{R}$–linear hull of $e_0, \dots, e_m$. Then $V \cap S^{kn+k-1} = S^m$, a unit sphere of dimension m. In each $x \in S^m$, $T_x S^m$ is perpendicular to $x \cdot \mathbb{K}$, and hence the restriction of the Hopf map H to S^m is an isometric and totally geodesic immersion. It factors through an isometric and totally geodesic embedding $P\mathbb{R}^m \to P_{\mathbb{R}} V = H(V \cap S^{kn+k-1})$.

Projective Planes: We show that $P\mathbb{K}^2$ is isometric to a round sphere of radius 1/2. Consider the atlas (κ_0, U_0), (κ_1, U_1) of $P\mathbb{K}^2$ as in (4.8). We have

$$\kappa_1 \circ \kappa_0^{-1}(x_1) = \kappa_1([1, x_1]) = x_1^{-1},$$
$$\kappa_1 \circ \kappa_1^{-1}(x_0) = \kappa_0([x_0, 1]) = x_0^{-1}.$$

This is the same coordinate transformation as for the atlas of the sphere S^k consisting of the stereographic projections from north and south pole. Hence $P\mathbb{K}^2$ is diffeomorphic to S^k.

We now compute the metric g_1 on $\mathbb{K}$ corresponding to the standard metric on U_1 with respect to κ_1. For $q \in S^{k-1} \subset \mathbb{K}$ and $\varphi \in \mathbb{R}$ consider

$$A_q = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \quad R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in G.$$

Now

$$(\kappa_1 \circ A_q \circ \kappa_1^{-1})(x) = xq^{-1}\,.$$

Since A_q acts isometrically on $\mathbb{K}P^2$, g_1 is invariant under right multiplication by q^{-1}. Since $q \in S^{k-1}$ is arbitrary, we conclude that g_1 is rotationally symmetric. In particular, $g_1(0)$ is a multiple of the Euclidean metric $\langle .,. \rangle$ of $\mathbb{K}$. On the other hand,

$$(\kappa_1 \circ R_\varphi \circ \kappa_1^{-1})(x) = \frac{\cos\varphi \cdot x - \sin\varphi}{\sin\varphi \cdot x + \cos\varphi} =: T_\varphi(x)\,,$$

hence g_1 is invariant under transformations T_φ on their domain of definition in $\mathbb{K}$. Now

$$dT_\varphi(0) \cdot v = (1 + |x|^2) \cdot v, \quad x = -\frac{\sin\varphi}{\cos\varphi}.$$

Hence in x,

$$g_1(x) = \frac{1}{(1 + |x|^2)} \cdot \langle .,. \rangle, \tag{4.18}$$

where $\langle .,. \rangle$ is the Euclidean metric on $\mathbb{K}$. Since g_1 is rotationally symmetric, we conclude that g_1 is of this form in all points $x \in \mathbb{K}$. On the other hand, the right hand side in (4.18) is equal to Riemann's metric with constant curvature 4, compare Examples 1.1.2 and 1.3.2 in [SRM].

By the same line of argument we get that the metric g_0 on $\mathbb{K}$ corresponding to the standard metric on U_0 is Riemann's metric. Hence $\mathbb{K}P^2$ is isometric to the sphere $S^k_{1/2}$ of radius 1/2.

Curvature Tensor: There are at least two efficient ways of determinig the curvature tensor of $\mathbb{K}P^n$. The first is via Riemannian submersions using O'Neill's Formula. In this case, the Riemannian submersion in question is the Hopf map.

The second way is via totally geodesic submanifolds, in this case the two types $P_{\mathbb{K}}V$ and $P_{\mathbb{R}}V$ discussed above. We choose the first way. For the second, we refer to the literature.

The horizontal space of the Hopf map $H : S^{kn+k-1} \to \mathbb{K}P^n$ in $p \in S^{kn+k-1}$ is given by the condition $(p, X) = 0$, the vertical space is $p \cdot \mathrm{Im}(\mathbb{K})$, where $\mathrm{Im}(\mathbb{K})$ denotes the imaginary part of $\mathbb{K}$. Hence for a horizontal field X and an arbitrary vector field Y on S^{kn+k-1},

$$(p, d_Y X(p)) = -(Y(p), X(p)).$$

In particular, if X and Y are horizontal, then

$$[X, Y]^{\mathcal{V}} = -2p \cdot \mathrm{Im}(X, Y).$$

If X, Y and Z are horizontal with $\|Y\| = 1$ and X and Z perpendicular to Y, then by O'Neill's Formula

$$\langle R(H_*X, H_*Y)H_*Y, H_*Z\rangle = \langle X, Z\rangle + 3 \cdot \mathrm{Re}(\mathrm{Im}(X, Y) \cdot \mathrm{Im}(Y, Z)). \tag{4.19}$$

If X, Y are perpendicular and orthonormal, then

$$K(\pi_*X \wedge \pi_*Y) = 1 + 3 \cdot |\,\mathrm{Im}(X, Y)|^2 \tag{4.20}$$

for the sectional curvature K of $\mathbb{K}P^n$. Hence K is between 1 and 4, depending on the angle between Y and $X \cdot \mathrm{Im}(\mathbb{K})$.

Sectional curvature 4 is realized on the totally geodesic projective lines $P\mathbb{K}^1$ defined by two-dimensional $\mathbb{K}$-linear subspaces of $\mathbb{K}^{n+1}$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$. (Note that in the case $\mathbb{K} = \mathbb{R}$, projective lines are closed geodesics of length π.) Sectional curvature 1 is realized by totally geodesic real projective subspaces defined by totally real subspaces of $\mathbb{K}^{n+1}$. This remark also hints at how to compute the curvature tensor using totally geodesic submanifolds.

5. Berger's Deformation

Consider a Riemannian submersion $f : M \to \bar{M}$. Define a new metric on M, depending on a parameter $\delta \neq 0$,

$$\langle X, Y\rangle_\delta = \langle X^{\mathcal{H}}, Y^{\mathcal{H}}\rangle + \delta \cdot \langle X^{\mathcal{V}}, Y^{\mathcal{V}}\rangle.$$

For $\delta = 1$, we obtain the given metric. For $\delta \to 0$, the metric tends to 0 in the vertical direction. The map $f : M \to \bar{M}$ is still a Riemannian submersion with the same horizontal and vertical distributions as for $\langle ., .\rangle$. For the curvature we have

$$\langle \bar{R}(f_*H, f_*K)f_*K, f_*H\rangle = \langle R^\delta(H, K)K, H\rangle + \frac{3}{4} \cdot \langle [H, K]^{\mathcal{V}}, [H, K]^{\mathcal{V}}\rangle,$$

where R^δ denotes the curvature tensor of $< ., . > \delta$.

It is an interesting feature of the standard deformation that the Levi-Civita connection D^δ converges smoothly for $\delta \to 0$. This follows from the Koszul formula by straightforward computations. On the horizontal distribution, the curvature of the limiting connection corresponds exactly to the curvature of $\bar{M}$. We leave the details to the reader.

Acknowledgments

I would like to thank Alexander Lytchak for proof-reading.

References

[SE] W. Ballmann: *Basic geometry of submanifolds in Euclidean spaces.* Available at: www.math.uni-bonn.de/people/ballmann/notes.html.

[SRM] W. Ballmann: *Semi-Riemannian Metrics.* Available at: www.math.uni-bonn.de/people/ballmann/notes.html.

[Be] A. L. Besse: *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[Ka] H. Karcher: Submersions via projections. *Geometriae Dedicata* 74 (1999), 249–260.

Mathematisches Institut, Universität Bonn, Beringstrasse 1, D-53115 Bonn,