BASIC DIFFERENTIAL GEOMETRY: CONNECTIONS AND GEODESICS

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Introduction

I discuss basic features of connections on manifolds: torsion and curvature tensor, geodesics and exponential maps, and some elementary examples. In one of the examples, I assume some familiarity with some elementary differential geometry as in SE. I refer to [VC] for a short expositon of the general theory of connections on vector bundles.

Contents

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Conventions

If $U \subset \mathbb{R}^m$ is open, V is a real (or complex) vector space (of finite dimension), and $\varphi : U \to V$ is a smooth function, then the partial derivative of φ with respect to x_i is denoted in the following different ways,

$$\varphi_i = \varphi_{x_i} = \frac{\partial \varphi}{\partial x^i} = d\varphi \cdot \frac{\partial}{\partial x^i}.$$

Analogous notation will be used for higher partial derivatives. There are other objects with indices, where the indices have a different meaning. But it seems that there is no danger of confusion.

Let M be a manifold. By $\mathcal{F}(M)$ and $\mathcal{V}(M)$ we denote the spaces of smooth real valued functions and smooth vector fields on M, respectively. Recall that tangent vectors of M act as derivations on smooth maps with values in vector spaces, $\varphi : M \to V$. For $X \in \mathcal{V}(M)$, we use the notations $Xf = df \cdot X$ for the induced smooth function $M \ni p \mapsto X(p)(f) \in V$.

A frame of TM over a subset U of M consists of a tuple $\Phi = (X_1, \dots, X_m)$ of smooth vector fields of M over U such that $(X_1(p), \dots, X_m(p))$ is a basis of T_pM, for all $p \in U$. If X is a vector field of M over U, then the map $\xi : U \to \mathbb{R}^m$ with $X = \xi^i X_i$ is called the *principal part* of X with respect to Φ. In the last formula, the *Einstein convention* is in force. I will use it throughout: If in a term an index occurs as upper and lower index, then it is understood that the sum over that index is taken.

If U is open, Φ is a frame of TM over U, and X is a smooth vector field of M over U, then the principal part ξ of X is smooth. If $x : U \to U'$ is a coordinate chart of M, then

$$(X_1, \dots, X_m) := \Big(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\Big) \tag{0.1}$$

is a frame of TM over U. We call it the *frame associated to x*. For this frame, the principal part of a vector field X of M over U is given by $dx \cdot X$.

1. Connections on manifolds

We start with some basic features of connections on manifolds, that is, connections on their tangent bundles.

Definition 1.1. A *connection* or *covariant derivative* on M is a map

$$D : \mathcal{V}(M) \times \mathcal{V}(M) \longrightarrow \mathcal{V}(M), \quad D_X Y = DY \cdot X,$$

such that D is tensorial in X and a *derivation* in Y.

By the latter we mean that

$$D_X(\varphi \cdot Y) = X(\varphi) \cdot Y + \varphi \cdot D_X Y \tag{1.1}$$

for all $\varphi \in \mathcal{F}(M)$ and $X, Y \in \mathcal{V}(M)$.

Examples 1.2. 1) We view vector fields on $M = \mathbb{R}^m$ as maps $X : \mathbb{R}^m \to \mathbb{R}^m$. Then the standard derivative d defines a connection on $\mathbb{R}^m$: For smooth vector fields X, Y on $\mathbb{R}^m$, set

$$D_X Y(p) := dY_p \cdot X(p). \tag{1.2}$$

For reasons which will become clear below, this connection on $\mathbb{R}^m$ is called the *flat connection.*

2) Let $M \subset \mathbb{R}^n$ be a submanifold, and identify tangent spaces of M with linear subspaces of $\mathbb{R}^n$ in the usual way. Then a vector field X on M is a map $X : M \to \mathbb{R}^n$ such that $X(p) \in T_pM$ for all $p \in M$. For example, a vector field on the unit sphere S^m in $\mathbb{R}^{m+1}$ is a map $X : S^m \to \mathbb{R}^{m+1}$ such that $\langle p, X(p)\rangle = 0$ for all $p \in S^m$. For smooth vector fields X, Y on M, define

$$D_X Y(p) := \pi_p \cdot dY_p \cdot X(p), \quad p \in M, \tag{1.3}$$

where $\pi_p : \mathbb{R}^n \to T_pM$ denotes the orthogonal projection. This defines a connection on M, the *Levi-Civita connection*, compare [SE, SR, IS].

3) Consider $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^* = A^{-1}\}$, a submanifold of $\mathbb{R}^{n\times n}$ of dimension $m = n(n-1)/2$. Vector fields on $O(n)$ are maps $X : O(n) \to \mathbb{R}^{n\times n}$ such that, for all $A \in O(n)$, $X(A) = AB(A)$, where $B^*(A) = -B(A)$. We say that a vector field X on $O(n)$ is *left-invariant* if $X(A) = AB$ for some fixed $B \in \mathbb{R}^{n\times n}$ with $B^* = -B$.

If $(B_1, \dots, B_m)$ is a basis of the vector space of $\{B \in \mathbb{R}^{n\times n} \mid B^* = -B\}$, then smooth vector fields on $O(n)$ are of the form $Y(A) = \eta^i(A)\, AB_i$, where the *principal part* $\eta : O(n) \to \mathbb{R}^m$ of Y with respect to the chosen basis is smooth. Define a connection D on $O(n)$ by

$$D_X Y(A) := (d\eta^i_A \cdot X(A))\, AB_i. \tag{1.4}$$

This connection is called the *left-invariant* connection on $O(n)$. A similar construction works for all closed matrix groups.

From now on, we let D be a connection on M. Let $Y \in \mathcal{V}(M)$. Then

$$DY : \mathcal{V}(M) \to \mathcal{V}(M), \quad X \mapsto DY(X), \tag{1.5}$$

is tensorial in X. Therefore, by the argument of Lemma A.2, DY defines a family of maps $DY(p) : T_pM \to T_pM$ such that $DY(p) \cdot X(p) = D_X Y(p)$ for all $p \in M$ and $X \in \mathcal{V}(M)$, see Exercise 5) in Section 4. We call DY the *covariant derivative of* Y. We think of covariant differentiation as a generalization of directional or partial differentiation.

1.1. **Localization.** In our next observation we show that $D_X Y(p)$, $p \in M$, only depends on the restriction of Y to a neighborhood of p.

LEMMA 1.3. *Let $p \in M$ and $Y_1, Y_2 \in \mathcal{V}(M)$ be vector fields such that $Y_1 = Y_2$ in some neighborhood U of p. Then*

$$(D_X Y_1)(p) = (D_X Y_2)(p) \quad \text{for all } X \in \mathcal{V}(M).$$

Proof. Choose a smooth function $\varphi : M \to \mathbb{R}$ with $\operatorname{supp}(\varphi) \subset U$ and such that $\varphi = 1$ in a neighborhood $V \subset U$ of p. Then $\varphi \cdot Y_1 = \varphi \cdot Y_2$ on M, hence

$$D_X(\varphi \cdot Y_1) = D_X(\varphi \cdot Y_2).$$

On the other hand, by (1.1) and the choice of φ,

$$\begin{aligned} D_X(\varphi \cdot Y_i)(p) &= X_p(\varphi) \cdot Y_i(p) + \varphi(p) \cdot D_X Y_i(p) \\ &= 0 \cdot Y_i(p) + 1 \cdot D_X Y_i(p) = D_X Y_i(p) \end{aligned}$$

for $i = 1, 2$. Hence $(D_X Y_1)(p) = (D_X Y_2)(p)$ as claimed. □

Let $U \subset M$ be an open subset and $p \in U$. Recall from Lemma A.1 that for all smooth vector fields X, Y on U there are smooth vector fields $\tilde{X}, \tilde{Y}$ on M such that $X = \tilde{X}$ and $Y = \tilde{Y}$ in an open neighborhood $V \subset U$ of p. Define

$$D_X^U Y(p) := (D\tilde{Y} \cdot \tilde{X})(p). \tag{1.6}$$

By Lemma 1.3, $D_X^U Y(p)$ does not depend on the choice of $\tilde{X}$ and $\tilde{Y}$. It is now easy to verify that D^U is a connection on U. We call D^U the *induced connection.* By abuse of notation we simply write D instead of D^U. This simplification will not lead to confusion.

Let $\Phi = (X_1, \dots, X_m)$ be a frame of TM over U. Then there are smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, $1 \le i, j, k \le m$, such that

$$D_{X_i} X_j = \Gamma_{ij}^k X_k. \tag{1.7}$$

These functions Γ_{ij}^k are called *Christoffel symbols* of D with respect to Φ. If X, Y are smooth vector fields on U and $\xi, \eta : U \to \mathbb{R}^m$ are their principal parts with respect to Φ, $X = \xi^i X_i$ and $Y = \eta^i X_i$, then

$$\begin{aligned} D_X Y = D_X(\eta^j \cdot X_j) &= X\eta^j \cdot X_j + \eta^j \cdot D_X X_j \\ &= X\eta^j \cdot X_j + \eta^j \cdot D_{\xi^i \cdot X_i} X_j \\ &= X\eta^j \cdot X_j + \Gamma^k_{ij}\xi^i\eta^j \cdot X_k \\ &= \big(X\eta^k + \Gamma^k_{ij}\xi^i\eta^j\big) \cdot X_k. \end{aligned}$$

Thus the principal part of $D_X Y$ is

$$X\eta + \Gamma(\xi, \eta) = d\eta(X) + \Gamma(\xi, \eta), \tag{1.8}$$

where

$$\Gamma(\xi, \eta) := \big(\Gamma^1_{ij}\xi^i\eta^j, \ldots, \Gamma^m_{ij}\xi^i\eta^j\big). \tag{1.9}$$

The above formalism holds, in particular, for the frame $(X_1, \ldots, X_m)$ associated to a coordinate chart (x, U).

Let $x : U \to U'$ and $\hat{x} : \hat{U} \to \hat{U}'$ be coordinate charts for M. If X and Y are vector fields on $U \cap \hat{U}$, then their principal parts with respect to x and $\hat{x}$ are related by

$$\hat{\xi} = d\hat{x}(X) = a \cdot \xi, \quad \text{and} \quad \hat{\eta} = d\hat{x}(Y) = a \cdot \eta,$$

where $a(p) := d(\hat{x} \circ x^{-1})(x(p))$, $p \in U \cap \hat{U}$. Similarly, for the corresponding principal parts of $D_X Y$ we have

$$\begin{aligned} a\big(d\eta(X) + \Gamma(\xi, \eta)\big) &= d\hat{x}(D_X Y) \\ &= d\hat{\eta}(X) + \hat{\Gamma}(\hat{\xi}, \hat{\eta}) \\ &= d(a \cdot \eta)(X) + \hat{\Gamma}(\hat{\xi}, \hat{\eta}) \\ &= b(\xi, \eta\big) + d\varphi_x \cdot \eta(X) + \hat{\Gamma}(\hat{\xi}, \hat{\eta}), \end{aligned}$$

where $b(p) := d^2(\hat{x} \circ x^{-1})(x(p))$. Now $dx(X) = \xi$ and hence

$$\hat{\Gamma}(a \cdot \xi, a \cdot \eta) = a \cdot \Gamma(\xi, \eta) - b(\xi, \eta), \tag{1.10}$$

the transformation rule for Christoffel symbols under a change of coordinates. The transformation rule involves second derivatives of $\hat{x} \circ x^{-1}$.

1.2. **Symmetry.** The Lie bracket of vector fields X, Y on $\mathbb{R}^m$ is given by $[X, Y] = d_X Y - d_Y X$. For connections on manifolds, this equality does not need to hold anymore. However, we are interested in having as much similarity to the standard differential calculus in $\mathbb{R}^m$ as possible — this leads to the notion of symmetric connections. We say that a connection D for M is *symmetric* if

$$D_X Y - D_Y X = [X, Y] \tag{1.11}$$

for all $X, Y \in \mathcal{V}(M)$. With respect to a coordinate chart (x, U) of M, that is, with respect to the frame associated to (x, U), this amounts to the symmetry of the lower indices of the corresponding Christoffel symbols,

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \quad 1 \le i, j, k \le m. \tag{1.12}$$

A measure of the symmetry is the *torsion tensor*

$$T : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M), \quad T(X, Y) = D_X Y - D_Y X - [X, Y]. \tag{1.13}$$

PROPOSITION 1.4. *The torsion tensor $T = T(X, Y)$ is tensorial and skew symmetric in X and Y.*

Proof. The skew symmetry of T follows from the skew symmetry of the Lie bracket and the definition of T. Additivity in X and Y is clear. As for $\mathcal{F}(M)$-homogeneity in X, we compute

$$\begin{aligned} T(\varphi \cdot X, Y) &= D_{\varphi \cdot X} Y - D_Y(\varphi \cdot X) - [\varphi \cdot X, Y] \\ &= \varphi \cdot D_X Y - Y(\varphi) \cdot X - \varphi \cdot D_Y X + Y(\varphi) \cdot X - \varphi \cdot [X, Y] \\ &= \varphi \cdot T(X, Y). \end{aligned}$$

Now $\mathcal{F}(M)$-homogeneity in Y follows from skew symmetry. □

EXAMPLES 1.5. 1) As explained in the beginning of this subsection, the flat connection on $\mathbb{R}^m$ is a symmetric connection.

2) Let $M \subset \mathbb{R}^n$ be a submanifold and $X, Y : M \to \mathbb{R}^n$ be smooth vector fields on M, compare Example 1.2.2. Let D be the Levi-Civita connection on M as defined in (1.3). Then since the Lie bracket $[X, Y]$ is tangential to M,

$$\begin{aligned} D_X Y - D_Y X &= \pi \cdot dY \cdot X - \pi \cdot dX \cdot Y \\ &= \pi \cdot \big(dY \cdot X - dX \cdot Y\big) = \pi \cdot [X, Y] = [X, Y], \end{aligned}$$

where we suppress the dependence on $p \in M$. Hence D is symmetric.

3) Consider the left-invariant connection D on $O(n)$ as in (1.4). Let $X(A) = AB$ and $Y(A) = AC$ be left-invariant vector fields on $O(n)$. Then we have $D_X Y = D_Y X = 0$, hence

$$T(X, Y)(A) = -[X, Y](A) = -A(BC - CB),$$

and hence D is not symmetric.

1.3. **Curvature.** For smooth vector fields X, Y on M and a smooth map $\varphi : M \to \mathbb{R}$ we have $XY(\varphi) - YX(\varphi) = [X, Y](\varphi)$, by the definition of the Lie bracket. For connections, the failure of the corresponding commutation formula is measured by the curvature tensor.

DEFINITION 1.6. The *curvature tensor* of D is the map

$$\begin{aligned} &R : \mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M), \\ &R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \end{aligned}$$

A connection is called *flat* if its curvature tensor $R = 0$.

Proposition 1.7. *The curvature tensor R is tensorial in X, Y and Z and skew symmetric in X and Y, $R(X,Y)Z = -R(Y,X)Z$.*

Proof. Skew symmetry in X and Y follows from the definition of R and the skew symmetry of the Lie bracket. Addidivity in X, Y and Z is immediate from the additivity of covariant derivative and Lie bracket. As for homogeneity over $\mathcal{F}(M)$, we compute:

$$\begin{aligned} D_X D_Y(\varphi \cdot Z) &= D_X\big(Y(\varphi)\cdot Z + \varphi \cdot D_Y Z\big) \\ &= XY(\varphi)\cdot Z + Y(\varphi)\cdot D_X Z + X(\varphi)\cdot D_Y Z + \varphi\cdot D_X D_Y Z. \end{aligned}$$

An analogous formula holds for $D_Y D_X(\varphi \cdot Z)$. Now

$$D_{[X,Y]}(\varphi\cdot Z) = [X,Y](\varphi)\cdot Z + \varphi \cdot D_{[X,Y]}Z$$

and hence

$$R(X,Y)(\varphi\cdot Z) = \varphi \cdot R(X,Y)Z.$$

The proof of homogeneity over $\mathcal{F}(M)$ in X and Y is simpler. □

By the argument of Lemma A.2, the curvature R of a connection D is given by a family of trilinear maps $R_p : T_pM \times T_pM \times T_pM \to T_pM$ such that

$$(R(X,Y)Z)(p) = R_p(X(p), Y(p))Z(p)$$

for all $p \in M$ and $X, Y, Z \in \mathcal{V}(M)$, compare Exercise 5) in Section 4.

Let $\Phi = (X_1, \dots, X_m)$ be a frame of TM over an open subset $U \subset M$, and let $\Gamma = (\Gamma^k_{ij})$ be the corresponding Christoffel symbols of D. Let $X, Y, Z \in \mathcal{V}(M)$ with principal parts ξ, η, ζ with respect to Φ. Then the principal part of $R(X,Y)Z$ with respect to Φ is given by

$$\begin{aligned} &X(Y\zeta + \Gamma(\eta,\zeta)) + \Gamma(\xi, Y\zeta + \Gamma(\eta,\zeta)) - Y(X\zeta + \Gamma(\xi,\zeta)) \\ &\quad - \Gamma(\eta, X\zeta + \Gamma(\xi,\zeta)) - [X,Y]\zeta - \Gamma(X\eta - Y\xi, \zeta) \\ &\quad = (X\Gamma)(\eta,\zeta) - (Y\Gamma)(\xi,\zeta) + \Gamma(\xi,\Gamma(\eta,\zeta)) - \Gamma(\eta,\Gamma(\xi,\zeta)), \end{aligned} \tag{1.14}$$

with $X\Gamma = (d\Gamma^k_{ij}\cdot X)$. This formula shows again that $R(X,Y)Z$ is tensorial in X, Y and Z, it involves the principal parts of X, Y and Z in a linear way.

Suppose now that $(X_1, \dots, X_m)$ is the frame associated to a coordinate chart $x : U \to U'$. Define smooth functions $R^l_{ijk} : U \to \mathbb{R}$ by

$$R(X_i, X_j)X_k = R^l_{ijk} X_l. \tag{1.15}$$

Then, by the definition of Christoffel symbols in (1.7),

$$R^l_{ijk} = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \big(\Gamma^l_{ih}\Gamma^h_{jk} - \Gamma^l_{jh}\Gamma^h_{ik}\big), \tag{1.16}$$

where $\Gamma^l_{jk,i}$ denotes the i-th partial derivative of Γ^l_{jk}.

Examples 1.8. 1) A straightforward calculation shows that the flat connection on $\mathbb{R}^m$ is flat in the sense of definition 1.6, that is, its curvature tensor $R = 0$.

2) Let $M \subset \mathbb{R}^m$ be a submanifold and D be its Levi-Civita connection as in (1.3). The curvature of this connection is intimately related to the geometry of M. This is a long and interesting story, a story behind more or less everything we discuss, and will be pursued further in [SR] and [IS].

3) Let D be the left-invariant connection on $O(n)$ as in (1.4). Since the curvature tensor is tensorial, it suffices to compute $R(X, Y)Z$ for left-invariant vector fields on $O(n)$. Now $DY = 0$ for all left-invariant vector fields Y on $O(n)$. Hence the curvature tensor $R = 0$.

2. Covariant derivative along maps

Let $f : N \to M$ be a smooth map. A *vector field along* f is a map $X : N \to TM$ with $\pi \circ X = f$, where $\pi : TM \to M$ is the projection to the foot point. The vector space of vector fields along f is denoted $\mathcal{V}(f)$ or $\mathcal{V}_f$.

Let $\Phi = (X_1, \dots, X_m)$ be a local frame of TM over an open set $U \subset M$. For $X \in \mathcal{V}(f)$, there exist smooth functions $\xi^i : f^{-1}(U) \to \mathbb{R}$ such that

$$X(p) = \xi^i(p) X_i(f(p)) \quad \text{for all } p \in f^{-1}(U). \tag{2.1}$$

We write (2.1) also more shortly as $X = \xi^i \cdot X_i \circ f$, and call $\xi = (\xi^1, \dots, \xi^m)$ the *principal part of* X *with respect to* Φ.

Example 2.1. If X is a vector field on N, then $f_* X : N \to TM$,

$$f_* X(p) := f_{*p} X(p), \quad p \in M,$$

is a vector field along M. Such vector fields along f will be called *tangential*.

Let D be a connection on M. We want to induce a covariant derivative on vector fields along f. To that end, let $\Phi = (X_1, \dots, X_m)$ be a local frame of TM over an open set $U \subset M$, and let $\Gamma = (\Gamma_{ij}^k)$ be the corresponding Christoffel symbols. Let Y be a smooth vector field along f with principal part $\eta = (\eta^1, \dots, \eta^m)$ with respect to Φ. For a smooth vector field X over $f^{-1}(U)$, define

$$D_X^f Y(p) = \left\{ X_p(\eta^k) + \Gamma_{ij}^k(f(p)) \xi^i(p) \eta^j(p) \right\} \cdot (X_k(f(p)), \quad p \in U, \tag{2.2}$$

where ξ is the principal part of $f_* X$. In short, the principal part of $D_X^f S$ with respect to the chosen frame is

$$X(\eta) + (\Gamma \circ f)(\xi, \eta). \tag{2.3}$$

This formula shows that $D_X^f Y$ is smooth. We have not checked yet that $D_X^f Y$ is well defined. For this, let $\Psi = (Y_1, \dots, Y_m)$ be another local frame of TM over an open subset $V \subset M$, and let $a = (a_i^j)$ be the matrix of functions on $U \cap V$ describing the change of frame, $X_i = a_i^j Y_j$. Let $W = f^{-1}(U \cap V)$. On W, the principal parts η_Φ and η_Ψ of Y with respect to Φ and Ψ, respectively, are related by $\eta_\Psi = (a \circ f) \cdot \eta_\Phi$. For the proposed principal parts of $D_X S$ we have

$$\begin{aligned} X(\eta_\Psi) + (f^*\omega_\Psi)(X) \cdot \eta_\Psi &= X\big((a \circ f) \cdot \eta_\Phi\big) + \omega_\Psi(f_* X) \cdot \eta_\Psi \\ &= X(a \circ f) \cdot \eta_\Phi + (a \circ f) \cdot X(\eta_\Phi) + \omega_\Psi(f_* X) \cdot (a \circ f) \cdot \eta_\Phi \\ &= X(a \circ f) \cdot \eta_\Phi + (a \circ f) \cdot X(\eta_\Phi) \\ &\qquad + (a \circ f) \cdot \omega_\Phi(f_* X) \cdot \eta_\Phi - X(a \circ f) \cdot \eta_\Phi \\ &= (a \circ f) \cdot \big(X(\eta_\Phi) + (f^*\omega_\Phi)(X) \cdot \eta_\Phi\big) . \end{aligned}$$

This shows that $D^f_X Y$ is well defined. For convenience, we simply write D instead of D^f. The following proposition is immediate from the local expressions in (2.2) or (2.3).

Proposition 2.2. *The covariant derivative $D = D^f$ along f,*

$$D : \mathcal{V}(N) \times \mathcal{V}(f) \to \mathcal{V}(f), \quad D_X Y = D_X \cdot Y,$$

is tensorial in X and a derivation in Y.

Example 2.3. Consider $\mathbb{R}^m$ with the flat connection d, and let $c : I \to \mathbb{R}^m$ be a smooth curve. Smooth vector fields along c correspond to smooth maps $Y : I \to \mathbb{R}^m$, and the covariant derivative of such a field Y is given by the usual derivative.

For any smooth vector field Y of M, $Y \circ f$ is a smooth vector field along f. The induced covariant derivative for sections along f is consistent with the original covariant derivative in the following sense.

Proposition 2.4 (Chain Rule). *If Y is a smooth vector field of M, then*

$$D(Y \circ f) \cdot X = DY \cdot (f_* X)$$

for all smooth vector fields X of N. □

2.1. **Torsion and curvature.** It is important that torsion and curvature tensor behave well under covariant differentiation along maps.

Proposition 2.5. *Let X, Y be smooth vector fields on N and Z be a smooth vector field along f. Then*

$$\begin{aligned} T(f_* X, f_* Y) &= D_X f_* Y - D_Y f_* X - f_*[X, Y], \\ R(f_* X, f_* Y) Z &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \end{aligned}$$

Proof. We check the assertion about the curvature tensor, the proof of the assertion about the torsion tensor is similar.

Let $\Phi = (X_1, \ldots, X_m)$ be a local frame of TM over an open subset $U \subset M$. Let $\Gamma = \Gamma^k_{ij}$ be the corresponding Christoffel symbols. Since the right hand side of the asserted equation is tensorial in Z, that is, additive and $\mathcal{F}(N)$-homogeneous in Z, it suffices to consider the case $Z = X_i \circ f$. Then the principal part of the right hand side of the asserted formula is

$$\begin{aligned} X(Y\zeta + (\Gamma \circ f)(\eta, \zeta)) + (\Gamma \circ f)(\xi, Y\zeta + (\Gamma \circ f)(\eta, \zeta)) - Y(X\zeta + (\Gamma \circ f)(\xi, \zeta)) \\ - (\Gamma \circ f)(\eta, X\zeta + (\Gamma \circ f)(\xi, \zeta)) - [X, Y]\zeta - (\Gamma \circ f)(X\eta - Y\xi, \zeta) \\ = (X(\Gamma \circ f))(\eta, \zeta) - (Y(\Gamma \circ f))(\xi, \zeta) \\ + (\Gamma \circ f)(\xi, (\Gamma \circ f)(\eta, \zeta)) - (\Gamma \circ f)(\eta, (\Gamma \circ f)(\xi, \zeta)), \end{aligned}$$

where ξ and η denote the principal part of f_*X and f_*Y, respectively, and where we note that $X\eta - Y\xi$ is the principal part of $f_*[X, Y]$ with respect to Φ. The right hand side in the above equation is equal to the principal part of $R(f_*X, f_*Y)Z$, compare (1.14). □

COROLLARY 2.6. *Let $W \subset \mathbb{R}^2$ be open and $f : W \to M$ be a smooth map. Denote by D_s and D_t the covariant derivtives in the coordinate directions s and t of $\mathbb{R}^2$. Then*

$$\begin{aligned} D_s f_t &= D_t f_s + T(f_s, f_t), \\ D_s D_t Z &= D_t D_s Z + R(f_s, f_t)Z, \end{aligned}$$

where Z is a smooth vector field along f.

Proof. The coordinate vector fields in $\mathbb{R}^2$ commute. □

2.2. **Parallel translation along curves.** The most important case is the covariant derivative along a curve $c = c(t)$ in M. If Y is a smooth vector field along c, then we set

$$Y' := DY \cdot \frac{\partial}{\partial t}. \tag{2.4}$$

If η is the principal part of Y with respect to a local frame $\Phi = (X_1, \ldots, X_m)$ of TM over U, then the principal part of Y' over $V = c^{-1}(U)$ is given by

$$\eta' + (\Gamma \circ c)(\xi, \eta), \tag{2.5}$$

where ξ is the principal part of c' with respect $\Phi = (X_1, \ldots, X_m)$.

REMARK 2.7. Note that $Y'(t)$ might be non-zero even if $c'(t) = 0$. For example, if c is a constant curve, $c(t) \equiv p$, and Y is a smooth vector field along c, that is, Y is a smooth map into T_pM, then Y' is the usual derivative of Y as a map into the fixed vector space T_pM.

DEFINITION 2.8. We say that a vector field Y along $f : N \to M$ is *parallel* if $D_X Y = 0$ for all vector fields X of N.

In general there are no parallel vector fields along a map f. However, for smooth curves there are always such fields, that is, fields which satisfy $Y' = 0$. In terms of (2.5), they correspond to solutions of the linear ordinary differential equation

$$\eta' + (\Gamma \circ c)(\xi, \eta) = 0. \tag{2.6}$$

¿From the standard theorems on ordinary differential equations we obtain the following assertion.

Corollary 2.9. *Let $c : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v \in T_{c(t_0)}M$. Then there is a unique parallel vector field Y along c with $Y(t_0) = v$.* □

Let $c : I \to M$ be a smooth curve, $t_0, t_1 \in I$, and set $p_0 = c(t_0)$, $P_1 = c(t_1)$. The map $P : T_{p_0}M \to T_{p_1}M$, which associates to $v \in T_{p_0}M$ the value $Y(t_1) \in T_{p_1}M$ of the unique parallel field Y along c with $Y(t_0) = v$, is called *parallel translation* along c from p_0 to p_1. The map P is a linear isomorphism: by uniqueness the inverse map is parallel translation along c from p_1 to p_0. In other words, if $(v_1, \dots, v_m)$ is a basis of $T_{p_0}M$ and E_i is the parallel vector field along c with $E_i(t_0) = v_i$, $1 \le i \le m$, then $(E_1(t), \dots, E_m(t))$ is a basis of $T_{c(t)}M$ for all $t \in I$. Such a frame along c will be called *parallel*.

Example 2.10. Consider $\mathbb{R}^m$ with the flat connection d. A vector field Y along a smooth curve $c : I \to \mathbb{R}^m$ is parallel if and only if Y is constant.

Parallel frames along curves are very useful: Let $\Phi = (X_1, \dots, X_m)$ be a parallel frame along c. If Y is a vector field along c, then there is a map $\eta : I \to \mathbb{R}^m$, the principal part of Y with respect to Φ, such that $Y = \eta^i X_i$. By Proposition 2.2, $Y' = (\eta^i)' X_i$ — covariant differentiation along c is reduced to standard differentiation.

3. Geodesics and exponential map

Let M be a manifold with a connection D. For a curve $c : I \to M$, the covariant derivative of the vector field c' along c is denoted c''.

Definition 3.1. A smooth curve $c : I \to M$ is called a *geodesic* if $c'' = 0$.

Let $x : U \to U'$ be a coordinate chart for M, and $(X_1, \dots, X_m)$ be the associated frame of TM over U. For a curve $c : I \to M$ set $c^i = x^i \circ c$ on $J = c^{-1}(U) \subset I$. On J, the coefficients of the principal part of c'' with respect to x are

$$(c^k)'' + \Gamma^k_{ij}(c^i)'(c^j)'.$$

Therefore, c is a geodesic on $J = c^{-1}(U) \subset I$ if and only if the tuple $(c^1, \dots, c^m)$ satisfies the *geodesic equation*

$$(c^k)'' + \Gamma^k_{ij}(c^i)'(c^j)' = 0. \tag{3.1}$$

This is a *system* of differential equations for the coefficient functions c^k. We consider the vector function $(c^1, \dots, c^m)$ as the independent variable and, therefore, simply speak of a differential equation.

Examples 3.2. 1) With respect to the flat connection d, geodesics of $\mathbb{R}^m$ are parameterized lines, $c(t) = p + tv$.

2) Consider the unit sphere S^m in $\mathbb{R}^{m+1}$. With respect to the Levi-Civita connection as in (1.3), unit speed geodesics on S^m are precisely the curves of the form $c(t) = \cos t\, x + \sin t\, y$, where $x, y \in \mathbb{R}^{m+1}$ are perpendicular unit vectors.

3) On $O(n)$ with the left-invariant connection as in (1.4), geodesics on $O(n)$ are precisely the curves of the form $A \exp(tB)$ with $A \in O(n)$ and where $B \in \mathbb{R}^{n\times n}$ satisfies $B^* = -B$.

Proposition 3.3. *Let $c : I \to M$ be a geodesic and $\alpha, \beta \in \mathbb{R}$. Then $t \mapsto c(\alpha t+\beta)$ is a geodesic on $\{t \in \mathbb{R} \mid \alpha t + \beta \in I\}$.*

Proof. With respect to a local coordinate chart $x : U \to U'$, let $c^i = x^i \circ c$. Then $(c^1(\alpha t+\beta), \dots, c^m(\alpha t+\beta))$ satisfies the geodesic equation 3.1 if $(c^1(t), \dots, c^m(t))$ does. □

3.1. **Geodesic flow.** The Geodesic Equation 3.1 is a non-linear ordinary differential equation of second order. Since the coefficients Γ^k_{ij} are smooth functions on U, the standard theorems on ordinary differential equations have the following consequences.

Proposition 3.4 (Uniqueness). *Let $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$ be geodesics such that $c_1'(t_0) = c_2'(t_0)$ for some $t_0 \in I := I_1 \cap I_2$. Then $c_1|I = c_2|I$.*

Proof. The set $J \subset I$ of $t \in I$ with $c_1'(t) = c_2'(t)$ is closed in I. By assumption, $J \neq \emptyset$.

Let $t \in J$. Choose a coordinate chart $x : U \to U'$ about $c_1(t) = c_2(t)$. Then both tuples $c_1^i = x^i \circ c_1$ and $c_2^i = x^i \circ c_2$ are solutions of the geodesic equation 3.1. Moreover, $c_1^i(t) = c_2^i(t)$ and $(c_1^i)'(t) = (c_2^i)'(t)$. Hence $x \circ c_1 = x \circ c_2$ in a neighborhood of t in I, and, therfore, also $c_1 = c_2$ in a neighborhood of t in I. Hence J is open in I and therefore $J = I$. □

Let $v \in TM$. According to Proposition 3.4, there is a maximal interval $I_v \subset \mathbb{R}$ containing 0 such that there is a geodesic $c = c_v : I_v \to \mathbb{R}$ with initial velocity $c_v'(0) = v$. Since geodesics are solutions of differential equations, I_v is open. We set $\mathcal{G} := \{(v, t) \in TM \times \mathbb{R} \mid t \in I_v\}$.

Proposition 3.5 (Smoothness). *The set $\mathcal{G}$ is an open subset of $TM \times \mathbb{R}$ and contains $\mathbb{R} \times \{0_p \mid p \in M\}$. The map $\mathcal{G} \ni (v, t) \mapsto c_v(t) \in M$ is smooth.*

Proof. We first prove the following weaker version WV of the proposition: Let $v \in TM$. Then there is an open neighborhood W of v in TM and an $\varepsilon > 0$ such that for all $w \in W$ there is a geodesic $c_w : (-\varepsilon, \varepsilon) \to M$ with $c_w'(0) = w$, and the map $W \times (-\varepsilon, \varepsilon) \to M$, $(w, t) \mapsto c_w(t)$ is smooth.

To prove this, choose a coordinate chart $x : U \to U'$ about the foot point p of v. Let $\hat{x} := (x \times dx) : TM|U \to U' \times \mathbb{R}^m$ be the associated coordinate chart for TM. Then $\hat{x}(v) = (x(p), \xi)$, where $\xi = (\xi^1, ..., \xi^m)$ is the principal part of v with respect to x.

By the standard theorems on ordinary differential equations, there is a neighborhood W' about $(x(p), \xi)$ in $U' \times \mathbb{R}^m$ and an $\varepsilon > 0$ such that, for all $(u, \eta) \in W'$, there is a unique solution $(c^1, \ldots, c^m)_{(y,\eta)} : (-\varepsilon, \varepsilon) \to U'$ of (3.1) with $(c^1, \ldots, c^m)_{(y,\eta)}(0) = y$ and $(c^1, \ldots, c^m)'_{(y,\eta)}(0) = \eta$. Moreover, the map

$$F : W' \times (-\varepsilon, \varepsilon) \to U', \quad F(y, \eta, t) = u_{(y,\eta)}(t),$$

is smooth. Now $W = \hat{x}^{-1} W'$ is a neighborhood as claimed for the given ε: For $w \in W$, $c_w = x^{-1} \circ (c^1, \ldots, c^m)_{(y,\eta)}$, where $(y, \eta) = \hat{x}(w)$, is a geodesic with initial velocity w. The map $W \times (-\varepsilon, \varepsilon) \to M$, $(w, t) \mapsto c_w(t)$ is smooth because it is the composition $(w, t) \mapsto x^{-1}(F(\hat{x}(w), t))$. This proves WV.

The claim of the proposition is an easy consequence of WV. For $(v, t) \in \mathcal{G}$ we need to show that there is a neighborhood W of v in TM and an $\varepsilon > 0$ such that $c_w(s)$ is defined for all $w \in W$ and $s \in (t - \varepsilon, t + \varepsilon)$ and such that $(w, s) \mapsto c_w(s)$ is smooth on $W \times (t - \varepsilon, t + \varepsilon)$. The case $t = 0$ is the assertion. For notational convenience, we assume $t > 0$. The case $t < 0$ is handled similarly.

Since $[0, t]$ is compact, WV implies that there exist an $\varepsilon > 0$ and a subdivision

$$0 = t_0 < t_1 < \ldots < t_k = t \quad \text{of } [0, t]$$

with $t_{i+1} < t_i + \varepsilon$ such that $v_i := c'_w(t_i)$ has an open neighborhood V_i in TM such that

$$V_i \times (-\varepsilon, \varepsilon) \to M, \quad (w, s) \mapsto c_w(s)$$

is defined and smooth, $0 \le i < k$. For $i \in \{0, k-1\}$ we assume inductively that v has an open neighborhhood W_i in TM such that

$$W_i \times (-\varepsilon, t_i + \varepsilon) \to M, \quad (w, s) \mapsto c_w(s)$$

is defined and smooth. Then $c'_w(t_{i+1})$ depends smoothly on $w \in W_i$. Since $v_{i+1} = c'_v(t_{i+1})$,

$$W_{i+1} = \{w \in W_i \mid c'_w(t_{i+1}) \in V_{i+1}\}.$$

is an open neighborhood of v in TM. For $w \in W_{i+1}$ and $s \in (-\varepsilon, t_i + \varepsilon)$ we have

$$c_w(s) = c_{w_{i+1}}(s - t_{i+1}), \quad \text{where} \quad w_{i+1} = c'_w(t_{i+1}),$$

by (3.1). We conclude that c_w, $w \in W_{i+1}$, is defined on $(-\varepsilon, t_{i+1} + \varepsilon)$. Moreover, (3.1) implies that

$$W_{i+1} \times (-\varepsilon, t_{i+1} + \varepsilon) \to M, \quad (w, s) \mapsto c_w(s)$$

is defined and smooth. Set $W = W_k$, then $W \times (t - \varepsilon, t + \varepsilon) \subset \mathcal{G}$. □

The *geodesic flow* associated to D is the map

$$\mathcal{G} \to TM, \quad (t, v) \mapsto c'_v(t). \tag{3.2}$$

We say that D is *complete* if $\mathcal{G} = \mathbb{R} \times TM$. If D is complete, the geodesic flow is a 1-parameter group of diffeomorphisms on TM.

3.2. **Geodesic variations and Jacobi fields.** For simplicity, we assume from now on that D is a symmetric connection on M. The general case can be handled in a similar way.

Let $c : I \to M$ be a geodesic. We say that a vector field V along c is a *Jacobi field* if it satisfies the *Jacobi equation*

$$V'' + R(V, c')c' = 0. \tag{3.3}$$

The Jacobi equation is a linear ordinary differential equation of second order with smooth coefficients.

Let $E_1, \dots, E_m$ be a parallel frame along c. Then a smooth vector field V along c can be written as a linear combination $V = v^i E_i$ with smooth functions $v^i : I \to \mathbb{R}$. Since the fields E_i are parallel, we have

$$V'' + R(V, c')c' = (v^i)'' E_i + v^i\, R(E_i, c')c'.$$

Now we can also express the smooth vector field $t \mapsto R(E_i(t), c'(t))c'(t)$ along c as a linear combination of the E_i, $R(E_i, c')c' = R_i^j E_j$. Then

$$V'' + R(V, c')c' = (v^i)'' E_i + v^i R_i^j E_j = \left((v^i)'' + R_j^i v^j\right) E_i.$$

Hence V is a Jacobi field if and only if the coefficients v_i satisfy the linear system of second order, ordinary differential equations

$$(v^i)'' + R_j^i v^j = 0, \quad 1 \le i \le m. \tag{3.4}$$

It follows that Jacobi fields are smooth and that linear combinations of Jacobi fields are again Jacobi fields. Hence the set of Jacobi fields $\mathcal{J}_c$ along c is a real vector space. Since a solution of the Jacobi equation is determined by its initial value and initial (covariant) derivative at some time $t_0 \in I$, we get

$$\dim \mathcal{J}_c = 2m. \tag{3.5}$$

We say that a smooth map $H : (-\varepsilon, \varepsilon) \times I \to M$ of c is a *geodesic variation* of c if $c_0 = H(0, \cdot) = c$ and if $c_s = H(s, \cdot)$ is a geodesic for each $s \in (-\varepsilon, \varepsilon)$. Recall that for a variation H of c, the variation field of H is the vector field $V = H_s(0, \cdot)$ along c.

Proposition 3.6. *1) The variation field of a geodesic variation is a Jacobi field. 2) Let $t_0 \in I$ and suppose that there is a neighborhood U of v in TM such that the geodesic c_u with $c_u'(t_0) = u$ is defined on all of I, for all $u \in U$. Then any Jacobi field V along $c := c_v : I \to M$ is the variation field of a geodesic variation of c.*

Remark 3.7. By Proposition 3.5, a neighborhood U as in 2) exists if I is a compact interval, $I = [a, b]$.

Proof of Proposition 3.6. 1) Let H be a geodesic variation of c, and denote the covariant derivative in the coordinate directions by D_s and D_t, respectively. Now

D is symmetric, hence

$$D_t D_t H_s = D_t D_s H_t = D_s D_t H_t - R(H_s, H_t) H_t.$$

Since the variation is geodesic, we have $D_t H_t = 0$. Therefore,

$$D_t D_t H_s = -R(H_s, H_t) H_t.$$

Now in $s = 0$, we have $H_t = c'$, $H_s = V$, and $D_t D_t H_s = V''$, where V is the variation field of H.

2) Let $\alpha = \alpha(s)$ be a smooth curve in M with $\alpha(0) = c(0)$ and $\alpha'(0) = V(t_0)$. Let X be a smooth vector field along α with $X(0) = c'(t_0)$ and $D_s X(0) = V'(t_0)$. Let $H(s,t) = c_s(t)$, where c_s is the geodesic with $c_s(t_0) = \alpha(s)$ and $c_s'(t_0) = X(s)$. By our assumption, c_s is defined on I for all $s \in (-\varepsilon, \varepsilon)$, if we choose $\varepsilon > 0$ is sufficiently small. Furthermore,

$$H(s,0) = \alpha(s), \quad H_s(0,0) = V(0), \quad H_t(s,0) = X(s)$$

and, since D is symmetric,

$$D_t H_s(0,0) = D_s H_t(0,0) = V'(0).$$

Hence the variation field $H_s(0,\cdot)$ of H has the same initial conditions at $t = 0$ as the given Jacobi field V. Now H is a geodesic variation and hence $H_s(0,\cdot)$ is a Jacobi field. Therefore $V = H_s(0,\cdot)$, hence V is the variation field of H. □

Examples 3.8. 1) For any geodesic c and constants $a, b \in \mathbb{R}$, the vector field $V(t) = (at+b)c'(t)$, $t \in I$, is a Jacobi field.

2) Let $M = S^m \subset \mathbb{R}^{m+1}$ be the unit sphere and D be its Levi-Civita connection as in (1.3). Let $c : \mathbb{R} \to S^m$ be a great circle parametrized by arc length, $c(t) = \cos t\, x + \sin t\, y$, where x and y are perpendicular unit vectors in $\mathbb{R}^{m+1}$. Let $v \in \mathbb{R}^{m+1}$ be a further unit vector, and suppose that v is perpendicular to x and y. Then

$$H(s,t) = \cos t\, x + \sin t\, (\cos s\, y + \sin s\, v)$$

is a geodesic variation of c. Note that the constant vector field $E(t) = v$ is parallel along c and that the variation field of H is $V(t) = \sin t\, E(t)$. A nice application: The Jacobi equation implies that $R(u, c'(t))c'(t) = u$ for any $t \in \mathbb{R}$ and $u \in T_{c(t)} S^m$ perpendicular to $c'(t)$.

3.3. **Exponential map.** The set $\mathcal{E} = \{v \in TM \mid (1,v) \in \mathcal{G}\}$ is open with $\{0_p \mid p \in M\} \subset \mathcal{E}$. The *exponential map* $\exp : \mathcal{E} \to M$ is defined by

$$\mathcal{E} \ni v \mapsto \exp(v) := c_v(1). \tag{3.6}$$

By Proposition 3.5, the exponential map is smooth.

Let $p \in M$. Then $\mathcal{E}_p = \mathcal{E} \cap T_p M$ contains 0_p, is star-shaped with respect to 0_p and open. The restriction of the exponential map to $\mathcal{E}_p$ is denoted $\exp_p$. For any $v \in T_p M$, $\exp_p(tv) = c_{tv}(1) = c_v(t)$ for all $t \in \mathbb{R}$ with $tv \in \mathcal{E}_p$.

PROPOSITION 3.9. *Let $p \in M$.*
1) The differential of $\exp_p$ *in* 0_p *is the identity,* $(\exp_p)_{*0_p}(v) = v$.
2) The differential of $\pi \times \exp : TM \to M \times M$ *in* 0_p*, where* $\pi : TM \to M$ *is the projection, is an isomorphism.*

Proof. For any $v \in T_pM$ and $t \in \mathbb{R}$ sufficiently small, we have $tv \in \mathcal{E}_p$ and $\exp_p(tv) = c_v(t)$. Hence $(\exp_p)_{*0_p}(v) = c_v'(0) = v$. This proves the first assertion.

To prove the second, note that by the first assertion, any tangent vector in $T_pM \times T_pM$ of the form $(0, v)$ is in the image of $(\pi \times \exp)_{*0_p}$. On the other hand, if c is a smooth curve through p with $c'(0) = u$, and $u(t) := 0_{c(t)}$, then u is a smooth curve in TM with $\big((\pi \times \exp) \circ u\big)'(0) = (u, u)$. This implies that $(\pi \times \exp)_{*0_p}$ is surjective. On the other hand, $\dim TM = 2m = 2 \dim M$. Hence $(\pi \times \exp)_{*0_p}$ is an isomorphism. □

It follows that there are open neighborhoods U' of 0_p in T_pM and U of p in M such that $\exp_p : U' \to U$ is a diffeomorphism. Hence, up to an isomorphism of T_pM with $\mathbb{R}^m$, we can consider $\exp_p^{-1} : U \to U'$ as a coordinate chart of M about p. In this coordinate chart, geodesics through p correspond to lines in T_pM.

It also follows that $\pi \times \exp$ is a diffeomorphism from a neighborhood V of 0_p in TM to a product neighborhood $U \times U$ of (p, p) in $M \times M$. In particular, for any pair of points (q_0, q_1) in $U \times U$, there is a unique tangent vector $u \in V$ with $\pi(v) = q_0$ and $\exp(v) = q_1$, and u depends smoothly on q_0 and q_1.

PROPOSITION 3.10. *Let $p \in M$, $v \in \mathcal{E}_p$ and $w \in T_vT_pM \cong T_pM$. Then*

$$(\exp_p)_{*tv}(tw) = V(t), \quad 0 \le t \le 1,$$

where V is the Jacobi field along the geodesic $c(t) = \exp_p(tv)$, $0 \le t \le 1$, with $V(0) = 0$ and $V'(0) = w$.

Proof. Let $H(s, t) = \exp_p(t(v + sw))$, $0 \le t \le 1$. Since $\mathcal{E}_p$ is open, H is defined for all s sufficiently small and

$$(\exp_p)_{*tv}(tw) = H_s(0, t).$$

Now $c_s = H(s, \cdot)$ is a geodesic for each s. Hence the variation field V of H is a Jacobi field along c. We have $V(0) = H_s(s, 0) = 0$ since $H(s, 0) = p$ for all s. Furthermore,

$$V'(0) = D_t H_s(0, 0) = D_s H_t(0, 0) = D_s(v + sw)\big|_{s=0} = w.$$

□

4. Exercises

1) Discuss the representation of the torsion tensor with respect to local frames. What is a decisive difference between a general local frame of TM and the one associated to a coordinate chart?

2) Let D be a symmetric connection on M and $f : M \to \mathbb{R}$ be a smooth function. Define the second covariant derivative $D^2 f$ by

$$D^2 f(X,Y) := XY(f) - D_X Y(f), \quad X,Y \in \mathcal{V}(M).$$

Show that $D^2 f$ is tensorial and symmetric in X and Y.

3) Let D be a symmetric connection on M and Z be a smooth vector field on M. Define the second covariant derivative $D^2 X$ by

$$D^2 Z(X,Y) := D_X D_Y Z - D_{D_X Y} Z, \quad X,Y \in \mathcal{V}(M).$$

Show that $D^2 Z$ is tensorial in X and Y. What about the symmetry of $D^2 X$?

4) Let D be a symmetric connection on M. Show that its curvature tensort R satisfies the *first Bianchi identity*:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

for all $X, Y, Z \in \mathcal{V}(M)$.

5) A smooth tensor field of type $(k,0)$ or $(k,1)$, respectively, is a k-linear map

$$L : \mathcal{V}(M) \times \ldots \times \mathcal{V}(M) \to \mathcal{V}, \quad (X_1, \ldots, X_k) \mapsto L(X_1, \ldots, X_k),$$

with $\mathcal{V} = \mathcal{F}(M)$ or $\mathcal{V} = \mathcal{V}(M)$, respectively, which is $\mathcal{F}(M)$-homogeneous in each variable X_i. Use the φ^2-argument from the proof of Lemma A.2 to show that any such tensor field is given by a family of k-linear maps $L_p : T_pM \times \ldots \times T_pM \to \mathbb{R}$ or $L_p : T_pM \times \ldots \times T_pM \to T_pM$, respectively, such that

$$L(X_1, \ldots, X_k)(p) = L_p(X_1(p), \ldots, X_k(p))$$

for all $p \in M$ and $X_1, \ldots, X_k \in \mathcal{V}(M)$. Discuss the representation of tensor fields with respect to local frames.

6) Let D be a connection on M and L be a tensor field of type $(k,0)$ or $(k,1)$, respectively. Define its covariant derivative DL by

$$DL(X_0, \ldots, X_k) := D_{X_0}\big(L(X_1, \ldots, X_k)\big) - \sum L(X_0, \ldots, D_{X_0} X_i, \ldots, X_k).$$

Here we use the notation $D_X f := Xf$ for smooth functions f. Show that DL is a tensor field of type $(k+1,0)$ or $(k+1,1)$, respectively.

Appendix A. Two technical lemmas

There are some lemmas of a more technical nature which we use over and over again. We also need modifications of these lemmas, the arguments underlying the proofs are useful tools.

Lemma A.1. *1) Given $p \in M$ and a neighborhood U of p in M, there exists a function $\varphi \in \mathcal{F}(M)$ such that $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset U$ and such that $\varphi(q) = 1$ for all q in a (small) neighborhood of p.*
2) Given $p \in M$ and a tangent vector $v \in T_pM$, there is a smooth vector field X on M with $X(p) = v$.
3) Let $W \subset M$ be open, $X \in \mathcal{V}(W)$, and $p \in W$. Then there is $Y \in \mathcal{V}(M)$ with $X(q) = Y(q)$ for all q in a neighborhood of p.

Proof. 1) By replacing U by a smaller neighborhood of p if necessary, we can and will assume that there is a coordinate chart $x : U \to U'$. We arrange x such that $x(p) = 0$ and let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a smooth function with $0 \le \psi \le 1$, $\operatorname{supp}\psi \subset U'$ and such that $\psi(u) = 1$ for all u in a (small) neighborhood of 0. Then $\varphi : M \to \mathbb{R}$, $\varphi = \psi \circ \psi$ on U and $\psi = 0$ otherwise satisfies the assertions. Then $\varphi = 0$ in a neighborhood of the boundary of U, hence φ is smooth.

2) Choose a coordinate chart $x : U \to U'$ about p. Let $X_1 \dots, X_m$ be the associated frame and $\xi \in \mathbb{R}^m$ be the principal part of v with respect to x, $v = \xi^i X_i(p)$. Choose φ as in 1) and define

$$Y(q) = \begin{cases} \varphi\xi^i X_i(q) & \text{for } q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = 0$ in a neighborhood of the boundary of U, hence Y is smooth.

3) We use the same argument as in 2): Choose the coordinate chart x such that $U \subset W$. Now the principal part of X with respect to x is a map $\xi : U \to \mathbb{R}^m$, all we do now is to replace, in the definition of Y, the constants ξ^i by the values $\xi(q)$. □

Lemma A.2. *Let $\Phi : \mathcal{V}(M) \to \mathcal{F}(M)$ be a linear map which is* tensorial, *i.e.,*

$$\Phi(\varphi X) = \varphi\Phi(X) \quad \textit{for all } \varphi \in \mathcal{F}(M) \textit{ and } X \in \mathcal{V}(M).$$

Then there is a smooth 1-form ω with

$$\Phi(X)(p) = \omega_p(X(p)) \quad \textit{for all } X \in \mathcal{V}(M) \textit{ and } p \in M.$$

Proof. Let $p \in M$ and X and Y be smooth vector fields on M such that $X(q) = Y(q)$ for all q in a neighborhood V of p. It suffices to show that $\Phi(X)(p) = \Phi(Y)(p)$.

In a neighborhood $U \subset V$ of p, choose a local frame $(X_1, \dots, X_m)$ of TM and a function $\varphi \in \mathcal{F}(M)$ with $\varphi(p) = 1$ and $\operatorname{supp}(\varphi) \subset U$. Then $\varphi \cdot X_i$ is a smooth vector field when extended by zero outside U, $1 \le i \le m$.

By assumption, the principal parts of X and Y coincide, $X = \xi^i X_i = Y$ on U. The functions $\varphi\xi^i$ are smooth on M when extended by zero outside U and

$$\varphi^2 \cdot X = (\varphi\xi^i) \cdot (\varphi X_i) = \varphi^2 \cdot Y.$$

Hence

$$\Phi(X)(p) = \varphi^2(p) \cdot \Phi(X)(p) = \Phi(\varphi^2 \cdot X)(p) = \Phi(Y)(p).$$

□

REMARK A.3. In the text, we need some variations of Lemma A.2, for example in the localization arguments concerning covariant derivatives and curvature tensors. On the other hand, the φ^2-argument is easy to adapt and, therefore, we leave the proof of the corresponding assertions to the reader, see Exercise 5) in Section 4.

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