BASIC GEOMETRY OF SUBMANIFOLDS

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Introduction

I discuss the geometry of submanifolds in $\mathbb{R}^n$, which motivated most of the results and techniques of present day differential geometry. This is not a historical account of differential geometry. With today's point of view I explain some of the main insights into differential geometry before the time of Gauss and Riemann.

There are two aspects of the geometry of submanifolds, intrinsic geometry and extrinsic geometry. In our setting, intrinsic differential geometry describes the geometry inside the submanifolds — the only role of the ambient space $\mathbb{R}^n$ is to induce a way of measuring angles and lengths of geometric objects contained in the submanifolds. Extrinsic geometry deals with the shape of submanifolds as subsets of the ambient space.

All considerations are local and the arguments go through for immersions as well. Since many of the elementary examples come as immersions, the setting in this chapter is the following: M is a manifold of dimension m, and $f : M \to \mathbb{R}^n$ is an immersion. The data is the pair (M, f).

I assume that the reader is familiar with the theory of curves in $\mathbb{R}^n$. I also assume some familiarity with the theory of manifolds. The reader who is not on good terms with manifolds yet is advised to think of them as open subsets or submanifolds of Euclidean spaces.

0.1. **Notation.** If $U \subset \mathbb{R}^m$ is open, V is a real vector space (of finite dimension), and $\varphi : U \to V$ is a smooth function, then the partial derivative of φ with respect to x_i is denoted in the following different ways,

$$\varphi_i = \varphi_{x_i} = \frac{\partial \varphi}{\partial x^i} = d\varphi \cdot \frac{\partial}{\partial x^i}.$$

Analogous notation will be used for higher partial derivatives. There are other objects with indices, where the indices have a different meaning. But it seems that there is no danger of confusion.

We let $\mathcal{F}(M)$ and $\mathcal{V}(M)$ be the spaces of smooth real valued functions and smooth vector fields on M, respectively. Recall that tangent vectors of M act as derivations on smooth maps with values in vector spaces, $\varphi : M \to V$. For $X \in \mathcal{V}(M)$, we use the notations $Xf = df \cdot X$ for the induced smooth function $M \ni p \mapsto X(p)(f) \in V$.

1. Intrinsic Geometry

The intrinsic geometry of M with respect to the given map f is concerned with the measurements of objects inside M. The only way in which Euclidean space $\mathbb{R}^n$ enters is through the restriction of the inner product to the tangent spaces of (M, f).

Definition 1.1. For $p \in M$, the *tangent space* T_pf and the *normal space* N_pf of (M, f) are the linear subspaces of $\mathbb{R}^n$ defined by

$$T_pf := \operatorname{im} df(p) \quad \text{and} \quad N_pf := [\operatorname{im} df(p)]^\perp.$$

Since f is an immersion, $df(p) : T_pM \to T_pf$ is an isomorphism for all $p \in M$. In the case where M is a submanifold and f is the inclusion, $df(p)$ is the usual identification of T_pM with a linear subspace of $\mathbb{R}^n$. In this case we write T_pM and N_pM instead of T_pf and N_pf. In the general case of an immersion, we also think of $df(p)$ as a natural identification of T_pM with T_pf. For all $p \in M$, the tuple $(f_1(p), \ldots, f_m(p))$ of partial derivatives of f is a basis of T_pf. The *codimension* of (M, f) is $n - m$, it is equal to the dimension of N_pf.

Example 1.2. Let $r > 0$ and $S_r^m = \{x \in \mathbb{R}^{m+1} \mid \|x\|^2 = r^2\}$ be the round sphere of dimension m and radius r. Then S_r^m is a submanifold of $\mathbb{R}^{m+1}$. Here f is the inclusion, $T_xS^m = \{y \in \mathbb{R}^{m+1} \mid \langle x, y\rangle = 0\}$, and $N_xM = \mathbb{R} \cdot x$.

Let $W \subset M$. For a map $X : W \to \mathbb{R}^n$ and $p \in W$, let $X(p) = X^T(p) + X^N(p)$ be the decomposition according to the splitting $\mathbb{R}^n = T_pf + N_pf$. We call X^T the *tangential* and X^N the *normal part* of X. We start with a technical lemma, which we want to have out of the way.

Lemma 1.3. *1) If $W \subset M$ is open, and $X : W \to \mathbb{R}^n$ is smooth, then X^T and X^N are smooth.*
2) For any point $p \in M$ there are an open neighborhood W of p in M and smooth maps $X_i : W \to \mathbb{R}^n$, $1 \le i \le n$, such that $(X_1(q), \ldots, X_m(q))$ is an orthonormal basis of T_qf and $(X_{m+1}(q), \ldots, X_m(q))$ an orthonormal basis of N_qf, for all $q \in U$.

Proof. Let $U \to U'$ be a coordinate chart of M about p. Choose $v_{m+1}, \ldots, v_n \in \mathbb{R}^n$ such that $(f_1(p), \ldots, f_m(p), v_{m+1}, \ldots, v_n)$ is a basis of $\mathbb{R}^n$. By continuity, there is an open neighborhood $W \subset U$ of p such that $(f_1(q), \ldots, f_m(q), v_{m+1}, \ldots, v_n)$ is still a basis of $\mathbb{R}^n$. Now apply Gram-Schmidt orthonormalization to this basis to obtain an orthonormal basis $X_1(q), \ldots, X_n(q)$ of $\mathbb{R}^n$. By the formulas defining Gram-Schmidt orthonormalization, we see that the maps X_i are smooth.

Gram-Schmidt orthonormalization has the characteristic property that it keeps linear hulls of any initial subsequence. In particular, the hull of $X_1(q), \ldots, X_m(q)$ is the same as that of $f_1(q), \ldots, f_m(q)$. Hence $(X_1(q), \ldots, X_n(q))$ is an orthonormal basis of T_qf. It follows that $(X_{m+1}(q), \ldots, X_m(q))$ is an orthonormal basis of N_qf. This proves Assertion 2). Assertion 1) is a direct consequence of 2). □

1.1. **First fundamental form.** In this section we discuss length measurements, that is, we are concerned with the lengths of curves in M. Let $c : [a, b] \to M$ be a piecewise smooth curve, that is, there is a subdivision

$$a = t_0 < t_1 < \ldots < t_k = b,$$

such that $c|[t_{i-1}, t_i]$ is smooth, $1 \le i \le k$. We define the *length* $L(c)$ of c by

$$L(c) = \int_a^b \|(f \circ c)'\| \, dt = \int_a^b \|df(c(t)) \cdot c'(t)\| \, dt. \tag{1.1}$$

We recall that the length of c is invariant under reparameterizations of c.

The inner product and the norm of Euclidean space enter the definition of length only via their restriction to the tangent spaces $T_{c(t)}f$. To emphasize that the ambient space does not enter in any other way, we define a family g_p, $p \in M$, of inner products on the respective tangent spaces of M as follows.

Definition 1.4. The *first fundamental form* g_p of (M, f) at p is the inner product on T_pM given by

$$g_p(v, w) := \langle df(p) \cdot v, df(p) \cdot w \rangle, \quad v, w \in T_pM.$$

Note that the formula defines an inner product indeed since f is an immersion. We denote the first fundamental form g_p also by $\langle . , . \rangle_p$. Because there is no danger of confusion, we will mostly delete the index p. In terms of the first fundamental form, the length of the curve c above is given by

$$L(c) = \int_a^b \|c'(t)\| \, dt, \tag{1.2}$$

where $\|.\| = \|.\|_p$ denotes the norm associated to g_p. By definition, $L(c) = L(f \circ c)$.

So far, the first fundamental form looks a bit abstract and the question arises how we can handle this object. The most straightforward way is by coordinate charts: Let $x : U \to U'$ be a coordinate chart of M. For all $p \in U$,

$$\big(\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^m}(p)\big)$$

is a basis of T_pM, and the coefficients $g_{ij} = g_{ij}(p)$ of the first fundamental form with respect to this basis are given by

$$g_{ij}(p) := \big\langle df(p) \cdot \frac{\partial}{\partial x^i}(p), df(p) \cdot \frac{\partial}{\partial x^j}(p) \big\rangle = \langle f_i(p), f_j(p) \rangle. \tag{1.3}$$

Since $U \ni p \mapsto f_i(p)$ is smooth for all i, it follows that the coefficients g_{ij} are smooth functions on U.

Examples 1.5. 1) *Graphs*: Let $W \subset \mathbb{R}^m$ be an open subset and $h : W \to \mathbb{R}$ be a smooth function. Define $f : W \to \mathbb{R}^{m+1}$ by $f(x) = (x, h(x))$. Then f is an injective immersion with $f_i = (e_i, h_i)$, where e_i is the i-th standard vector in $\mathbb{R}^m$. For all $p \in M$, the tangent space T_pf is the linear hull of the linearly independent vectors $f_i(p)$. If $m = 2$, N_pf is the line determined by the vector product $f_1(p) \times f_2(p)$. In higher dimension, there is a similar formula for a generator of the normal space. The entries of the matrix of the first fundamental form are

$$g_{ij} = \langle f_i, f_j \rangle = \delta_{ij} + h_i h_j.$$

2) *Surfaces of revolution*: Let $(r, h) = (r(t), h(t))$, $t \in I$, be a regular curve in the (x, z)-plane and suppose $r > 0$. Define a smooth map $f : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$f(t, \varphi) = (r(t) \cos \varphi, r(t) \sin \varphi, h(t)).$$

The curves $\varphi = \text{const}$ are called *profile curves* or *meridians*, the curves $t = \text{const}$ *parallels* of f. The partial derivatives of f are

$$f_t = (r' \cos \varphi, r' \sin \varphi, h'), \quad f_\varphi = (-r \sin \varphi, r \cos \varphi, 0).$$

Since f_t and f_φ are non-zero and perpendicular to each other, they are linearly independent. Hence f is an immersion with $T_p f$ generated by $f_t(p)$ and $f_\varphi(p)$. The normal space $N_p f$ is the line generated by the vector product $f_t(p) \times f_\varphi(p)$. The entries of the matrix of the first fundamental form are

$$g_{tt} = \langle f_t, f_t \rangle = (r')^2 + (h')^2, \quad g_{t\varphi} = g_{\varphi t} = \langle f_\varphi, f_t \rangle = 0, \quad g_{\varphi\varphi} = \langle f_\varphi, f_\varphi \rangle = r^2.$$

If c is parameterized by arc length, then $g_{tt} = 1$, and then the matrix of the first fundamental form of f (in the given coordinates (t, φ)) is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$.

Suppose now that X and Y are smooth vector fields on M. Let $\xi = (\xi^1, \cdots, \xi^m)$ and $\eta = (\eta^1, \ldots, \eta^m)$ be the principal parts of X and Y with respect to the coordinate chart x, that is, for each $p \in U$ we have[1]

$$X(p) = \xi^i(p) \frac{\partial}{\partial x^i}(p) \quad \text{and} \quad Y(p) = \eta^i(p) \frac{\partial}{\partial x^i}(p).$$

Applying the first fundamental form, we get

$$\langle X(p), Y(p) \rangle = \langle \xi^i(p) \frac{\partial}{\partial x^i}(p), \eta^j(p) \frac{\partial}{\partial x^j}(p) \rangle = g_{ij}(p) \xi^i(p) \eta^j(p). \tag{1.4}$$

In particular, $\langle X(p), Y(p) \rangle$ depends smoothly on $p \in U$, that is, $\langle X, Y \rangle$ is a smooth function on U and hence on M. Mostly we suppress the dependence on p and write Equation 1.4 as

$$\langle X, Y \rangle = g_{ij} \xi^i \eta^j \tag{1.5}$$

with the understanding that this is an equality of functions, i.e., that it holds at each point $p \in U$ as in Equation 1.4.

Recall that $\xi^i = dx^i(X)$, $\eta^i = dx^i(Y)$. Hence Equations 1.4 and 1.5 imply that on U,

$$g = g_{ij} dx^i dx^j. \tag{1.6}$$

This is the expression of the first fundamental form in a coordinate chart.

[1]Here and below we use Einstein's summation convention.

Below we will frequently need the inverse matrix of the matrix (g_{ij}). Its coefficients are denoted g^{ij}. By definition, we have

$$g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k. \tag{1.7}$$

1.2. **Intrinsic Distance.** We come now to one of the main objects of intrinsic geometry, namely the *intrinsic distance* d associated to (M, f). For points $p, q \in M$ set

$$d(p, q) = \inf L(c), \tag{1.8}$$

where the infimum is taken over all piecewise smooth curves c in M from p to q.

Clearly the *extrinsic distance* $\|f(p)-f(q)\| \leq d(p,q)$, for all $p, q \in M$. However, the extrinsic distance involves the geometry of the ambient space $\mathbb{R}^n$ — shortest curves with respect to the extrinsic distance are straight lines in $\mathbb{R}^n$ and, in general, not contained in the image of f.

Theorem 1.6. *The intrinsic distance d is a metric on M and induces the given topology of M.*

Remark 1.7. Note that we allow for $d(p, q) = \infty$. This occurs precisely if p and q lie in different path components of M.

Proof of Theorem 1.6. The only non-trivial assertions are that $d(p,q) > 0$ if $p \neq q$ (in the case where f is not injective) and that d induces the given topology of M. To show these, choose a coordinate chart $x : U \to U'$ of M about a given point p_0. Then there exist constants $\varepsilon > 0$ and $\rho > 0$ such that the closed (Euclidean) ball B of radius ρ about $x(p_0)$ is contained in U' and such that

$$\varepsilon^2 \sum (\xi^i)^2 \leq g_{ij}(p)\xi^i\xi^j \leq \varepsilon^{-2} \sum (\xi^i)^2$$

for all $p \in x^{-1}(B)$ and $\xi \in \mathbb{R}^m$. Since M is Hausdorff, it follows that any path in M starting in p_0 and leaving $x^{-1}(B)$ runs from p_0 to $x^{-1}(\partial B)$ before leaving $x^{-1}(B)$. Therefore, any such path has length at least $\varepsilon\,\rho$. Hence $d(p_0, q) \geq \varepsilon\,\rho > 0$ for all $q \in M \setminus x^{-1}(B)$. Inside $x^{-1}(B)$, lengths of curves satisfy the estimate

$$\varepsilon\, L(c) \leq L_E(x \circ c) \leq L(c)/\varepsilon,$$

where $L_E(x \circ c)$ denotes the Euclidean length of $x \circ c$. Hence

$$\varepsilon \|x(p) - x(q)\| \leq d(p,q) \leq \|x(p) - x(q)\|/\varepsilon$$

for all $p, q \in x^{-1}(B)$. $\square$

Example 1.8. Let $M = S^m$ be the standard sphere of radius 1 in $\mathbb{R}^{m+1}$, compare Example 1.2. We will prove now, although it is rather clear, that the interior distance is given by $d(x, y) = \angle(x, y)$.

To that end, we fix $x \in S^m$ and parameterize points $y \in S^m \setminus \{x, -x\}$ by

$$y = \cos\varphi\, x + \sin\varphi\, z$$

with $\varphi = \varphi(y) \in (0, \pi)$ and $z = z(y)$ contained in the equator of S^m with respect to x, that is, $\langle x, z\rangle = 0$ and $\langle z, z\rangle = 1$. On $S^m \setminus \{x, -x\}$ we define a vector field V of norm one by

$$V(\cos\varphi\, x + \sin\varphi\, z) = -\sin\varphi\, x + \cos\varphi\, z.$$

Suppose now that $c : [a, b] \to S^m$ is a piecewise smooth curve with $c(a) = x$. We claim that $L(c) \geq \varphi(c(b))$. For the proof of this claim we can assume that $a = a_0 := \sup\{t \in [a, b] \mid c(t) = x\}$ since we can restrict c to the interval $[a_0, b]$ otherwise. Similarly, we can assume $b = \inf\{t \in [a, b] \mid \varphi(c(t)) = \varphi(c(b))\}$. Then $c(t) \in S^m \setminus \{x, -x\}$ for all $t \in (a, b)$. Now $\|V\| = 1$ and hence

$$\|c'(t)\| \geq \langle c'(t), V(c(t))\rangle$$

for all $t \in (a, b)$. We obtain

$$L(c) = \int_a^b \|c'(t)\|\, dt \geq \int_a^b \langle c'(t), V(c(t))\rangle\, dt.$$

On (a, b), we write $c(t) = \cos\varphi(t)\, x + \sin\varphi(t)\, y(t)$. Then

$$c' = -\sin\varphi\, \varphi'\, x + \cos\varphi\, \varphi'\, y + \sin\varphi\, y',$$

where we suppress the parameter t. Now $y = y(t)$ is a curve in the equator, hence $\langle y', V\rangle = 0$. Since $\lim_{t\to a} \varphi(t) = 0$ we get

$$L(c) \geq \int_a^b \varphi'\, dt = \varphi(b).$$

This proves our claim on $L(c)$. It follows that $d(x, y) \geq \angle(x, y)$. Now the great circle arc between x and y has length $\angle(x, y)$, therefore $d(x, y) = \angle(x, y)$ as claimed.

We leave it as an exercise to the reader to show that equality holds in the above estimate on $L(c)$ if and only if there is a non-negative function $a = a(t)$ such that $c'(t) = a(t) \cdot V(c(t))$ for all $t \in (a, b)$, that is, if and only if c is a weakly monotonic reparameterization of a great circle arc starting from x.

1.3. **First Variation of Arc Length and Geodesics.** We discuss now necessary conditions for a piecewise smooth curve to be a minimal connection of its end points. We fix a piecewise smooth curve $c : [a, b] \to M$ and let $p = c(a)$ and $q = c(b)$ be the end points of c. The idea is to look at families c_s of curves with $c = c_0$ and with the same endpoints. If c is a minimal connection of its end points, then the length function $s \mapsto L(c_s)$ has a minimum in $s = 0$. This will lead to a differential equation for the curve c.

Definition 1.9. A *(piecewise smooth) variation* of c is a map

$$h : (-\varepsilon, \varepsilon) \times [a, b] \to M \quad \text{with} \quad h(0, .) = c,$$

such that there is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ such that $h|(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$ is smooth, $1 \le i \le k$. We say that a variation h of c is *proper* if $h(s, a) = c(a)$ and $h(s, b) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$.

We call the curves $c_s := h(s, .)$ the *curves of the variation* and the piecewise smooth vector field $V = h_s(0, .)$ along c the *variation field* of h.

Let c be a piecewise smooth curve and h be a variation of c. By the definition of the first fundamental form we have

$$\|c_s'(t)\| = \|(f \circ h)_t(s, t)\|.$$

Suppose now that c is regular and that the subdivision $a = t_0 < t_1 < \cdots < t_k = b$ is chosen as in Definition 1.9. Then we see from the above expression that $\|c_s'(t)\|$ depends smoothly on s and t in the rectangles $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$, $1 \le i \le k$, at least if we pass to a sufficiently small ε such that all the curves c_s are regular as well. Hence $L(c_s)$ depends smoothly on s (for s sufficiently small) since we can differentiate under the integral sign.

Definition 1.10. For a regular curve $c : [a, b] \to M$ and a variation h of c as in Definition 1.9, the derivative $\dfrac{d(L(c_s))}{ds}(0)$ is called the *first variation of arc length* with respect to h. We say that c is a *geodesic* if c has constant speed and the first variation of arc length is 0 for every proper variation of c.

Example 1.11. Suppose p and q are points in M such that the Euclidean line segment between $f(p)$ and $f(q)$ is contained in M, more precisely, such that there is a curve $c : [a, b] \to M$ of constant speed such that $f \circ c$ parameterizes the line segment between $f(p)$ and $f(q)$. Then $f \circ c$ is a shortest curve between $f(p)$ and $f(q)$ in the ambient space $\mathbb{R}^m$, and therefore c is, a fortiori, a shortest connection, hence a geodesic, between p and q in M.

A class of examples where this occurs: Let $c : I \to \mathbb{R}^3$ and $v : I \to \mathbb{R}^3$ be two curves. The associated *ruled surface* is defined as

$$f(s, t) = c(t) + s \cdot v(t).$$

A particular ruled surface is the *helicoid* with $c(t) = (0, 0, t)$ and $v = (\cos t, \sin t, 0)$. Another example is the *Möbius band*

$$f(s, t) = \big((1 + s \cdot \cos \frac{t}{2}) \cos t, (1 + s \cdot \cos \frac{t}{2}) \sin t, \sin \frac{t}{2}\big).$$

The partial derivatives of f are

$$f_s = v \quad \text{and} \quad f_t = c' + s \cdot v'.$$

It follows that f is an immersion in an open neighborhood U of $\{s = 0\} \subset \mathbb{R} \times I$ if $v(t)$ and $c'(t)$ are linearly independent, for all $t \in I$. The straight lines $t = \text{const}$ are called *rulings* of f. They are geodesics, by the above argument.

We now compute the first variation of arc length under the assumption that the given piecewise smooth curve $c : [a, b] \to M$ has constant non-zero speed.

First Variation Formula 1.12. *Let $c : [a,b] \to M$ be a piecewise smooth curve such that $\|c'(t)\| = const \neq 0$. Then*

$$\frac{d(L(c_s))}{ds}(0) = \frac{1}{\|c'\|}\left[\sum_{i=1}^{k} \langle V, c'\rangle\Big|_{t_{i-1}}^{t_i} - \int_a^b \langle df\cdot V, (f\circ c)''\rangle\, dt\right]$$
$$= \frac{1}{\|c'\|}\left[\langle V, c'\rangle\Big|_a^b + \sum_{i=1}^{k-1} \langle V(t_i), \Delta_i\rangle - \int_a^b \langle df\cdot V, (f\circ c)''\rangle\, dt\right]$$

with $\Delta_i = c'(t_i-) - c'(t_i+)$, $1 \leq i \leq k-1$.

Remark 1.13. The assumption on the speed is not essential since arc length is independent under reparameterizations.

Proof of First Variation Formula.

$$\frac{d(L(c_s))}{ds} = \frac{d}{ds}\Big(\int_a^b \|c_s'(t)\|\, dt\Big)$$
$$= \int_a^b \frac{d}{ds}\sqrt{\langle (f\circ h)_t, (f\circ h)_t\rangle}\, dt.$$

Recall that differentiation under the integral sign is legitimate since $\|c_s'(t)\|$ is smooth in (s,t) in the rectangles $(-\varepsilon, \varepsilon)\times[t_{i-1}, t_i]$, $1 \leq i \leq k$. Since c has constant non-zero speed, we get

$$\frac{d(L(c_s))}{ds}(0)$$
$$= \frac{1}{\|c'\|}\int_a^b \big\langle (f\circ h)_{ts}, (f\circ h)_t\big\rangle(0,t)\, dt$$
$$= \frac{1}{\|c'\|}\int_a^b \big\langle (f\circ h)_{st}, (f\circ h)_t\big\rangle(0,t)\, dt$$
$$= \frac{1}{\|c'\|}\int_a^b \frac{\partial}{\partial t}\big\langle (f\circ h)_s, (f\circ h)_t\big\rangle(0,t)\, dt - \frac{1}{\|c'\|}\int_a^b \big\langle (f\circ h)_s, (f\circ h)_{tt}\big\rangle(0,t)\, dt$$
$$= \frac{1}{\|c'\|}\sum_{i=1}^{k} \big\langle (f\circ h)_s, (f\circ h)_t\big\rangle(0,t)\Big|_{t_{i-1}}^{t_i} - \frac{1}{\|c'\|}\int_a^b \big\langle (f\circ h)_s, (f\circ h)_{tt}\big\rangle(0,t)\, dt.$$

In $s = 0$ we have

$$(f\circ h)_s = df\cdot V, \quad (f\circ h)_t = df\cdot c' \quad \text{and} \quad (f\circ h)_{tt} = (f\circ c)''.$$

Now the asserted formula follows from the definition of the first fundamental form. □

We arrive at a result, which was obtained — in a somewhat different but equivalent formulation — by Johann Bernoulli in 1698 (unpublished), compare

[HT, pages 117 ff.] one of the first results in differential geometry. It goes without saying that Johann Bernoulli only considered surfaces in Euclidean space $\mathbb{R}^3$.

THEOREM 1.14 (Johann Bernoulli). *A smooth curve $c : [a, b] \to M$ is a geodesic iff the second derivative of $f \circ c$ is normal to (M, f), that is, iff*

$$(f \circ c)''(t) \in N_{c(t)} f \quad \text{for all } t \in [a, b].$$

Proof. Suppose first that $(f \circ c)''$ is normal. Then, by the definition of the first fundamental form,

$$\langle c', c' \rangle' = \langle (f \circ c)', (f \circ c)' \rangle' = 2\langle (f \circ c)', (f \circ c)'' \rangle = 0,$$

and hence c has constant speed.

Consider a proper variation h of c as above. Since h is proper, $V(a) = 0$ and $V(b) = 0$. Since c is smooth, $\Delta_i = 0$, $1 \le i \le k-1$. Therefore, the only term remaining on the right hand side of the first variation formula is the integral. Now the integrand is zero since $df \cdot V$ is in $T_p f$ and $(f \circ c)''$ is perpendicular to $T_p f$. Hence c is a geodesic.

We now prove the converse. By Lemma 1.3, we need only to assure that $(f \circ c)''(t)$ is normal for all t in the open interval (a, b). To arrive at a contradiction, suppose that there is a $t_0 \in (a, b)$ such that $(f \circ c)''(t_0)$ is not normal. Then there is a tangent vector $v \in T_{c(t_0)} M$ with

$$\langle df(c(t_0)) \cdot v, (f \circ c)''(t_0) \rangle > 0.$$

Choose a coordinate chart $x : U \to U'$ for M about $c(t_0)$ and let $\xi \in \mathbb{R}^m$ be the principal part of v with repect to x. We have

$$v = \xi^i \frac{\partial}{\partial x^i}(c(t_0)) \quad \text{and} \quad df(c(t_0)) \cdot v = \xi^i f_i(c(t_0)).$$

By continuity, there is an $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0 + \varepsilon) \subset (a, b)$ and such that

$$\left\langle \xi^i f_i(c(t)), (f \circ c)''(t_0) \right\rangle > 0$$

for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function with

$$0 \le \phi \le 1, \quad \operatorname{supp} \phi \subset (t_0 - \varepsilon, t_0 + \varepsilon), \quad \text{and} \quad \phi(t_0) = 1.$$

Set

$$h(s, t) = \begin{cases} \left(f \circ x^{-1}\right)\left(x(c(t)) + s\,\phi(t)\,\xi\right) & \text{for } t \in (t_0 - \varepsilon, t_0 + \varepsilon), \\ c(t) & \text{otherwise.} \end{cases}$$

Then h is a proper smooth variation of c and, by construction, the variation field $V = V(t) = (f \circ h)_s(0, t)$ satisfies

$$df \cdot V = \phi\, \xi^i f_i.$$

Now the first variation formula implies that the first variation of arc length of h is positive, hence c is not a geodesic. $\square$

Let $c : I \to M$ be a smooth curve. In order to determine the tangential part of $(f \circ c)''$ more explicitely, we let $x : U \to U'$ be a coordinate chart and assume, by restricting to a subinterval of I if necessary, that the image of c is contained in U. We set $c^i := x^i \circ c$, $1 \le i \le m$. Now $f \circ c = (f \circ x^{-1}) \circ (x \circ c)$ and hence

$$(f \circ c)'' = (f_i \cdot (c^i)')' = f_i \cdot (c^i)'' + f_{ij} \cdot (c^i)'(c^j)',$$

where we note that $(c^i)'(t)$, $(c^j)'(t)$, and $(c^i)''(t)$ are scalars and $f_i(c(t)) \in T_{c(t)}f$, for all $t \in I$. Hence the tangential part of $(f \circ c)''$ is

$$[(f \circ c)'']^T = f_i \cdot (c^i)'' + [f_{ij}]^T \cdot (c^i)'(c^j)'.$$

By definition, the tangential part at $f(p)$ is a linear combination of the basis $(f_1(p), \ldots, f_m(p))$ of $T_p f$,

$$[f_{ij}(p)]^T =: \Gamma^k_{ij}(p) f_k(p). \tag{1.9}$$

The coefficients $\Gamma^k_{ij} : U \to \mathbb{R}$ are called *Christoffel symbols.* By Lemma 1.3, the Christoffel symbols are smooth, see also Theorem 1.17 below. Now c is a geodesic if and only if $(f \circ c)''$ does not have a tangential part. Thus c is a geodesic if and only if it satisfies the system of differential equations

$$(c^k)'' + \Gamma^k_{ij}(c^i)'(c^j)' = 0. \tag{1.10}$$

This is a system of ordinary differential equations of second order for the coordinate curves c^k, $1 \le k \le m$.

Corollary 1.15. *1) If $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$ are geodesics with $c_1|I_1 \cap I_2 = c_2|I_1 \cap I_2$, then their union $c : I_1 \cup I_2 \to M$ is also a geodesic.*
2) Given $a, b \in \mathbb{R}$ and a geodesic $c = c(t)$, then $\tilde{c}(s) = c(as+b)$ is also a geodesic.
3) Given $t \in \mathbb{R}$, $p \in M$, and $v \in T_pM$ a tangent vector at p, there is precisely one maximal geodesic $c : I \to M$ such that $c(t) = p$ and $c'(t) = v$. □

Here *maximal* means that the domain of definition of any other geodesic with the same initial conditions is contained in the (open) interval I.

Example 1.16. Let x, y be unit vectors in $\mathbb{R}^{m+1}$. Then

$$c = c(t) := (r \cos t \, x, r \sin t \, y)$$

is a geodesic on S^m_r, and any geodesic on S^m_r is of this form.

Since arc length of curves in M only depends on the first fundamental form of (M, f), it is reasonable to expect that the differential equation 1.10 is determined by the first funfdamental form. Indeed we have the following statement.

Theorem 1.17. *The Christoffel symbols only depend on the coefficients of the first fundamental form and their first derivatives. More precisely,*

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk}(g_{jk,i} + g_{ik,j} - g_{ij,k}).$$

Proof. Let $x : U \to U'$ be a coordinate chart of M and $g_{ij} = \langle f_i, f_j \rangle : U \to \mathbb{R}$ be the coefficients of the matrix of the first fundamental form with respect to x. Then the partial derivative

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} = \langle f_i, f_j \rangle = \langle f_{ik}, f_j \rangle + \langle f_i, f_{jk} \rangle.$$

Corresponding formulas hold for $g_{ik,j}$ und $g_{jk,i}$. Since f_k is tangential,

$$\langle f_{ij}, f_k \rangle = \langle [f_{ij}]^T, f_k \rangle,$$

and hence

$$2\,\Gamma^l_{ij} g_{lk} = 2\,\langle\,\Gamma^l_{ij} f_l, f_k \rangle = 2\,\langle f_{ij}, f_k \rangle = g_{jk,i} + g_{ik,j} - g_{ij,k}.$$

Therefore,

$$\begin{aligned} 2\,\Gamma^l_{ij} &= 2\,\Gamma^\mu_{ij}\,\delta^l_\mu = 2\,\Gamma^\mu_{ij}(g_{\mu k} g^{kl}) \\ &= 2\,\big(\Gamma^\mu_{ij} g_{\mu k}\big) g^{kl} = g^{kl}(g_{jk,i} + g_{ik,j} - g_{ij,k}). \end{aligned}$$

This is the asserted formula. □

1.4. **Parallel vector fields.** Let $c : I \to M$ be a smooth curve. A *vector field along* c is a map $X : I \to TM$ with $X(t) \in T_{c(t)}M$ for all $t \in I$. A first example is the tangent field $X = c'$.

Let X be a smooth vector field along c. Then $df \cdot X$ is a smooth map from I to $\mathbb{R}^n$. If we think of $df(c(t)) \cdot X(t)$ as a vector with footpoint in $f(c(t))$, then this field of vectors along c consists of parallel vectors in $\mathbb{R}^n$ if and only if $df \cdot X : I \to \mathbb{R}^n$ is constant, that is, if and only if $(df \cdot X)' = 0$. It turns out that the condition on the second derivative of $f \circ c$ for geodesics is closely related to the question of how parallel $df \cdot X$ can be. More precisely, let $t_0 \in I$ and suppose $X_0 \in T_{c(t_0)}M$ is given. Let X be a smooth vector field along c with $X(t_0) = X_0$. How small can $\|(df \cdot X)'(t_0)\|$ possibly be? This question is also related to examples in physics like the gyro compass.

Let $x : U \to U'$ be a coordinate chart of M about $p_0 = c(t_0)$. Without loss of generality we assume again that the image of c is contained in U. Let $X : I \to M$ be a vector field along c with $X(t_0) = X_0$. Let $\xi : I \to \mathbb{R}^m$ be the principal part of X with respect to x, that is

$$X(t) = \xi^i(t) \frac{\partial}{\partial x^i}(c(t)).$$

Then

$$(df \cdot X)(t) = df(c(t)) \cdot X(t) = \xi^i(t) f_i(c(t)).$$

With $c^i := x^i \circ c$ we get

$$(df \cdot X)'(t) = (\xi^i)'(t) f_i(c(t)) + (c^i)'(t) \xi^j(t) f_{ij}(c(t)).$$

The first term on the right hand side is tangential. The second term does not depend on derivatives of X. Hence the normal part $[(df \cdot X)'(t_0)]^N$ of $(df \cdot X)'(t_0)$ is determined by $X(t_0) = X_0$ and does not depend on the derivative of X in t.

Hence the norm of $(df \cdot X)''(t)$ is minimal if its tangential part is zero, that is, if the first term and the tangential part of the second term in the equation above cancel each other. Now by the definition of the Christoffel symbols, the tangential part of the second term is $(c^i)'\xi^j\Gamma^k_{ij}f_k$, where we suppress the dependence on the parameters. Comparison of coefficients leads to the system

$$(\xi^k)' + (c^i)'\xi^j(t)\Gamma^k_{ij} = 0 \tag{1.11}$$

of linear ordinary differential equations for the coefficients of the principal part of X. Recall that for any initial data, such equations have a solution on the whole interval of definition of the system of equations.

Definition 1.18. We say that a smooth vector field X along $c : I \to M$ is *parallel* along c, if $[(df \cdot X)'(t)]^T = 0$ for all $t \in I$. In coordinates, this amounts to the linear system of ordinary differential equations for the coefficients of the corresponding principal part ξ of X as in (1.11).

Corollary 1.19. *Let $c : I \to M$ be a curve. Let $t_0 \in I$ and $X_0 \in T_{c(t_0)}M$. Then there is a unique parallel vector field X along c with $X(t_0) = X_0$.* □

Example 1.20. Let S^2 be the round sphere of radius 1 in $\mathbb{R}^3$. Let $\theta \in (-\pi/2, \pi/2)$ and consider the circle $c = c(t) = (\cos\theta\cos(t/\cos\theta), \cos\theta\sin(t/\cos\theta), \sin\theta)$ on S^2. Then $v = v(t) = (-\sin\theta\cos(t/\cos\theta), -\sin\theta\sin(t/\cos\theta), \cos\theta)$ is a vector field along c, and $(c'(t), v(t))$ is an orthonormal basis of $T_{c(t)}S^2$, for each $t \in \mathbb{R}$. The vector fields

$$\sin(\tan\theta \cdot t)\, c' + \cos(\tan\theta \cdot t)\, v \quad \text{and} \quad \cos(\tan\theta \cdot t)\, c' + \sin(\tan\theta \cdot t)\, v$$

are parallel along c, and any parallel field along c is a linear combination of these.

We end this section with some definitions. The importance of these definitions will become clear later.

Definition 1.21. Let $X : I \to TM$ be a vector field along c. Then the *covariant derivative* X' is the unique vector field along c with

$$df \cdot X' = [(df \cdot X)']^T.$$

There is a conflict of notation, which we want to ignore: If we interpret X as a smooth curve in the manifold TM and forget about the curve c of reference, then the derivative of the curve X takes values in TTM, the tangent bundle of TM. Now the covariant derivative X' of X along c has the important technical advantage of still taking values in TM. In terms of the covariant derivative, X is parallel along c if $X' = 0$. This is all I can offer as motivation for the definition of the covariant derivative at this stage.

Our discussion above shows that with respect to a coordinate chart (x, U), the coefficients of the principal part of the covariant derivative X' are given by

$$(\xi^k)' + \Gamma^k_{ij}(c^i)'\xi^j, \tag{1.12}$$

where $c^i := x^i \circ c$. We see that with respect to local coordinates, covariant differentiation differs from ordinary differentiation by a correction term of order zero.

DEFINITION 1.22. Let $X, Y \in \mathcal{V}(M)$. Then the *covariant derivative* of Y in the direction of X is the unique vector field $D_X Y$ on M such that

$$df \cdot D_X Y := [XYf]^T.$$

Let $p \in M$. If c is a curve through p with $c'(0) = X_p$, then

$$XYf(p) = \big((df \circ c) \cdot (Y \circ c)\big)'(0).$$

Hence

$$D_X Y(p) = (Y \circ c)'(0), \tag{1.13}$$

the covariant derivative of the vector field $Y \circ c$ along c.

Let (x, U) be a coordinate chart of M, and let ξ and η be the principal parts of X and Y with respect to x. Then $Yf = \eta^i f_i$ and $XYf = (X\eta^i) f_i + \eta^i \xi^j f_{ij}$, hence

$$X\eta^k + \Gamma^k_{ij} \xi^i \eta^j \tag{1.14}$$

are the coefficients of the principal part of $D_X Y$ with respect to x. From this representation of $D_X Y$ we see immediately that $D_X Y$ is smooth, that is, we have

$$D : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M). \tag{1.15}$$

The covariant derivative is $\mathcal{F}(M)$-linear in X, $\mathbb{R}$-linear in Y, and is a *derivation* with respect to Y:

$$D_X(\varphi \cdot Y) = d_X\varphi \cdot Y + \varphi \cdot D_X Y \quad \text{for all } \varphi \in \mathcal{F}(M). \tag{1.16}$$

Hence D is what is called a *connection* on M, the so-called *Levi-Civita connection* of (M, f). I will not go into the theory of connections here. See [VC] for a short exposition.

2. Extrinsic Geometry

The tangent space $T_p f = \operatorname{im} df(p)$ is a first order approximation of the image of f about p. Since $T_p f$ is a linear space, it does not describe the bending of the image in the ambient space. For that reason, we now consider a second order approximation of f.

Recall that $N_p f$ denotes the orthogonal complement of $T_p f$ in $\mathbb{R}^n$. Let $\pi^T_p : \mathbb{R}^n \to T_p f$ and $\pi^N_p : \mathbb{R}^n \to N_p f$ be the orthogonal projections. Define a smooth map

$$x : M \to T_p f, \quad x(q) = \pi^T_p(f(q) - f(p)). \tag{2.1}$$

Then $x(p) = 0$ and $dx(p) = \pi^T_p \circ df(p) = df(p)$. For the latter, note that π^T_p is linear and that $\operatorname{im} df(p) = T_p f$. Therefore $dx(p) : T_p M \to T_p f$ is an isomorphism.

Applying the Inverse Mapping Theorem we get that there are open neighborhoods U of p in M and U' of 0 in T_pf such that

$$x : U \to U' \text{ is a diffeomorphism with } x(p) = 0 \text{ and } dx(p) = df(p). \tag{2.2}$$

Up to an isomorphism $T_pf \simeq \mathbb{R}^m$, the map x is a coordinate chart for M about p. It is the chart about p which is best adapted to the exterior geometry of the image of f about p. Here *exterior* refers to the shape of the image of f in the ambient space $\mathbb{R}^n$.

THEOREM 2.1 (Local Normal Form). *Let $h = \pi_p^N \circ (f - f(p)) \circ x^{-1} : U' \to N_pf$ be the N_pf-component of $(f - f(p)) \circ x^{-1}$. Then $h(0) = 0$, $dh(0) = 0$ and*

$$(f - f(p))(x^{-1}(u)) = u + h(u) \quad \textit{for all } u \in U'.$$

Proof. The only non-trivial assertion is $dh(0) = 0$. By definition,

$$dh(0) = \pi_p^N \cdot df(p) \cdot dx^{-1}(0).$$

Now $\operatorname{im} df(p) = T_pf$, hence $\pi_p^N \cdot df(p) = 0$. $\square$

By Theorem 2.1, the Taylor expansion of h about 0 is

$$h(u) = \frac{1}{2}H_p(u, u) + \text{terms of order} \geq 3 \text{ in } u$$

with $H_p := d^2h(0)$. Up to the shift by $f(p)$ and terms of third and higher order in u, the quadric

$$Q_p = \{u + v \in \mathbb{R}^n \mid u \in T_pf,\ v = \frac{1}{2}H_p(u, u) \in N_pf\} \tag{2.3}$$

describes (M, f) in a neighborhood of p. We call Q_p the *osculating paraboloid* of (M, f) at p. It is an m-dimensional submanifold of $\mathbb{R}^n$ and approximates $f - f(p)$ at p up to second order.

Let $c : I \to M$ be a curve with $c(t_0) = p$ for some $t_0 \in I$. Then about $t = t_0$, we have

$$(f - f(p)) \circ c = (f - f(p)) \circ x^{-1} \circ (x \circ c) = x \circ c + h \circ (x \circ c).$$

Now $dh(0) = 0$, hence

$$\begin{aligned}
(f \circ c)''(t_0) &= \big((f - f(p)) \circ c\big)''(t_0) \\
&= \big(x \circ c + h \circ (x \circ c)\big)''(t_0) \\
&= (x \circ c)''(t_0) + H_p\big((x \circ c)'(t_0), (x \circ c)'(t_0)\big) \\
&= (x \circ c)''(t_0) + H_p\big(df(p) \cdot c'(t_0), df(p) \cdot c'(t_0)\big).
\end{aligned}$$

The first term on the right hand side is in T_pf, the second in N_pf. Therefore, the right hand side is the orthogonal decomposition of the second derivative of $f \circ c$, at $t = t_0$, with respect to T_pf and N_pf. In particular, the first term is equal to $df(p) \cdot c''(t_0)$, where c'' denotes the covariant derivative of c' as above. Now we concentrate on the second term.

Definition 2.2. The *second fundamental form* of f at p is the symmetric bilinear form

$$S(v,w) = S_p(v,w) = H_p(df(p)\cdot v, df(p)\cdot w)$$

on T_pM with values in N_pf.

By definition and our computation above, the decomposition of the second derivative of $f \circ c$ into tangential and normal part corresponds to the covariant derivative and second fundamental form, that is,

$$(f\circ c)'' = df \cdot c'' + S(c', c'). \tag{2.4}$$

In particular, we do not need to compute the local normal form of f in order to determine the second fundamental form.

Examples 2.3. 1) The simplest case occurs when the dimension of M is 1, that is, $M = I$, an (open) interval, and $f = c : I \to \mathbb{R}^n$, a regular curve. Let $t \in I$. Then T_tc is spanned by the unit vector $c'(t)/\|c'(t)\|$. According to Theorem 2.1, there is a reparameterization x of c in some neighborhood of t such that $x(t) = 0$ and $x'(t) = \|c'(t)\|$ and such that

$$c(s) - c(t) = x(s)\frac{c'(t)}{\|c'(t)\|} + h(x(s)).$$

Here h is a smooth map from a neighborhood of 0 in $\mathbb{R}$ to N_tc with $h(0) = 0$ and $h'(0) = 0$. Since the dimension of T_tI is one, we have

$$S(c'(t), c'(t)) = H(t)\|c'(t)\|^2$$

with $H(t) = h''(0)$. We call $H(t)$ the *curvature vector* and $\kappa(t) = \|H(t)\|$ the *curvature* of c in t. From Formula 2.4 we get

$$H(t) = \frac{1}{\|c'(t)\|^2}\Big[c''(t) - \frac{\langle c''(t), c'(t)\rangle}{\langle c'(t), c'(t)\rangle}\, c'(t)\Big].$$

If $c'(t)$ and $c''(t)$ are linearly dependent, then $H(t) = 0$ and $\kappa(t) = 0$. If $c'(t)$ and $c''(t)$ are linearly independent, then $H(t) \neq 0$ and we call the plane spanned by $c'(t)$ and $c''(t)$ the *osculating plane* of c at t. If c is parametrized by arc length, then c' and c'' are perpendicular and $H = c''$.

A direct computation shows that a circle of radius r has curvature $1/r$ (compare with next example). Usually, this is taken as the definition of the curvature of circles and the curvature of a curve in a given point is defined via the *osculating circle* — the circle which approximates the curve up to second order in the given point.

2) Let $M = S^m_r \subset \mathbb{R}^{m+1}$ be the round sphere of radius r. In this example, f is the inclusion into $\mathbb{R}^{m+1}$. Let p be a point in S^m_r. Let $U = \{q \in S^m_r \mid \langle p, q\rangle > 0\}$. Then the orthogonal projection $\pi : U \to T_pS^m_r$ is a diffeomorphism onto its image

and can be chosen as coordinate chart (corresponding to x in our discussion above). The map h is given by

$$h(u) = \frac{1}{r}\big(\sqrt{r^2 - \|u\|^2} - r\big) \cdot p.$$

Hence the second fundamental form of $M = S_r^m$ in p is

$$S_p(v, w) = -\frac{\langle v, w\rangle}{r^2} \cdot p.$$

Let $p \in M$, $X, Y \in \mathcal{V}(M)$, and consider the coordinate chart $x = \pi_p^T \circ (f - f(p))$ and the map $h = \pi_p^T \circ (f - f(p)) \circ x^{-1}$ as above. Then

$$\begin{aligned}[XYf]^N(p) &= [X(df \cdot Y)]^N(p) = X(d(h \circ x) \cdot Y)(p) \\ &= d^2h(0)\big(df(p) \cdot X(p), df(p) \cdot Y(p)\big) = S_p(X(p), Y(p)),\end{aligned}$$

where we use that $dh(0) = 0$. The right hand side does not involve the choice of the adapted coordinate chart x about p, hence

$$S(X, Y) = [XYf]^N, \tag{2.5}$$

compare with (1.22). Now let $x : U \to U'$ be an arbitrary coordinate chart for M. By the above computation the entries[2] of the matrix of the second fundamental form with respect to x are

$$h_{ij} := S\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big) = \left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right]^N, \tag{2.6}$$

the normal parts of the second partial derivatives of f with respect to x. We defer the discussion of examples to the following subsection.

2.1. **Hypersurfaces.** We now consider the case of *hypersurfaces*, that is, the codimension $n - m = 1$. Let $f : M \to \mathbb{R}^{m+1}$ be an immersion. For any $p \in M$, there are precisely two unit vectors ii N_pf. We fix $p \in M$ and choose one of this vectors, n. Then the second fundamental form S_p can be written as $S_p = S_p^n \cdot n$ with

$$S_p^n(v, w) := \langle S(v, w), n\rangle. \tag{2.7}$$

We refer to S_p^n as the *second fundamental form of* (M, f) *at* p *with respect to* n.

With respect to a coordinate chart x about p, the coefficients of the matrix of S_p^n with respect to x are given by

$$h_{ij}(p) := S_p^n\big(\frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p)\big) = \big\langle \frac{\partial^2 f}{\partial x^i \partial x^j}(p), n\big\rangle,$$

Some of the celebrated classical results follow easily from our modern approach. Let $E \subset \mathbb{R}^n$ be an affine plane through $f(p)$, tangent to n and a unit vector $u \in T_pf$, $u = df(p) \cdot v$. Then in a neighborhood of p, E intersects (M, f) in a

[2] The entries are vectors.

curve c: with x as in Theorem 2.1, $c(t) = x^{-1}(tu)$, $-\varepsilon < t < \varepsilon$. We call c the *normal section* determined by u.

One of the first ideas in the theory of surfaces $f : M \to \mathbb{R}^3$ was to describe their shapes via the curvatures of the normal sections in $f(p)$. A *Theorem of Euler* asserts that, as a function of the unit vector $u \in T_p f$, minimum and maximum of the curvatures of the normal sections in $f(p)$ are achieved in perpendicular directions. Now for c as above, $\langle (f \circ c)'(0), (f \circ c)c''(0) \rangle = 0$ and hence the *oriented curvature* of $f \circ c$ in $t = 0$ satisfies

$$\kappa_o(u) := \langle (f \circ c)''(0), n \rangle = S_p^n(c'(0), c'(0)). \tag{2.8}$$

With our setup at hand, we can prove Euler's theorem as follows (for general m): S_p^n is a real valued symmetric bilinear form on $T_p M$. Consider the function

$$\{v \in T_p M \mid \|v\| = 1\} \to \mathbb{R}, \quad v \mapsto \kappa_o(df(p) \cdot v).$$

The critical values of this function are called *principal curvatures*, the corresponding directions *principal curvature directions* of (M, f) at p. By standard linear algebra, principal curvature directions to different principal curvatures are mutually perpendicular. In particular, there is an orthonormal basis of $T_p M$ with respect to which the matrix of S_p^n is in diagonal form. This proves Euler's theorem.

The *Theorem of Meusnier* from 1776 completes the result of Euler on normal sections by considering general smooth curves in M: Let $p \in M$ and $v \in T_p M$ be a vector of norm 1. Assume that $S_p^n(v, v) \neq 0$. Let S be the m-dimensional sphere of radius $r = 1/|S_p^n(v, v)|$ which is tangent to (M, f) in $f(p)$, that is, $f(p) \in S$ and $T_p f = T_{f(p)} S$. Let $c : I \to M$ be a smooth curve with $c(0) = p$ and $c'(0) = v$. The assumption $S_p^n(v, v) \neq 0$ implies that $(f \circ c)''(0)$ is not tangential. Meusnier's theorem asserts that the curvature of $f \circ c$ in $t = 0$ is equal to the curvature of the circle $E \cap S$, where E is the affine plane through $f(p)$ spanned by $u = (f \circ c)'(0)$ and $(f \circ c)''(0)$. If $\theta \in [0, \pi/2)$ denotes the angle between E and the line $N_p f$, then the curvature of $E \cap S$ is $|S_p^n(v, v)|/\cos\theta$. Since $(f \circ c)'(0)$ has norm one, the curvature of $f \circ c$ in $t = 0$ is $|(f \circ c)''(0)|$. Now

$$|(f \circ c)''(0)| \cos\theta = |\langle (f \circ c)''(0), n \rangle|,$$

hence Meusnier's theorem is immediate from Equation 2.8.

The *Weingarten map* is the endomorphism $L = L_p$ of $T_p M$ which satisfies

$$\langle Lv, w \rangle = -S_p^n(v, w) \quad \text{for all } v, w \in T_p M. \tag{2.9}$$

The reason for the minus sign will become clear in (2.11). A *Theorem of Rodrigues* from 1816 says that the principal curvatures are the eigenvalues of L and the principal directions the corresponding eigenspaces. This is clear from our identification of the curvature of the normal sections in (2.8).

Let $\kappa_1(p), \dots, \kappa_m(p)$ be the principal curvatures of (M, f) at p, where we count principal curvatures according to their multiplicities. The principal curvatures do not come in a prefered order, therefore symmetric functions of them are the main numerical invariants associated to L_p or S_p^n, respectively. The product $K(p) = \kappa_1(p) \cdot \ldots \cdot \kappa_m(p)$ is called *Gauß-Kronecker curvature*, *Gauß curvature* in the case $m = 2$, the arithmetic mean $H(p) = (\kappa_m(p) + \ldots + \kappa_m(p))/m$ the *mean curvature* of (M, f) at p. These are the most important invariants derived from the principal curvatures, other symmetric functions of the principal curvatures do not seem to play a comparable role.

Remark 2.4. The sign of the mean cuvature depends on the choice of normal vector n. If m is even, $K(p)$ is independent of the choice of n.

Proposition 2.5. *Let (x, U) be a coordinate chart of M about p. Then*

$$K(p) = \det(g^{-1}(p) \cdot h(p)) \quad \text{and} \quad H(p) = \frac{1}{m} \operatorname{tr}(g^{-1}(p) \cdot h(p)),$$

where $g = (g_{ij})$ is the matrix of the first fundamental form and $h(p) = (h_{ij}(p))$ is the matrix of the second fundamental form S_p^n with respect to x.

Proof. By definition, $g(p)$ and $h(p)$ are the matrices of the first fundamental form and second fundamental form on T_pM with respect to the basis

$$\Big(\frac{\partial}{\partial x^1}(p), \ldots \frac{\partial}{\partial x^m}(p)\Big).$$

If g_B respectively h_B denote the corresponding matrices for some other basis $B = (b_1, \ldots, b_m)$ of T_pM, then

$$g_B = A^* \cdot g(p) \cdot A \quad \text{and} \quad h_B = A^* \cdot h(p) \cdot A,$$

where A is the matrix of the base change and A^* denotes the transposed of A. It follows that the right hand sides in the asserted equations are equal to $\det(g_B^{-1} \cdot h_B)$ respectively $\operatorname{tr}(g_B^{-1} \cdot h_B)$. That is, these expressions are independent of the base. Now there is an orthonormal basis of T_pM which diagonalizes S_p^n; by definition, the corresponding diagonal entries are the principal curvatures of (M, f) at p. □

Let $W \subset M$ be open. A smooth map $n : W \to S^m \subset \mathbb{R}^{m+1}$ such that $n(p)$ is perpendicular to T_pf for all $p \in W$ is called a *Gauß map* for (M, f) over W. If $m = 2$ and $x : U \to U'$ is a coordinate chart of M, then the normalized vector product

$$n := \frac{f_1 \times f_2}{\|f_1 \times f_2\|} \tag{2.10}$$

is a *Gauß map* for (M, f) over U. In higher dimensions, there is a similar formula.

Remark 2.6. If M is connected, we say that (M, f) is *two-sided* if there is a *Gauß map* $n : M \to S^m$ for (M, f), otherwise we call (M, f) *one-sided.* The Möbius band in $\mathbb{R}^3$ is one-sided. Note that M is two-sided if and only if M is *orientable*: If n is a Gauß map and $p \in M$, say that a basis $B = (b_1, \dots, b_m)$ of T_pM is positively oriented iff

$$C = (df(p) \cdot b_1, \dots, df(p) \cdot b_m, n(p))$$

is a positively oriented basis of $\mathbb{R}^{m+1}$, where $\mathbb{R}^{m+1}$ is equipped with its standard orientation. Vice versa, if M is oriented and B is a positively oriented basis of T_pM, let $n(p)$ be the unique vector of norm 1 in N_pf such that C is a positively oriented basis of $\mathbb{R}^{m+1}$.

Suppose now that n is a *Gauß map* for (M, f) over W. For any vector field Y of M over W we have $\langle Yf, n\rangle \equiv 0$. Differentiating this in the direction of another vector field X of M over W we obtain

$$0 = X\langle Y(f), n\rangle = \langle XY(f), n\rangle + \langle Y(f), X(n)\rangle.$$

Now $\langle Y(f), n\rangle = S^n(X, Y)$ since $S(X, Y)$ is the normal component of $XY(f)$. Therefore

$$S^n(X, Y) = -\langle Yf, Xn\rangle. \tag{2.11}$$

Examples 2.7. 1) Let $c = (x, y) : I \to \mathbb{R}^2$ be a regular smooth curve in the plane. Then $\tilde{n} = (y', -x')$ is non-zero and perpendicular to c', and $n = \tilde{n}/\|\tilde{n}\|$ is a unit vector field normal to c such that (c', n) is a positvely oriented basis of $\mathbb{R}^2$. We call n the *oriented normal* and $\kappa_o := \langle c'', n\rangle/\langle c', c'\rangle$ the *oriented curvature* of c, compare (2.8).

2) Consider a ruled surface $f(s, t) = c(t) + s \cdot v(t)$ as in Example 1.11, where $c'(t)$ and $v(t)$ are linearly independent for all t. Then $f_s = v$, $f_t = c' + sv'$, hence the entries of the first fundamental form are

$$g_{ss} = \|v\|^2, \quad g_{st} = g_{ts} = \langle v, c' + sv'\rangle, \quad g_{tt} = \langle c' + sv', c' + sv'\rangle$$

As Gauß map n we choose $f_s \times f_t$, renormalized to a unit vector as in (2.10). Since the second partial derivative $f_{ss} = 0$, the rulings $t = \text{const}$ are *asymptote lines* of f, that is, $h_{ss} = \langle f_{ss}, n\rangle = 0$. In particular, the second fundamental form is not definite, hence the Gauß curvature is ≤ 0.

Since $h_{ss} = 0$, the Gauß curvature is identically zero iff the normal part of f_{st} vanishes, that is, iff f_{st} is in the tangent plane of f, spanned by f_s and f_t. Differentiating $\langle f_s, n\rangle = \langle f_t, n\rangle = 0$ and $\langle n, n\rangle = 1$ with respect to t, we get

$$\langle f_{st}, n\rangle = -\langle f_s, n_t\rangle, \quad \langle f_{tt}, n\rangle = -\langle f_t, n_t\rangle, \quad \langle n, n_t\rangle = 0.$$

Hence the Gauß curvature is identically zero iff n is constant along the rulings.

3) We continue our discussion of ruled surfaces. Let $(r, h) = (r(t), h(t))$, $t \in I$, be a regular curve in the (x, z)-plane with $r > 0$, and consider the corresponding ruled surface as in Example 1.5. The Gauß map n as in (2.10) is given by

$$n = \frac{f_t \times f_\varphi}{\|f_t \times f_\varphi\|} = \frac{1}{\sqrt{(r')^2 + (h')^2}}(-h' \cos\varphi, -h' \sin\varphi, r').$$

The second partial derivatives of f are

$$f_{tt} = (r'' \cos\varphi, r'' \sin\varphi, h''),$$
$$f_{t\varphi} = f_{\varphi t} = (-r' \sin\varphi, r' \cos\varphi, 0) = \frac{r'}{r} f_\varphi,$$
$$f_{\varphi\varphi} = (-r\cos\varphi, -r\sin\varphi, 0).$$

Therefore the coefficients of the second fundamental form are

$$h_{tt} = \langle f_{tt}, n\rangle = \frac{r'h'' - h'r''}{\sqrt{(r')^2 + (h')^2}},$$
$$h_{t\varphi} = h_{\varphi t} = \langle f_{\varphi t}, n\rangle = 0,$$
$$h_{\varphi\varphi} = \langle f_{\varphi\varphi}, n\rangle = \frac{rh'}{\sqrt{(r')^2 + (h')^2}}.$$

Since first and second fundamental form are in diagonal form, the coordinate lines — meridians and parallels — are lines of curvature, i.e. their tangents are principal curvature directions. The corresponding principal curvatures are

$$\kappa_r = \frac{r'h'' - h'r''}{((r')^2 + (h')^2)^{3/2}}, \qquad \kappa_\varphi = \frac{h'}{r\sqrt{(r')^2 + (h')^2}}.$$

If the profile curve (r, h) has constant speed ν, then the Gauß curvature is given by $K = -r''/r\nu^2$.

Remarks 2.8. 1) Dupin's fundamental theorem of hypersurfaces implies that, up to a motion of $\mathbb{R}^n$, a hypersurface is determined locally by its first and second fundamental form. Compare the discussion of helical surfaces in Section 3, providing an interesting example for the fact that a hypersurface is not determined by its first fundamental form.

2) The celebrated theorema egregium of Gauß asserts that the Gauß curvature K of a surface in $\mathbb{R}^3$ is an invariant of the inner geometry of (M, f). More precisely, Gauß derived a formula for K which involves the entries g_{ij} of the matrix of the first fundamental form and their first and second deivatives. This is one of the deepest insights in the history of differential geometry and marks the beginning of inner differential geometry. I will discuss this story in a later chapter.

3. More examples and some exercises

1) Let $(x(t), y(t))$, $t \in I$, be a regular curve in the (x, y)-plane and $a > 0$ be a constant. The *helical surface* determined by these data is

$$f(t, \varphi) = (x(t)\cos\varphi - y(t)\sin\varphi, x(t)\sin\varphi + y(t)\cos\varphi, a\varphi).$$

The curve (x, y) is called the *profile curve* of this surface. A particular example is the helicoid with $(x(t), y(t)) = (t, 0)$ and $a = 1$.

The partial derivatives of f are

$$f_t = (x' \cos\varphi - y' \sin\varphi, x' \sin\varphi + y' \cos\varphi, 0),$$
$$f_\varphi = (-x \sin\varphi - y\cos\varphi, x\cos\varphi - y\sin\varphi, a).$$

They are linearly independent since $f_t \neq 0$ and $a > 0$. The coefficients of the matrix of the first fundamental form are

$$g_{tt} = (x')^2 + (y')^2 > 0, \quad g_{t\varphi} = g_{\varphi t} = xy' - x'y, \quad g_{\varphi\varphi} = x^2 + y^2 + a^2.$$

Note the independence on the parameter φ.

We show that there is a surface of revolution with the same first fundamental form as the given helical surface f, at least up to a t-dependent shift of the parameter φ (a diffeomorphism of the parameter domain). That is, instead of f we consider

$$\tilde{f}(t,\varphi) = f(t, \varphi + \alpha(t)).$$

The partial derivatives of $\tilde{f}$ are

$$\tilde{f}_t(t,\varphi) = f_t(t,\varphi+\alpha) + \alpha' f_\varphi(t,\varphi+\alpha),$$
$$\tilde{f}_\varphi(t,\varphi) = f_\varphi(t,\varphi+\alpha),$$

and the coefficients of the matrix of the first fundamental form are

$$\begin{aligned}
\tilde{g}_{tt} &= (x')^2 + (y')^2 + 2\alpha'(xy' - x'y) + (\alpha')^2(x^2+y^2+a^2) \\
\tilde{g}_{t\varphi} &= \tilde{g}_{\varphi t} = xy' - x'y + \alpha'(x^2+y^2+a^2) \\
\tilde{g}_{\varphi\varphi} &= x^2 + y^2 + a^2.
\end{aligned}$$

Now $\tilde{f}$ is supposed to have the first fundamental form of a surface of revolution, therefore we must have $\tilde{g}_{t\varphi} = \tilde{g}_{\varphi t} = 0$ and $\tilde{g}_{\varphi\varphi} = r^2$, where the profile curve of the surface of revolution is denoted (r,h). Hence

$$\alpha' = \frac{x'y - xy'}{r^2} \quad \text{with} \quad r = \sqrt{x^2+y^2+a^2}.$$

We conclude that

$$\frac{(xx'+yy')^2}{r^2} + (h')^2 = (r')^2 + (h')^2 = \tilde{g}_{tt} = (x')^2 + (y')^2 - \frac{(xy'-x'y)^2}{r^2}$$

and obtain

$$h' = \pm\frac{a}{r}\sqrt{(x')^2+(y')^2}, \quad \text{hence} \quad h = \pm\int \frac{a}{r}\sqrt{(x')^2+(y')^2}.$$

We can choose the sign and the constant of integration for h, geometrically this corresponds to a flip of, respectively a shift along the z-axis. Note that the constant of integration for α can also be chosen. If the profile curve of f is parametrized by arc length — which can be achieved by reparametrization — then

$$h = \pm\int \frac{a}{r}\,dt.$$

Coming back to the particular example of the helicoid, we can choose $\alpha = 0$ (no shift!) and get (respectively choose)

$$r(t) = \sqrt{t^2+1} \quad \text{and} \quad h(t) = \operatorname{arsinh} t.$$

Then $t = \sinh h$ and $r = \cosh h$, so that the corresponding surface of revolution is a catenoid.

2) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and K be the group of all $\mathbb{K}$-linear maps of $\mathbb{K}^n$ preserving the standard bilinear form $(x, y) = \sum \bar{x}^i y^i$ of $\mathbb{K}^n$. Then the real part $\mathrm{Re}(\cdot,\cdot)$ is the standard inner product on $\mathbb{K}^n \cong \mathbb{R}^{kn}$, where $k = \dim_{\mathbb{R}} \mathbb{K} = 1, 2, 4$. By the usual representation of $\mathbb{K}$-linear maps of $\mathbb{K}^n$ by matrices,

$$K = \{A \in \mathbb{K}^{n\times n} \mid A^* = A^{-1}\}.$$

Hence $A \in K$ if and only if the tuple of columns of A is an orthonormal basis of $\mathbb{K}^n$ with respect to $(\cdot,\cdot)$. It follows easily that K is a compact subgroup of $Gl(n, \mathbb{K})$, the *orthogonal group* $O(n)$ for $\mathbb{K} = \mathbb{R}$, the *unitary group* $U(n)$ for $\mathbb{K} = \mathbb{C}$, and the *symplectic group* $Sp(n)$ for $\mathbb{K} = \mathbb{H}$.

We show now that K is a smooth submanifold of $\mathbb{K}^{n\times n}$. To that end, we introduce the following two $\mathbb{R}$-linear subspaces of $\mathbb{K}^{n\times n}$,

$$\begin{aligned} \mathfrak{k} = \mathfrak{k}(n, \mathbb{K}) &= \{C \in \mathbb{K}^{n\times n} \mid C^* = -C\}, \\ \mathfrak{p} = \mathfrak{p}(n, \mathbb{K}) &= \{C \in \mathbb{K}^{n\times n} \mid C^* = C\}. \end{aligned}$$

The dimensions of $\mathfrak{k}$ and $\mathfrak{p}$ are $(k-1)n + kn(n-1)/2$ and $n + kn(n+1)/2$,respectively, and $\mathfrak{p}$ is the $(\cdot,\cdot)$-orthogonal complement of $\mathfrak{k}$ in $\mathbb{K}^{n\times n}$. Conjugation by $A \in K$ leaves $\mathfrak{k}$ and $\mathfrak{p}$ invariant,

$$A \cdot \mathfrak{k} \cdot A^{-1} = \mathfrak{k}, \quad A \cdot \mathfrak{p} \cdot A^{-1} = \mathfrak{p}.$$

For $\mathbb{K} = \mathbb{R}$, $\mathfrak{k}$ is the space of skew symmetric and $\mathfrak{p}$ the space of symmetric matrices (of size $(n \times n)$ and with real entries); for $\mathbb{K} = \mathbb{C}$, $\mathfrak{k}$ is the space of skew Hermitian and $\mathfrak{p}$ the space of Hermitian matrices.

Consider the smooth map $F : \mathbb{K}^{n\times n} \to \mathfrak{p}$, $F(A) = A^*A$. By definition, K is the F-preimage of the identity matrix. For $A, C \in \mathbb{K}^{n\times n}$ we have

$$F(A + t \cdot AC) = A^*A + t \cdot A^*AC + t \cdot C^*A^*A + o(t)$$

and therefore, for $A \in K$,

$$dF_A(AC) = C + C^*.$$

It follows that $dF_A(AC) = 2C$ for $A \in K$ and $C \in \mathfrak{p}$. Hence F has maximal rank $\dim_{\mathbb{R}} \mathfrak{p}$ along K. Now by the Implicit Function Theorem, K is a smooth submanifold of $\mathbb{K}^{n\times n}$, and the dimension of K is $(k-1)n + kn(n-1)/2$. We can also read off the tangent to K: for $A \in K$

$$T_AK = \{AC \mid C \in \mathfrak{k}\} = A \cdot \mathfrak{k}.$$

The standard bilinear form on $K^{n\times n}$ is given by $(A,B) = \mathrm{tr}(A^*B)$. It follows that the normal space to K in $A \in K$ is

$$N_AK = \{AC \mid C \in \mathfrak{p}\} = A \cdot \mathfrak{p}.$$

An important observation related to K is that for any $A \in K$, left and right multiplication by A preserves $(\cdot,\cdot)$, that is,

$$(AC, AD) = (CA, DA) = (C, D) = \mathrm{tr}(C^*D)$$

for all $C, D \in \mathbb{K}^{n\times n}$.

It also follows from that 1-parameter subgroups are geodesics. More generally, for $A \in K$ and $C \in K$, consider the curve $B(t) = A \cdot e^{tC}$. Then $B'(t) = B(t) \cdot C$ and $B''(t) = B(t) \cdot C^2$. Now $C^2 \in \mathfrak{p}$ since C is in $\mathfrak{k}$; hence B'' is normal and therefore B is a geodesic. Similarly, $B(t) = e^{tC} \cdot A$ is a geodesic.

Since the curve B is a geodesic, $S_A(B'(0), B'(0)) = B''(0) = A \cdot C^2$. Polarizing, we get the second fundamental form,

$$S_A(AC, AD) = \frac{1}{2} A \cdot (CD + DC).$$

3)Let $f : M \to \mathbb{R}^3$ be an immersed surface, $p \in M$ and n a unit normal to (M, f) at p. We say that p is an *elliptic point* if $K(p) > 0$, a *hyperbolic point* if $K(p) < 0$ and a *parabolic point* otherwise. Draw figures and explain the names.

3) Let $f : M \to \mathbb{R}^{m+1}$ be an immersed hypersurface. Suppose that the function $\|f\| : p \mapsto \|f(p)\|$ has a local maximum in p_0. Show that $n = f(p_0)/\|f(p_0)\|$ is a normal vector for f in p_0. Show that all the principal curvatures of f in p_0 with respect to n are negative, more precisely, $\le -1/\|f(p_0)\|$.

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