VECTOR BUNDLES AND CONNECTIONS

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The exposition of vector bundles and connections below is taken from my lecture notes on differential geometry at the University of Bonn. I included more material than I usually cover in my lectures. On the other hand, I completely deleted the discussion of "concrete examples", so that a pinch of salt has to be added by the customer.

Standard references for vector bundles and connections are [GHV] and [KN], where the interested reader finds a rather comprehensive discussion of the subject.

I would like to thank Andreas Balser for pointing out some misprints. The exposition is still in a preliminary state. Suggestions are very welcome.

Contents

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1. **Vector Bundles**

A *bundle* is a triple (π, E, M), where $\pi : E \to M$ is a map. In other words, a bundle is nothing else but a map. The term bundle is used when the emphasis is on the preimages $E_p := \pi^{-1}(p)$ of the points $p \in M$; we call E_p the *fiber of* π *over* p and p the *base point of* E_p. We view the fiber E_p as sitting over p and M as organizing the different fibers into a family. We call M the *base space*, E the *total space* and π the *projection* of the bundle.

Frequently we pretend that the projection π is known and speak of the bundle E over M or, even more simply, of the bundle E.

For a bundle $\pi : E \to M$ and $N \subset M$, we set $E|N := \pi^{-1}(N)$. The restriction $\pi : E|N \to N$ is called the *restricted bundle*.

We say that a bundle is *smooth* if E and M are smooth manifolds and π is a smooth map. We assume throughout that all our bundles are smooth. If $\pi : E \to M$ is smooth and $U \subset M$ is open, then the restricted bundle $\pi : E|U \to U$ is also a smooth bundle.

We are interested in the case when the fibres are equipped with vector space structures over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. This leads to the notion of real and complex vector bundles. The case of quaternionic vector bundles can be handled in the same way. However, since $\mathbb{H}$ is not commutative, one has to be careful when writing down formulas.

The most important example is the tangent bundle, a real vector bundle. But there are many other interesting vector bundles.

1.0.1. Definition. A $\mathbb{K}$*–vector bundle* over M of *rank* k consists of a bundle $\pi : E \to M$ whose fibres are $\mathbb{K}$–vector spaces and such that about each point $p \in M$, there is an open neighborhood U in M and a diffeomorphism $\Phi : U \times \mathbb{K}^k \to E|U$ such that

(1) $\pi \circ \Phi = \pi_1$, where $\pi_1 : U \times \mathbb{R}^k \to U$ is the projection to the first factor and
(2) for each $q \in U$, the map $\Phi_q : \mathbb{K}^k \to E_q$, $\Phi_q(\xi) := \Phi(q, \xi)$, is a $\mathbb{K}$–linear isomorphism.

We also speak of *real vector bundles* and *complex vector bundles* depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Usually we suppress the dependence on the field $\mathbb{K}$ and simply speak of vector bundles. Vector bundles of rank one are also called *line bundles*.

A map Φ as in Definition 1.0.1 is called a *trivialization* of $E|U$ or also a *local trivialization* of E.

1.0.2. Examples. 1) The *trivial bundle* $\pi : M \times \mathbb{K}^k \to M$, $\pi(p,x) = p$. The vector space structure on the fibers $\{p\} \times \mathbb{K}^k$ is the natural one,

$$\begin{aligned} (p,x) + (p,y) &:= (p, x+y) && \text{for } x,y \in \mathbb{K}^k\,, \\ \kappa \cdot (p,x) &:= (p, \kappa \cdot x) && \text{for } \kappa \in \mathbb{K} \text{ and } x \in \mathbb{K}^k\,. \end{aligned}$$

In other words, we consider p as an index. Similar rules lead to natural vector space structures in many other cases.

2) The *tangent bundle* $\pi : TM \to M$, a real vector bundle of rank $\dim M$. We assume that the reader is familiar with the construction of the tangent bundle of a smooth manifold. In Lemma 1.3.4 below we discuss a general construction of which the usual construction of the tangent bundle is a special case.

3) Let $\mathbb{K}P^n$ be the projective space of $\mathbb{K}$–lines in $\mathbb{K}^{n+1}$ and let

$$\hat{\mathbb{K}}^{n+1} = \{(l,x) \mid l \in \mathbb{K}P^n, x \in l\}\,.$$

The natural map

$$\pi : \hat{\mathbb{K}}^{n+1} \to \mathbb{K}P^n\,, \quad \pi(l,x) = l\,,$$

is called the *canonical line bundle* over $\mathbb{K}P^n$. The fiber over the line $l \in \mathbb{K}P^n$ consists of all pairs (l,x) with $x \in l$. Since the fibers are $\mathbb{K}$–lines, we can introduce a natural $\mathbb{K}$–vector space structure on them,

$$\begin{aligned} (l,x) + (l,y) &:= (l, x+y) && \text{for } x,y \in l\,, \\ \kappa \cdot (l,x) &:= (l, \kappa \cdot x) && \text{for } x \in l \text{ and } \kappa \in \mathbb{K}\,. \end{aligned}$$

This turns the fibers of π into $\mathbb{K}$–vector spaces of dimension one. Note that the different lines $l \in \mathbb{K}P^n$ are not disjoint, they all intersect in $0 \in \mathbb{K}^{n+1}$. By considering tuples (l,x), we force them to be disjoint.

For a non–zero vector $x \in \mathbb{K}^{n+1}$, we denote by $[x]$ the $\mathbb{K}$–line spanned by x. For $1 \le i \le n+1$, let $U_i = \{[x] \in \mathbb{K}P^n \mid x_i \neq 0\}$ and consider the bijections

$$\Phi_i : U_i \times \mathbb{K} \to \hat{\mathbb{K}}^{n+1}|U_i\,, \quad \Phi_i([x],\xi) = \big([x], \frac{\xi}{x_i}x\big)\,.$$

It is easy to see that $\hat{\mathbb{K}}^{n+1}$ has a unique structure as a smooth manifold such that the canonical bundle $\pi : \hat{\mathbb{K}}^{n+1} \to \mathbb{K}P^n$ is a $\mathbb{K}$–line bundle for which the bijections Φ_i are local trivializations, see Lemma 1.3.4.

1.0.3. Definitions. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be $\mathbb{K}$–vector bundles. A pair (f, L) of smooth maps $f : M \to M'$ and $L : E \to E'$ is called a *bundle map* if $\pi' \circ L = f \circ \pi$ and if $L_p = L|E_p : E_p \to E'_{f(p)}$ is $\mathbb{K}$–linear for all $p \in M$.

If $M = M'$, then a map $L : E \to E'$ is called a *morphism* if (id_M, L) is a bundle map. A morphism L is called an *isomorphism* if it is invertible.

We may think of a bundle map as a smooth family of $\mathbb{K}$–linear maps L_p between the corresponding fibers E_p of E and $E'_{f(p)}$ of E'.

1.0.4. Example. Let $f : M \to M'$ be a smooth map. Then the pair (f, f_*), where $f_* : TM \to TM'$ is the differential of f, is a bundle map.

1.0.5. Definition. Let $\pi : E \to M$ be a $\mathbb{K}$–vector bundle of rank k. Then a submanifold $E' \subset E$ is called a *subbundle of rank k' of E* if the restriction $\pi : E' \to M$ is a $\mathbb{K}$–vector bundle of rank k'.

1.0.6. Examples. 1) Consider $S^n = \{p \in \mathbb{R}^{n+1} \mid |p| = 1\}$. Then the submanifold

$$\{(p, x) \in S^n \times \mathbb{R}^{n+1} \mid x \perp p\}$$

is a subbundle of the trivial bundle $S^n \times \mathbb{R}^{n+1}$. This subbundle is isomorphic to the tangent bundle of S^n.

2) The canonical bundle $\hat{\mathbb{K}}^{n+1} \to \mathbb{K}P^n$ is a subbundle of the trivial bundle $\mathbb{K}P^n \times \mathbb{K}^{n+1}$. Furthermore, the submanifold

$$\{(l, x) \in \mathbb{K}P^n \times \mathbb{K}^{n+1} \mid x \perp l\} \to \mathbb{K}P^n \,,$$

is also a subbundle of the trivial bundle $\mathbb{K}P^n \times \mathbb{K}^{n+1}$. The latter subbundle is isomorphic to the tangent bundle of $\mathbb{K}P^n$.

1.1. **Sections.** A vector field on a smooth manifold M is a map which associates to each point $p \in M$ a tangent vector at p. Similar maps are also important in the case of a general vector bundle $\pi : E \to M$.

1.1.1. Definition. A *section of E* is a map $S : M \to E$ with $\pi \circ S = \mathrm{id}_M$, i.e., we have $S(p) \in E_p$ for all $p \in M$. Similarly, a *section of E over $U \subset M$* is a map $S : U \to E$ with $\pi \circ S = \mathrm{id}_U$.

1.1.2. Examples. 1) The *zero section* is defined by $S(p) = 0_p \in E_p$, $p \in M$. The zero section is smooth.

2) Vector fields on manifolds are sections of the tangent bundle.

If S, S_1 and S_2 are sections of E and $\varphi : M \to \mathbb{K}$ is a function, then there are new sections $S_1 + S_2$ and $\varphi \cdot S$,

$$(S_1 + S_2)(p) := S_1(p) + S_2(p) \,,$$
$$(\varphi \cdot S)(p) := \varphi(p) \cdot S(p) \,,$$

obtained by pointwise addition and multiplication respectively.

By $\mathcal{V}(M)$ we denote the space of smooth vector fields on M. For a general vector bundle E over M, the space of smooth sections of E is denoted $\mathcal{S}(E)$.

We recall that $\mathcal{F}_{\mathbb{K}}(M)$ denotes the space of smooth functions $\varphi : M \to \mathbb{K}$. By using local trivializations it is easy to see that the above operations of addition and multiplication preserve smoothness. This we can express in the following more sophisticated way.

1.1.3. Proposition. *Pointwise addition of sections and multiplication of a function with a section turn $\mathcal{S}(E)$ into a module over $\mathcal{F}_{\mathbb{K}}(M)$.* □

We show next that vector bundles have many smooth sections.

1.1.4. Lemma. *Let $\pi : E \to M$ be a vector bundle over M and $p \in M$ be a point. Then*

(1) *for each $x \in E_p$ there is $S \in \mathcal{S}(E)$ with $S(p) = x$.*
(2) *for $U \subset M$ open with $p \in U$ and $S : U \to E$ a smooth section of E over U, there is a smooth section $S' \in \mathcal{S}(E)$ such that $S = S'$ in a neighborhood of p.*

Proof. Choose a trivialization $\Phi : W \times \mathbb{K}^k \to \pi^{-1}(W)$ of E with $p \in W$ and a smooth function $\varphi : M \to \mathbb{R}$ with $\operatorname{supp}(\varphi) \subset W$ and $\varphi(p) = 1$. Now $\Phi(p, \xi) = x$ for some (unique) $\xi \in \mathbb{K}^k$. Define a section S of E by

$$S(q) = \begin{cases} \Phi(q, \varphi(q) \cdot \xi) & \text{if } q \in W\,, \\ 0 & \text{if } q \notin W\,. \end{cases}$$

Then $S(p) = x$. Now Φ is a diffeomorphism, hence S is smooth on W. Since $\operatorname{supp}(\varphi) \subset W$, S is zero on a neighborhood of $M \setminus W$. Hence S is smooth on M. This proves the first assertion. The proof of the second assertion is similar and is left to the reader. □

1.2. **Frames.**

Let $\pi : E \to M$ be a vector bundle of rank k and $U \subset M$ be open.

1.2.1. Definition. A *frame of E over U* is a k–tuple $(S_1, \ldots, S_k)$ of smooth sections of E over U such that $S_1(p), \ldots, S_k(p)$ is a basis of E_p for all $p \in U$.

We also speak of *local frames* when the set U is not specified. The existence of local frames is guaranteed by the following proposition.

1.2.2. Proposition. *Let $(S_1, \ldots, S_k)$ be a frame of E over U. Then the map*

$$\Phi : U \times \mathbb{K}^k \to E|U\,, \quad \Phi(p, \xi) = \xi^i S_i(p)\,,$$

is a trivialization of $E|U$. *Vice versa, if* $\Phi : U \times \mathbb{K}^k \to E$ *is a trivialization of* $E|U$ *and* $e_1, \dots, e_k$ *is the standard basis of* $\mathbb{K}^k$*, then the* k*–tuple* $S_i = \Phi(\,.\,, e_i)$*,* $1 \le i \le k$*, is a frame of* E *over* U*.*

Proof. For each $p \in U$, Φ_p is an isomorphism, hence Φ is bijective. It remains to show that Φ has maximal rank. To that end, let Ψ be a local trivialization of E in a neighborhood $V \subset U$ of a point $p \in U$. Then we have

$$(\Psi^{-1}\Phi)(q, \xi) = (q, (\Psi_q^{-1}\Phi_q)(\xi))$$

for all $q \in V$ and $\xi \in \mathbb{K}^k$. Now for each $q \in V$, $\Phi_q, \Psi_q : \mathbb{K}^k \to E_q$ are isomorphisms. It follows easily that

$$\Psi^{-1}\Phi : V \times \mathbb{K}^k \to V \times \mathbb{K}^k$$

is a diffeomorphism. Hence Φ has maximal rank on $V \times \mathbb{K}^k$. Now $p \in U$ was arbitrary, hence Φ is a diffeomorphism. The other direction is trivial. □

The above proposition tells us that local trivializations and frames are two sides of the same coin. For that reason we often will not distinguish between the two.

Let $\Phi = (S_1, \dots, S_k)$ be a local trivialization/frame and S be a section of E over U. Then there is a map $\sigma = \sigma_\Phi : U \to \mathbb{K}^k$ such that $S(p) = \Phi(p, \sigma(p))$ for all $p \in U$, that is,

$$S = \sigma^i S_i\,, \quad \sigma = (\sigma^1, \dots, \sigma^k)\,. \tag{1.2.3}$$

We call σ the *principal part of* S *with respect to* Φ. Since Φ is a diffeomorphism, σ is smooth iff S is smooth.

Let $\Psi = (T_1, \dots, T_k)$ be another local trivialization of E, say over an open subset V of M. Then over $U \cap V$ we have

$$S_i = g_i^j T_j \tag{1.2.4}$$

with smooth functions $g_i^j : U \cap V \to \mathbb{K}$. For each $p \in U \cap V$, the $(k \times k)$–matrix $g(p) = (a_i^j(p))$ describes a change of basis, hence it is invertible. We obtain a smooth map $g : U \cap V \to Gl(k, \mathbb{K})$.

Let S be a section of E over $U \cap V$ and σ_Φ and σ_Ψ be the principal part of S with respect to Φ and Ψ. Then

$$\sigma_\Psi^j = g_i^j \sigma_\Phi^i \quad \text{respectively} \quad \sigma_\Psi = g \cdot \sigma_\Phi\,. \tag{1.2.5}$$

In this sense we may interpret sections of vector bundles as families of maps $\sigma_\Phi : U \to \mathbb{K}^k$ *which transform in the right way*, that is, according to (1.2.5). This point of view is important in computations.

1.3. **Constructions.** The following situation arises frequently: We are given a family E_p, $p \in M$, of *pairwise disjoint* $\mathbb{K}$–vector spaces, an open cover (U_α) of M and, for each index α and point $p \in U_\alpha$, a $\mathbb{K}$–linear isomorphism

$$\Phi_{\alpha,p} : \mathbb{K}^k \to E_p \,. \tag{1.3.1}$$

We face the problem to turn the (disjoint) union $E = \cup E_p$ together with the canonical map $\pi : E \to M$ with fibers E_p into a (smooth) $\mathbb{K}$–vector bundle such that the given bijections

$$\Phi_\alpha : U_\alpha \times \mathbb{K}^k \to E|U_\alpha \,, \quad \Phi_\alpha(p,\xi) = \Phi_{\alpha,p}(\xi) \,, \tag{1.3.2}$$

are local trivializations. A necessary condition is that all the maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(k,\mathbb{K}) \,, \quad g_{\alpha\beta}(p) := \Phi_{\alpha,p}^{-1}\Phi_{\beta,p} \,, \tag{1.3.3}$$

are smooth. We prove now that this condition is also sufficient. The arguments in the proof are the same as those used in the construction of the tangent bundle.

1.3.4. Lemma. *Suppose that all the maps $g_{\alpha\beta}$ are smooth. Then E carries a unique structure as a smooth manifold such that $\pi : E \to M$ is a $\mathbb{K}$–vector bundle for which the bijections Φ_α are local trivializations.*

Proof. We obtain a topology on E by declaring $O \subset E$ to be open iff, for all α, $\Phi_\alpha^{-1}(O)$ is open in $U_\alpha \times \mathbb{K}^k$. Now the maps $g_{\alpha\beta}$ are smooth, hence the maps

$$\Phi_\alpha^{-1}\Phi_\beta : (U_\alpha \cap U_\beta) \times \mathbb{K}^k \to (U_\alpha \cap U_\beta) \times \mathbb{K}^k$$

are diffeomorphisms. Therefore $O \subset E$ is open iff, for any $p \in M$, there is an index α with $p \in U_\alpha$ such that $\Phi_\alpha^{-1}(O)$ is open in $U_\alpha \times \mathbb{K}^k$. It follows easily that the topology is Hausdorff with a countable base.

If $\varphi : U \to U'$ is a local parameterization of M with $U' \subset U_\alpha$ for some α, then we obtain a local parameterization $\hat\varphi : U \times \mathbb{K}^k \to E|U'$ by setting

$$\hat\varphi(u,\xi) = \Phi_{\alpha,\varphi(u)}(\xi) \,.$$

With respect to the above topology, $\hat\varphi$ is a homeomorphism and the set of such local parameterizations — or rather the set of their inverse maps — is a C^∞–atlas of E. With respect to the corresponding smooth structure of E, $\pi : E \to M$ is a $\mathbb{K}$–vector bundle for which the bijections Φ_α are local trivializations. The uniqueness of the smooth structure is clear since the maps Φ_α are required to be diffeomorphisms onto their image. □

1.3.5. Examples. We discuss several constructions by which we obtain new bundles out of old ones.

1) (Dual bundle) The dual bundle of the tangent bundle TM is the cotangent bundle T^*M. There is a similar construction for a $\mathbb{K}$–vector bundle $\pi : E \to M$:

For $p \in M$, let E_p^* be the dual space of E_p, that is, $E_p^* = \mathrm{Hom}_{\mathbb{K}}(E_p, \mathbb{K})$. For any local trivialization $\Phi : U \times \mathbb{K}^k \to E|U$ and point $p \in U$ define an isomorphism

$$\hat{\Phi}_p : \mathbb{K}^k \cong (\mathbb{K}^k)^* \to E_p^*$$

by $\hat{\Phi}_p(\varphi) := \varphi \circ \Phi_p^{-1}$. By a straightforward computation it follows that the maps $\hat{g}_{\Phi,\Psi}$ given by $\hat{g}_{\Phi,\Psi}(p) = (\hat{\Phi}_p)^{-1}\hat{\Psi}_p$ are smooth in p. Hence the condition of Lemma 1.3.4 is satisfied. Hence the disjoint union $E^* = \cup E_p^*$ carries a unique smooth structure such that the canonival projection $\pi^* : E^* \to M$ with fibers E_p^* is a $\mathbb{K}$–vector bundle for which the bijections $\hat{\Phi}$ are local trivializations. We call $\pi^* : E^* \to M$ the *dual bundle* of $\pi : E \to M$.

2) (Whitney sum) Let $\pi : E' \to M$ and $\pi'' : E'' \to M$ be $\mathbb{K}$–vector bundles over M of rank k' and k'' respectively. For any pair of local trivializations

$$\Phi : U \times \mathbb{K}^{k'} \to E'|U\,, \quad \Phi'' : U \times \mathbb{K}^{k''} \to E''|U$$

and point $p \in U$ define an isomorphism

$$\Phi_p : \mathbb{K}^{k'+k''} \cong \mathbb{K}^{k'} \oplus \mathbb{K}^{k''} \to E'_p \oplus E''_p\,,$$

by

$$\Phi_p(\xi', \xi'') = (\Phi'_p(\xi'), \Phi''_p(\xi''))\,.$$

It is easy to see that the maps $g_{\Phi,\Psi}$ given by $g_{\Phi,\Psi}(p) = \Phi_p^{-1}\Psi_p$ are smooth in p. Hence the condition of Lemma 1.3.4 is satisfied. Hence the disjoint union $E' \oplus E'' = \cup E'_p \oplus E''_p$ carries a unique smooth structure such that the canonical projection $E' \oplus E'' \to M$ with fibers $E'_p \oplus E''_p$ is a $\mathbb{K}$–vector bundle for which the bijections Φ are local trivializations. We call $E' \oplus E'' \to M$ the *direct sum* or *Whitney sum* of E' and E''.

3) Let $\pi' : E' \to M$ and $\pi'' : E'' \to M$ be $\mathbb{K}$–vector bundles over M of rank k' and k'' respectively and let $m \geq 0$. For each $p \in M$ let $\Lambda^m(E'_p, E''_p)$ be the $\mathbb{K}$–vector space of alternating m–linear maps

$$E'_p \times \cdots \times E'_p \to E''_p\,.$$

This space is isomorphic to the space $\Lambda^m(k', k'')$ of alternating m–linear maps

$$\mathbb{K}^{k'} \times \cdots \times \mathbb{K}^{k'} \to \mathbb{K}^{k''}\,.$$

For any pair of local trivializations

$$\Phi' : U \times \mathbb{K}^{k'} \to E'|U\,, \quad \Phi'' : U \times \mathbb{K}^{k''} \to E''|U$$

and point $p \in U$ define an isomorphism

$$\Phi_p : \mathbb{K}^{\binom{k'}{m}\cdot k''} \cong \Lambda^m(k', k'') \to \Lambda^m(E'_p, E''_p)$$

by

$$\Phi_p(\phi)(x_1, \ldots, x_m) = \Phi''_p(\phi((\Phi'_p)^{-1}x_1, \ldots, (\Phi'_p)^{-1}x_m))\,.$$

It is easy to see that the maps $g_{\Phi,\Psi}$ given by $g_{\Phi,\Psi}(p) = \Phi_p^{-1}\Psi_p$ are smooth in p. Hence the condition of Lemma 1.3.4 is satisfied. Hence the disjoint union $\Lambda^m(E', E'') = \cup\Lambda^m(E'_p, E''_p)$ carries a unique smooth structure such that the natural projection $\Lambda^m(E', E'') \to M$ is a $\mathbb{K}$–vector bundle with fibers $\Lambda^m(E'_p, E''_p)$.

The case $E' = TM$ is particularly important (here $\mathbb{K} = \mathbb{R}$). Sections of $\Lambda^m(TM, E'') \to M$ are called *m–forms with values in E''*.

Another special case which deserves attention is the case $m = 1$. We denote $\Lambda^1(E', E'') = \mathrm{Hom}_{\mathbb{K}}(E', E'')$, a vector bundle of rank $k'k''$ with fibers the spaces of homomorphisms $\mathrm{Hom}_{\mathbb{K}}(E'_p, E''_p) = \Lambda^1(E'_p, E''_p)$. Sections of this bundle are morphisms $E' \to E''$.

1.3.6. Exercises. One essential feature in the above examples is that the corresponding construction from linear algebra is canonical, that is, it does not depend on the choice of a basis.

1) Again let $\pi' : E' \to M$ and $\pi'' : E'' \to M$ be $\mathbb{K}$–vector bundles over M of rank k' and k'' respectively. Follow the construction of the bundle $E' \oplus E''$ and define the *tensor bundle* $E' \otimes E'' \to M$, a $\mathbb{K}$–vector bundle of rank $k'k''$ with fibers $E'_p \otimes E''_p$.

2) Generalize other standard constructions from linear algebra and show that the natural isomorphisms from linear algebra hold. For example, show that $(E' \oplus E'') \oplus E''' \cong E' \oplus (E'' \oplus E''')$.

1.4. **Pull Back.** Let $\pi : E \to M$ be a vector bundle over M and $f : N \to M$ be a smooth map. For $p \in N$ define a vector space

$$(f^*E)_p = \{(p, x) \mid x \in E_{f(p)}\}\,,$$

where the vector space structure is inherited from $E_{f(p)}$,

$$(p, x) + (p, y) := (p, x + y)\,, \quad \kappa \cdot (p, x) := (p, \kappa \cdot x)\,.$$

Furthermore, for each local trivialization $\Phi : U \times \mathbb{K}^k \to E|U$ and point $p \in V = f^{-1}(U)$, define a map $(f^*\Phi)_p : V \times \mathbb{K}^k \to (f^*E)_p$ by

$$(f^*\Phi)_p(\xi) := (p, \Phi(f(p), \xi))\,.$$

It is easy to see that the family of pairwise disjoint vector spaces $(f^*E)_p$ and maps $(f^*\Phi)_p$ satisfies the assumption of Lemma 1.3.4. We obtain a new vector bundle $f^*E = \cup(f^*E)_p$, the *pull back* of E via f. Note that $f^*E \subset N \times E$ and that the projection is the restriction of π_1, the projection to the first factor in $f^*E \subset N \times E$.

A map $S : N \to f^*E$ is a section of f^*E iff there is a map $s : N \to E$, the so–called *principal part of* S, with $\pi \circ s = f$ and such that $S(p) = (p, s(p))$ for all $p \in N$. Note that a section S is smooth iff the corresponding principal part s is smooth.

A map $s : N \to E$ satisfying $\pi \circ s = f$ is called a *section of E along* f. Often it is more convenient to deal with sections along f instead of dealing with sections of f^*E, the simple reason being that the first component of a section of f^*E contains no information.

We denote the space of smooth sections of E along f by $\mathcal{S}_f(E)$.

1.4.1. Example. Let I be an interval and $c : I \to M$ be a smooth curve. Then the differential $\dot{c} : I \to TM$ is a smooth section of TM along c. More generally, if $X(t)$ is a tangent vector of M with foot point $c(t)$, $t \in I$, then $X : I \to TM$ is a section of TM along c.

1.5. **The Fundamental Lemma on Morphisms.** Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over M. and $L : E \to E'$ be a morphism. Then the map $\mathcal{L} : \mathcal{S}(E) \to \mathcal{S}(E')$ defined by $\mathcal{L}(S) = L \circ S$ is *tensorial*, i.e., linear over $\mathcal{F}_{\mathbb{K}}(M)$,

$$\mathcal{L}(S_1 + S_2) = \mathcal{L}(S_1) + \mathcal{L}(S_2) \quad \text{and} \quad \mathcal{L}(\varphi \cdot S) = \varphi \cdot \mathcal{L}(S) \tag{1.5.1}$$

for all $S, S_1, S_2 \in \mathcal{S}(E)$ and $\varphi \in \mathcal{F}_{\mathbb{K}}(M)$. In Corollary 1.5.3 below, we characterize morphisms by this property. The fundamental property of tensorial maps is captured in the following fundamental lemma.

1.5.2. Lemma. *Let $\mathcal{L} : \mathcal{S}(E) \to \mathcal{S}(E')$ be tensorial. Let $p \in M$ and $S, \tilde{S}$ be smooth sections of E with $S(p) = \tilde{S}(p)$. Then*

$$\mathcal{L}(S)(p) = \mathcal{L}(\tilde{S})(p)\,.$$

Proof. In a neighborhood U of p, choose a local frame $\Phi = (S_1, \dots, S_k)$ of E and a smooth function φ on M with $\varphi(p) = 1$ and $\operatorname{supp}(\varphi) \subset U$. Then $\varphi \cdot S_i$ is a smooth section of E when extended by zero outside U. Hence $\mathcal{L}(\varphi S_i)$ is defined.

Let σ and $\tilde{\sigma}$ be the principal parts of S and $\tilde{S}$ with respect to Φ. Then $\varphi\sigma$ and $\varphi\tilde{\sigma}$ are smooth on M when extended by zero outside U and $\sigma(p) = \tilde{\sigma}(p)$. By definition we have $S = \sigma^i S_i$, $\tilde{S} = \tilde{\sigma}^i \tilde{S}_i$ and

$$\varphi^2 \cdot S = (\varphi\sigma^i) \cdot (\varphi S_i)\,, \quad \varphi^2 \cdot \tilde{S} = (\varphi\tilde{\sigma}^i) \cdot (\varphi S_i)\,.$$

Hence

$$\begin{aligned}\mathcal{L}(S)(p) &= \varphi^2(p)\cdot\mathcal{L}(S)(p) = \mathcal{L}(\varphi^2\cdot S)(p)\\ &= \mathcal{L}\big((\varphi\sigma^i)\cdot(\varphi S_i)\big)(p)\\ &= \big((\varphi\sigma^i)\cdot\mathcal{L}(\varphi S_i)\big)(p)\\ &= \sigma^i(p)\cdot\mathcal{L}(\varphi S_i)(p)\\ &= \tilde{\sigma}_i(p)\cdot\mathcal{L}(\varphi S_i)(p) = \mathcal{L}(\tilde{S})(p)\,,\end{aligned}$$

the asserted identity. □

We discuss two applications of Lemma 1.5.2. Further on there will be similar applications of Lemma 1.5.2, we will not repeat the (easy) arguments however.

1.5.3. Corollary. *Let $\mathcal{L}:\mathcal{S}(E)\to\mathcal{S}(E')$ be tensorial. Then there is a morphism $L:E\to E'$ such that $\mathcal{L}(S)=L\circ S$ for all $S\in\mathcal{S}(E)$.*

Proof. By Proposition 1.1.4, for each $p\in M$ and $x\in E$, there is $S\in\mathcal{S}(E)$ with $S(p)=x$. Define $L_p(x)=\mathcal{L}(S)(p)$. By Lemma 1.5.2, L_p is well defined and $\mathcal{L}(S)(p)=L_p(S(p))$ for all $S\in\mathcal{S}(E)$. Clearly, L_p is linear for each $p\in M$. We obtain a map $L:E\to E'$ by setting $L|E_p := L_p$.

It remains to show smoothness of L. To that end, consider a point $p\in M$, a local frame $(S_1,\dots,S_k)$ of E in an open neighborhood U of p and a smooth function φ with $\operatorname{supp}(\varphi)\subset U$ and such that $\varphi(q)=1$ for all q in an open neighborhood V of p. Then $\varphi S_1,\dots,\varphi S_k$ are smooth sections of E when extended by zero outside U. Hence $\mathcal{L}(\varphi S_i)$ is defined and is a smooth section of E'. If $(S_1',\dots,S_l')$ is a frame of E' in a neighborhood U' of p, then for $q\in V\cap U'$,

$$L(S_i(q)) = L((\varphi S_i)(q)) = \mathcal{L}(\varphi S_i)(q) = a_i^j(q)S_j'(q)\,.$$

Now the functions a_i^j are smooth in q since $\mathcal{L}(\varphi S_i)\in\mathcal{S}(E')$. □

1.5.4. Corollary. *Let $\mathcal{L}:\mathcal{V}(M)\times\cdots\times\mathcal{V}(M)\to\mathcal{S}(E)$ be m–linear, alternating, and tensorial in each variable X_i,*

$$\mathcal{L}(X_1,\dots,\varphi X_i,\dots,X_m) = \varphi\mathcal{L}(X_1,\dots,X_m)\,.$$

Then there is an m–form ω with values in E such that

$$\mathcal{L}(X_1,\dots,X_m) = \omega(X_1,\dots,X_m)\,.$$

Proof. Let $p\in M$ and $X_1,\dots,X_m$, $\tilde{X}_1,\dots,\tilde{X}_m$ be smooth vector fields with $X_i(p)=\tilde{X}_i(p)$, $0\le i\le m$. By induction over $i\le m$ it follows that

$$\mathcal{L}(X_1,\dots,X_m)(p) = \mathcal{L}(\tilde{X}_1,\dots\tilde{X}_m)(p)\,.$$

Define

$$\omega_p(X_1(p), \dots, X_m(p)) = \mathcal{L}(X_1, \dots, X_m)(p) .$$

It is clear that ω is an m–form with values in E. Smoothness of ω follows as in the previous proof. □

2. **Connections**

In what follwos, $\pi : E \to M$ is a $\mathbb{K}$–vector bundle of rank k.

2.0.1. Definition. A *connection* or *covariant derivative* on E is a map

$$D : \mathcal{V}(M) \times \mathcal{S}(E) \longrightarrow \mathcal{S}(E), \quad D(X, S) = D_X S = DS(X),$$

such that D is tensorial in X and a *derivation* in S.

By the definition of derivation, a connection D satisfies

$$D_X(\varphi \cdot S) = X(\varphi) \cdot S + \varphi \cdot D_X S \tag{2.0.2}$$

for all $\varphi \in \mathcal{F}_{\mathbb{K}}(M)$ and $S \in \mathcal{S}(E)$.

From now on, we let D be a connection on E. Let $S \in \mathcal{S}(E)$. Then by Lemma 1.5.2, DS is a morphism of E, that is, a 1–form with values in E in the language of Example 1.3.5. We call DS the *covariant derivative of* S.

Let $X \in \mathcal{V}(M)$. We call $D_X S$ the *covariant derivative of* S *in the direction of* X. We think of covariant differentiation as a generalization of directional or partial differentiation in Euclidean space $\mathbb{R}^m$.

2.0.3. Examples. 1) Let $E = M \times \mathbb{K}^k$ be the trivial vector bundle of rank k over M. A smooth section of E is a map S of the form $S(p) = (p, \sigma(p))$, where $\sigma = (\sigma^1, \dots, \sigma^k) : M \to \mathbb{K}^k$ is the *principal part of* S. Define

$$(D_X S)(p) := (p, X_p(\sigma)) ,$$

where

$$X(\sigma) = (X(\sigma^1), \dots, X(\sigma^k)) = d\sigma \cdot X$$

is the usual derivative of σ in the direction of X. It is straightforward to check that D is a connection on E, the so–called *trivial connection.*

2) Let $E = M \times \mathbb{K}^k$ be the trivial bundle as above and ω be a 1–form on M with values in $\mathrm{Mat}(k \times k, \mathbb{K})$, that is, $\omega = (\omega_i^j)$ is a matrix of $\mathbb{K}$–valued 1–forms ω_i^j on M, $1 \le i, j \le k$. For a section $S = (\mathrm{id}, \sigma)$ as above define

$$(D_X S)(p) := \big(p, X_p(\sigma) + \omega_p(X_p) \cdot \sigma(p)\big) .$$

In other words, if σ is the principal part of S, then

$$X(\sigma) + \omega(X) \cdot \sigma$$

is the principal part of $D_X S$, where in index notation

$$\omega(X) \cdot \sigma = \left(\omega_i^1(X)\sigma^i, \ldots, \omega_i^k(X)\sigma^i\right).$$

Once again it is straightforward to check that D is a connection on E. In fact, every connection on the trivial bundle is of this form as we will see below.

2.0.4. Proposition. *Let D be a connection on E and ω be a 1–form with values in* $\mathrm{Hom}(E,E)$. *Then*

$$D'_X S := D_X S + \omega(X) \cdot S$$

is a connection on E. Vice versa, if D and D' are connections on E, then

$$\omega(X) \cdot S := D'_X S - D_X S$$

defines a 1–form ω with values in $\mathrm{Hom}(E,E)$. □

2.1. **Local Data.** We now show that a connection D on a vector bundle $\pi : E \to M$ is determined by local data. It is clear from Lemma 1.5.2 that for any section S of E and any point $p \in M$, the covariant derivative $D_X S$ only depends on $X(p)$. In our first observation we use (2.0.2) to show that $D_X S(p)$ only depends on the restriction of S to a neighborhood of p.

2.1.1. Lemma (Localization). *Let $p \in M$ and $S_1, S_2 \in \mathcal{S}(E)$ be smooth sections such that $S_1 = S_2$ in some neighborhood U of p. Then*

$$(D_X S_1)(p) = (D_X S_2)(p) \quad \text{for all } \ X \in \mathcal{V}(M).$$

Proof. Choose a smooth function $\varphi : M \to \mathbb{R}$ with $\mathrm{supp}(\varphi) \subset U$ such that $\varphi = 1$ in a neighborhood $V \subset U$ of p. Then $\varphi \cdot S_1 = \varphi \cdot S_2$ on M, hence

$$D_X(\varphi \cdot S_1) = D_X(\varphi \cdot S_2).$$

On the other hand, by (2.0.2) and the choice of φ we have

$$\begin{aligned} D_X(\varphi \cdot S_i)(p) &= X_p(\varphi) \cdot S_i(p) + \varphi(p) \cdot D_X S_i(p) \\ &= 0 \cdot S_i(p) + 1 \cdot D_X S_i(p) = D_X S_i(p) \end{aligned}$$

for $i = 1, 2$. Hence $(D_X S_1)(p) = (D_X S_2)(p)$ as claimed. □

Now let $U \subset M$ be an open subset. Let $p \in U$. Recall from Lemma 1.1.4 that for any smooth vector field X on U and smooth section S of E over U, there is a smooth vector field X' on M and a smooth section S' of E over M such that $X = X'$ and $S = S'$ in an open neighborhood $V \subset U$ of p. Define

$$D^U_X S(p) := D(X', S')(p). \tag{2.1.2}$$

By Lemma 2.1.1, $D_X S(p)$ does not depend on the choice of X' and S'. It is now easy to verify that (2.1.2) D^U is a connection on $E|U$. We call D^U the *induced connection* on $E|U$.

By abuse of notation we simply write D instead of D^U. This simplification does not lead to a conflict with the previous notation: if X is the restriction of a smooth vector field X' on M to U and S is the restriction of a smooth section S' of E over M to U, then $D^U(X,S)$ is the restriction of $D(X',S')$ to U.

Let $U \subset M$ be open and $\Phi = (S_1,\dots,S_k)$ be a frame of E over U. Let X be a smooth vector field of M over U. Then since $D = D^U$ is a connection of $E|U$, $D_X S_1,\dots,D_X S_k$ are smooth sections of $E|U$. Hence their principal parts with respect to Φ, denoted $\omega_1(X),\dots,\omega_k(X)$, are smooth maps from U to $\mathbb{K}^k$ and satisfy

$$D_X S_i = \omega_i^j(X) \cdot S_j \,. \tag{2.1.3}$$

It is immediate from the definition of covariant derivative that

$$\omega_i^j(\varphi X + \psi Y) = \varphi \cdot \omega_i^j(X) + \psi \cdot \omega_i^j(Y)$$

for all i,j, all smooth functions φ,ψ and smooth vector fields X,Y on U. Hence ω_i^j is a 1–form on U, $1 \le i,j \le k$. As in Example 2.0.3, we organize the different ω_i^j in a $(k\times k)$–matrix $\omega = (\omega_i^j)$. Then by definition, ω is a 1–form on U with values in $\mathrm{Mat}(k\times k,\mathbb{K})$. We call ω the *connection form* of D with respect to Φ. Sometimes we indicate the dependence on Φ by using Φ as an index.

Now let S be an arbitrary smooth section of E over U and $\sigma = (\sigma^1,\dots,\sigma^k)$ be the principal part of S with respect to Φ. If X is a smooth vector field on U, then by (2.0.2)

$$\begin{aligned} D_X S = D_X\{\sigma^i S_i\} &= \big\{X(\sigma^i)\cdot S_i + \sigma^i \cdot D_X S_i\big\} \\ &= \big\{X(\sigma^i)\cdot S_i + \sigma^i \omega_i^j(X)\cdot S_j\big\} \\ &= \big\{X(\sigma^j) + \omega_i^j(X)\sigma^i\big\}\cdot S_j \,. \end{aligned}$$

Let $\omega = (\omega_i^j)$ be the connection form on U with respect to Φ. Then the result of the above computation can be formulated as follows.

2.1.4. Proposition. *If σ is the principal part of a section S with respect to Φ, then*

$$X(\sigma) + \omega(X)\cdot\sigma$$

is the principal part of $D_X S$ with respect to Φ. □

Let $V \subset M$ be another open set and $\Psi = (T_1,\dots,T_k)$ be a local frame of E over V. On $U\cap V$, there is a smooth map $g = (g_i^j)$ to

$Gl(k,\mathbb{K})$ such that

$$S_i = g_i^j T_j \,,$$

see (1.2.4). If $\sigma_\Phi, \sigma_\Psi : U \cap V \to \mathbb{K}^k$ are the principal parts of a smooth section S of E over $U \cap V$ with respect to Φ and Ψ respectively, then

$$\sigma_\Psi = g \cdot \sigma_\Phi \,.$$

as in (1.2.5). This discussion applies also to the principal parts of $D_X S$ with respect to Φ and Ψ, where X is a smooth vector field on $U \cap V$. Hence we must have

$$\begin{aligned} g \cdot \big(X(\sigma_\Phi) + \omega_\Phi(X) \cdot \sigma_\Phi\big) &= X(\sigma_\Psi) + \omega_\Psi(X) \cdot \sigma_\Psi \\ &= X(g \cdot \sigma_\Phi) + \omega_\Psi(X) \cdot \sigma_\Psi \\ &= X(g) \cdot \sigma_\Phi + g \cdot X(\sigma_\Phi) + \omega_\Psi(X) \cdot \sigma_\Psi \,. \end{aligned}$$

The terms $g \cdot X(\sigma_\Phi)$ cancel. Substituting $\sigma_\Phi = g^{-1} \cdot \sigma_\Psi$ we get

$$\omega_\Psi(X) = g \cdot \omega_\Phi(X) \cdot g^{-1} - X(g) \cdot g^{-1} \,. \tag{2.1.5}$$

This equation describes the transformation rule of the connection form ω_Φ under a change of frame.

2.2. **Induced Connections.** As we saw before, there are many new bundles which we can construct from given ones. We discuss now shortly, how connections on given bundles induce connections on the new bundles. The guiding principle are linearity of connections and the product rule.

2.2.1. Examples. 1) Let $\pi^* : E^* \to M$ be the dual bundle of the vector bundle $\pi : E \to M$. Suppose that D is a connection on E. Then define the covariant derivative of a section T of E^* by

$$(D^*_X T)(S) = X(T(S)) - T(D_X S)\,,$$

where S is a section of E. Note that this is the product rule if we consider $(T, S) \mapsto T(S)$ as a product. This is justified by the expression for $T(S)$ in terms of local frames.

2) Let $E' \oplus E'' \to M$ be the direct sum of vector bundles $E' \to M$, $E'' \to M$ and suppose that D' and D'' are connections on E' and E'' respectively. By a slight abuse of notation, we write sections of $E' \oplus E''$ in the form $S' + S''$, where S' is a section of E' and S'' is a section of E''. Then the rule

$$D_X(S' + S'') = D'_X(S') + D''_X S''$$

defines a connection D on $E' \oplus E''$. Here the required additivity of connections leaves us no choice in the definition of D.

3) Again let $E' \to M$, $E'' \to M$ be vector bundles with connections D' and D'' respectively and let $m \geq 0$. On $\Lambda^m(E', E'')$ define a connection D by

$$(D_X T)(S_1, \dots, S_m) = D''_X(T(S_1, \dots, S_m)) - \sum T(S_1, \dots, D'_X S_i, \dots, S_m) .$$

Here it is again the product rule which leads to the definition of D.

The last example has a very interesting variation. Suppose $E \to M$ is a vector bundle with a connection D. Then there is a *covariant exterior derivative* on the space of alternating m–forms with values in the given bundle E,

$$d^D : \mathcal{S}(\Lambda^m(TM, E)) \to \mathcal{S}(\Lambda^{m+1}(TM, E)) , \tag{2.2.2}$$

defined by

$$d^D T(X_0, \dots, X_m) = \sum_{i=0}^{m} (-1)^i D_{X_i}(T(X_0, \dots, \hat{X}_i, \dots, X_m)) + \sum_{0 \leq i < j \leq m} (-1)^{i+j} T([X_i, X_j], X_0, \dots, \hat{X}_i \dots, \hat{X}_j, \dots, X_m) . \tag{2.2.3}$$

The exterior covariant derivative is connected to the curvature tensor which we discuss further below, see Definition 3.0.1.

2.2.4. Exercises. 1) Compute the connection forms in the above examples with respect to local frames as given in Examples 1.3.5.

2) Using the product rule, define a connection on the tensor product $E' \otimes E''$ of two vector bundles $E' \to M$, $E'' \to M$ with connections D' and D'' respectively.

3) For a section S of a vector bundle E with connection D, compute $d^D d^D S$ and compare the result with the curvature tensor as in Definition 3.0.1. More generally, for an m–form T with values in E, compute the second covariant exterior derivative $d^D d^D T$.

2.3. **Pull Back.** Let D be a connection on the vector bundle $\pi : E \to M$. Our goal is to show that for a smooth map $f : N \to M$, there is a natural connection f^*D on f^*E which is compatible with the natural bundle map $L : f^*E \to E$. Clearly it suffices to define the covariant derivative of a smooth section S along f. To that end, let $\Phi = (S_1, \dots, S_k)$ be a frame of E over an open set $U \subset M$ and let $\omega = (\omega_i^j)$ be the connection form of D as in (2.1.3). On $V = f^{-1}(U)$, let $\sigma = (\sigma^1, \dots, \sigma^k)$ be the *principal part of S with respect to $\Phi \circ f$*, that is,

$$S = \sigma^i \cdot (S_i \circ f) .$$

For a smooth vector field X over V, define

$$D^f_X S = \{X(\sigma^j) + \omega^j_i(f_*X) \cdot \sigma^i\} \cdot (S_j \circ f)\,. \tag{2.3.1}$$

In other words, the principal part of $D^f_X S$ with respect to the chosen frame is

$$X(\sigma) + (f^*\omega)(X) \cdot \sigma\,. \tag{2.3.2}$$

This formula shows that $D^f_X S$ is smooth. We have not checked yet that $D^f_X S$ is well defined. For this, let $\Psi = (T_1, \ldots, T_k)$ be a local frame of E over an open subset $V \subset M$ and let (g^j_i) be the matrix of functions on $U \cap V$ describing the change of frame as in (1.2.4). Let $W = f^{-1}(U \cap V)$. On W, the principal parts σ_Φ and σ_Ψ of S with respect to the frames Φ and Ψ are related by

$$\sigma_\Psi = (g \circ f) \cdot \sigma_\Phi\,,$$

see (1.2.5). For the proposed principal parts of $D_X S$ we have by (2.1.5),

$$\begin{aligned}
X(\sigma_\Psi) + (f^*\omega_\Psi)(X) \cdot \sigma_\Psi &= X\big((g \circ f) \cdot \sigma_\Phi\big) + \omega_\Psi(f_*X) \cdot \sigma_\Psi \\
&= X(g \circ f) \cdot \sigma_\Phi + (g \circ f) \cdot X(\sigma_\Phi) + \omega_\Psi(f_*X) \cdot (g \circ f) \cdot \sigma_\Phi \\
&= X(g \circ f) \cdot \sigma_\Phi + (g \circ f) \cdot X(\sigma_\Phi) \\
&\qquad + (g \circ f) \cdot \omega_\Phi(f_*X) \cdot \sigma_\Phi - X(g \circ f) \cdot \sigma_\Phi \\
&= (g \circ f) \cdot \big(X(\sigma_\Phi) + (f^*\omega_\Phi)(X) \cdot \sigma_\Phi\big)\,.
\end{aligned}$$

This shows that $D^f_X S$ is well defined. For convenience, we usually simply write D instead of D^f.

The following proposition is immediate from the local expressions in (2.3.1) or (2.3.2).

2.3.3. Proposition. *The covariant derivative $D = D^f$ along f,*

$$D : \mathcal{V}(N) \times \mathcal{S}_f(E) \to \mathcal{S}_f(E)\,, \quad D(X, S) = D_X S = DS(X)\,,$$

is tensorial in X and a derivation in S. In other words, if we identify sections of f^*E with sections of E along f, then the map

$$f^*D : \mathcal{V}(N) \times \mathcal{S}(f^*(E)) \to \mathcal{S}(f^*(E))\,, \quad (f^*D)(X, S) = D_X S\,,$$

is a connection on f^*E.

If ω is the connection form of D with respect to a frame Φ of $E|U$, then the connection form of f^*D with respect to the frame $\Phi \circ f$ of $f^*E|f^{-1}(U)$ is the pull back $f^*\omega$. □

For any smooth section $S : M \to E$, $S \circ f$ is a smooth section of E along f. The induced covariant derivative for sections along f is consistent with the original covariant derivative in the following sense.

2.3.4. Proposition (Chain Rule). *If $S : M \to E$ is a smooth section, then*

$$D(S \circ f)(v) = DS(f_{*q}v)$$

for any point $q \in N$ and tangent vector $v \in T_qN$. □

The most important case of this construction is the covariant derivative along a curve $c = c(t)$ in M. If S is a smooth section of E along c, then we set

$$D_tS := DS(\partial_t)\,, \tag{2.3.5}$$

where ∂_t is the canonical coordinate vector field on $\mathbb{R}$. If σ is the principal part of S with respect to a local frame $\Phi = (S_1, \dots, S_k)$ of E over U, then the principal part of D_tS over $V = c^{-1}(U)$ is given by

$$\dot\sigma + \omega(\dot c) \cdot \sigma\,. \tag{2.3.6}$$

2.3.7. Remark. Note that $(D_tS)(t)$ might be non–zero even if $\dot c(t) = 0$. For example, if c is a constant curve, $c(t) \equiv p$, and S is a smooth section along c, then $D_tS = \partial_tS$, the usual derivative of S as a map into the fixed vector space E_p.

3. **Curvature**

For smooth vector fields X, Y on M and a smooth map $\sigma : M \to \mathbb{K}^k$ we have $XY(\sigma) - YX(\sigma) = [X, Y](\sigma)$ by the definition of the Lie bracket. An analogous formula holds for second covariant derivatives of sections of the trivial bundle with respect to the trivial connection as in Example 2.0.3. For arbitrary connections, the failure of the corresponding formula is measured by the curvature tensor.

3.0.1. Definition. The *curvature tensor* of D is the map

$$\begin{aligned} &R : \mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{S}(E) \to \mathcal{S}(E)\,, \\ &R(X, Y)S = D_XD_YS - D_YD_XS - D_{[X,Y]}S\,. \end{aligned}$$

We quickly check that the curvature tensor is a tensor field. Note that this is also immediate from Proposition 3.0.5 below.

3.0.2. Proposition. *The curvature tensor R is tensorial in X, Y and S and skew symmetric in X and Y, $R(X, Y)S = -R(Y, X)S$.*

Proof. Skew symmetry in X and Y follows from the definition of R and the skew symmetry of the Lie bracket. Addidivity in X, Y and S is immediate from the additivity of covariant derivative and Lie bracket. As for homogeneity over $\mathcal{F}(M)$, we compute:

$$\begin{aligned} D_X D_Y(\varphi \cdot S) &= D_X\big(Y(\varphi)\cdot S + \varphi \cdot D_Y S\big) \\ &= XY(\varphi)\cdot S + Y(\varphi)\cdot D_X S + X(\varphi)\cdot D_Y S + \varphi \cdot D_X D_Y S\,. \end{aligned}$$

An analogous formula holds for $D_Y D_X(\varphi \cdot S)$. Now

$$D_{[X,Y]}(\varphi \cdot S) = [X,Y](\varphi)\cdot S + \varphi \cdot D_{[X,Y]}S$$

and hence

$$R(X,Y)(\varphi \cdot S) = \varphi \cdot R(X,Y)S\,.$$

The proof of homogeneity over $\mathcal{F}(M)$ in X and Y is simpler. $\square$

Now Corollary 1.5.3 implies that for any fixed vector fields X and Y on M, we may consider $R(X,Y)$ as a section of $\mathrm{Hom}_{\mathbb{K}}(E,E)$. By Corollary 1.5.4 we may consider R also as a 2–form with values in $\mathrm{Hom}_{\mathbb{K}}(E,E)$. In fact, by Lemma 1.5.2, there is a family of 3–linear maps

$$R_p : T_pM \times T_pM \times E_p \to E_p\,,$$

the *curvature tensor of D at $p \in M$*.

Let $\Phi = (S_1, \ldots, S_k)$ be a frame of E over an open set $U \subset M$ and let ω be the corresponding connection form as in (2.1.3). Recall that the exterior derivative of ω is defined by

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\,. \tag{3.0.3}$$

For convenience, we also set

$$[\omega,\omega](X,Y) := [\omega(X),\omega(Y)] = \omega(X)\cdot\omega(Y) - \omega(Y)\cdot\omega(X). \tag{3.0.4}$$

3.0.5. Proposition. *If S is a smooth section of E over U with principal part σ with respect to Φ and if X, Y are vector fields on U, then the principal part of $R(X,Y)S$ with respect to Φ is*

$$\big(d\omega(X,Y) + [\omega(X),\omega(Y)]\big)\cdot\sigma\,.$$

Proof. By Proposition 2.1.4, the principal part of $D_X D_Y S$ is

$$\begin{aligned} &X\big(Y(\sigma) + \omega(Y)\cdot\sigma\big) + \omega(X)\cdot\big(Y(\sigma) + \omega(Y)\cdot\sigma\big) = \\ &XY(\sigma) + X(\omega(Y))\cdot\sigma + \omega(Y)\cdot X(\sigma) + \omega(X)\cdot Y(\sigma) + \omega(X)\cdot\omega(Y)\cdot\sigma\,. \end{aligned}$$

An analogous formula holds for the principal part of $D_Y D_X S$. The principal part of $D_{[X,Y]}S$ is

$$[X,Y](\sigma) + \omega([X,Y])\cdot\sigma = XY(\sigma) - YX(\sigma) + \omega([X,Y])\cdot\sigma\,.$$

The claim follows. $\square$

We call $\Omega = d\omega + [\omega, \omega]$ the *curvature form* of D with respect to Φ. The curvature form is a 2–form with values in $\mathrm{Mat}(k \times k, \mathbb{K})$.

We now let $f : N \to M$ be a smooth map and consider the pull back connection f^*D on the pull back bundle f^*E of E. Associated to our frame $\Phi = (S_1, \dots, S_k)$ on $E|U$ above we have the frame $\Phi \circ f$ along f on $f^{-1}(U)$. By abuse of notation, we also consider $\Phi \circ f$ as a local frame of the pull back bundle f^*E.

3.0.6. Corollary. *With respect to the frame $\Phi \circ f$, the curvature form of f^*D is*

$$f^*\Omega = f^*(d\omega + [\omega, \omega]) .$$

Proof. Recall that $f^*\omega$ is the connection form of f^*D. Now

$$d(f^*\omega)(X, Y) = f^*(d\omega)(X, Y) = d\omega(f_*X, f_*Y) ,$$

hence the assertion. □

3.0.7. Proposition. *Let $f : N \to M$ be smooth and $S \in \mathcal{S}_f(E)$. Then for all $X, Y \in \mathcal{V}N$,*

$$D_X D_Y S - D_Y D_X S - D_{[X,Y]} S = R(f_*X, f_*Y)S .$$

Proof. Let $\Phi = (S_1, \dots, S_k)$ be a frame of E over the open subset $U \subset M$ and let ω be the connection form with respect to this frame. Let σ be the principal part of S on $V = f^{-1}(U)$ with respect to $\Phi \circ f$. Then by Corollary 3.0.6, the principal part of $R(X, Y)S$ with respect to $\Phi \circ f$ is

$$\begin{aligned} &\big(d(f^*\omega)(X, Y) + [(f^*\omega)(X), (f^*\omega)(Y)]\big) \cdot \sigma \\ &\qquad = \big(d\omega(f_*X, f_*Y) + [\omega(f_*X), \omega(f_*Y)]\big) \cdot \sigma \end{aligned}$$

Now by Proposition 3.0.5, the right hand side is the principal part of $R(f_*X, f_*Y)S$. □

3.0.8. Corollary. *Let $U \subset \mathbb{R}^2$ be open and $f = f(x, y)$ be a smooth map from U to M. If S is a section of E along f, then*

$$D_x D_y S - D_y D_x S = R(f_x, f_y)S ,$$

where f_x and f_y denote the partial derivatives of f with respect to x and y.

Proof. The Lie bracket of the coordinate vector fields ∂_x and ∂_y in $\mathbb{R}^2$ vanishes. □

3.0.9. Exercise. Prove the *Bianchi identity* $d^D R = 0$.

3.1. **Parallel Translation and Curvature.** Let $c : I \to M$ be a smooth curve. We consider sections along c.

3.1.1. Definition. We say that a smooth section S along c is *parallel along* c if $D_t S = 0$.

Note that the differential equation $D_t S = 0$ is linear. In particular, linear combinations $\alpha S + \beta T$, $\alpha, \beta \in \mathbb{K}$, of parallel sections S and T along c are again parallel.

Let $\Phi = (S_1, \ldots, S_k)$ be a frame of E over an open set $U \subset M$, ω the corresponding connection form as in (2.1.3) and σ the principal part of S with respect to Φ. Then S is parallel along the part of c contained in U iff

$$\dot{\sigma} + \omega(\dot{c}) \cdot \sigma = 0 \,. \tag{3.1.2}$$

This is a first order linear differential equation for σ with smooth coefficients. Hence we have the following result.

3.1.3. Lemma. *Let $t_0 \in I$ and $x \in E_{c(t_0)}$. Then there is exactly one parallel section S along c with $S(t_0) = x$.* □

Now let $t_0, t_1 \in I$ and set $p = c(t_0)$ and $q = c(t_1)$. For $x \in E_p$ let $P(x) = S(t_1) \in E_q$, where S is the unique parallel section along c with $S(t_0) = x$. Then P_c is called the *parallel translation along* c from p to q. There is a little ambiguity in this terminology since there might be other parameters t with $c(t) = p$ or $c(t) = q$. However, in real life this does not lead to any problems.

Since linear combinations of parallel sections are parallel, P is a linear map. Moreover, P is invertible: parallel translation along c from q to p is the inverse map. This shows the existence of parallel frames as follows.

3.1.4. Lemma. *Let $t_0 \in I$ and $S_1, \ldots, S_k$ be parallel sections along c. Suppose that $S_1(t_0), \ldots, S_k(t_0)$ is a basis of $E_{c(t_0)}$. Then $S_1(t), \ldots, S_k(t)$ is a basis of $E_{c(t)}$ for all $t \in I$.*

Proof. By definition, $S_i(t) = PS_i(t_0)$, where P is parallel translation along c from $p = c(t_0)$ to $q = c(t)$. By what we said before, P is invertible. □

3.1.5. Definition. We say that a k–tuple $\Phi = (S_1, \ldots, S_k)$ of sections along c is a *frame of E along* c if $S_1(t), \ldots, S_k(t)$ is a basis of $E_{c(t)}$ for all $t \in I$. We say that a frame $\Phi = (S_1, \ldots, S_k)$ of E along c is *parallel along* c if $S_1, \ldots, S_k$ are parallel along c.

Let $\Phi = (S_1, \dots, S_k)$ be a parallel frame of E along c. Let S be a smooth section along c. Then there is a smooth map $\sigma : I \to \mathbb{K}^k$, the *principal part of S with respect to Φ*, such that

$$S = \sigma^i S_i \,.$$

Since the sections S_i are parallel,

$$D_t S = (\partial_t \sigma^i) S_i \,. \tag{3.1.6}$$

In other words, $\partial_t \sigma$ is the principal part of $D_t S$. Hence by using a parallel frame along c, we reduce covariant derivatives to standard derivatives (of the principal part). Yet another way of expressing this fact is formulated in the following lemma.

3.1.7. Lemma. *Let $t_0 \in I$ and P_t be parallel translation along c from $c(t)$ to $p = c(t_0)$. Let $S \in \mathcal{S}_c(E)$. Then*

$$P_t D_t S = \partial_t P_t S \,.$$

Proof. Choose a parallel frame $\Phi = (S_1, \dots, S_k)$ of E along c and let σ be the principal part of S with respect to Φ. Then since Φ is parallel,

$$P_t S = \sigma^i S_i(t_0) \quad \text{and} \quad P_t D_t S = (\partial_t \sigma^i) S_i(t_0) \,.$$

Hence $P_t D_t S = \partial_t P_t S$ as claimed. □

3.1.8. Lemma. *Let $\phi : J \to I$ be smooth and $S \in \mathcal{S}_c(E)$ be parallel along c. Then $S \circ \phi$ is parallel along $c \circ \phi$.*

Proof. This is immediate from the differential equation (3.1.2). □

It follows that parallel translation along c does not depend on the parameterization of c.

Now let $c : I \to M$ be piecewise smooth and S be a piecewise smooth section along c. Choose a subdivision

$$\dots < t_{i-1} < t_i < t_{i+1} \dots$$

of I such that c and S are smooth on the subintervals $[t_{i-1}, t_i]$. Then we say that S is *parallel along c* if $D_t S = 0$ along these subintervals. Lemmas 3.1.3 and 3.1.4 extend to the piecewise smooth situation.

3.1.9. Corollary. *Let $t_0 \in I$. Then we have:*

1. *For any $x \in E_{c(t_0)}$, there is exactly one parallel section S along c with $S(t_0) = x$.*
2. *If $S_1, \dots, S_k$ are parallel sections along c and $S_1(t_0), \dots, S_k(t_0)$ is a basis of $E_{c(t_0)}$, then $S_1(t), \dots, S_k(t)$ is a basis of $E_{c(t)}$ for all $t \in I$.* □

We now discuss the relation between parallel translation and curvature. Let I be an interval. Consider a piecewise smooth homotopy

$$c : I \times [a,b] \to M\,, \quad c(s,t) = c_s(t)\,.$$

Let $P_{s,t}$ be parallel translation along c_s from $c(s,t)$ to $c(s,b)$ and set

$$R_{s,t} = P_{s,t} \circ R(\partial_t c(s,t), \partial_s c(s,t)) \circ P_{s,t}^{-1}\,. \tag{3.1.10}$$

3.1.11. Lemma. *Let S be a piecewise smooth section along c such that $D_t S = 0$ and $D_s S(\,.\,,a) = 0$. Then*

$$D_s S(s,b) = \Big(\int_a^b R_{s,t}\,dt\Big) \cdot S(s,b)$$

Proof. By Corollary 3.0.8 and our assumption on S we have

$$D_t D_s S = D_s D_t S + R(\partial_t c, \partial_s c) S = R(\partial_t c, \partial_s c) S$$

and hence by (3.1.7),

$$\begin{aligned} \partial_t P_{s,t} D_s S(s,t) &= P_{s,t} D_t D_s S(s,t) = P_{s,t} R(\partial_t c(s,t), \partial_s c(s,t)) S(s,t) \\ &= P_{s,t} R(\partial_t c(s,t), \partial_s c(s,t)) P_{s,t}^{-1} S(s,b) = R_{s,t} S(s,b)\,. \end{aligned}$$

Now $P_{s,b} = \mathrm{id}$ by the definition of $P_{s,t}$ and $D_s S(\,.\,,a) = 0$ by our assumption on S. Therefore

$$\begin{aligned} D_s S(s,b) &= P_{s,b} D_s S(s,b) - P_{s,a} D_s S(s,a) \\ &= \int_a^b R_{s,t} S(s,b)\,dt = \Big(\int_a^b R_{s,t}\,dt\Big) \cdot S(s,b)\,. \end{aligned}$$

This is the asserted equation. $\square$

One of the interesting applications of the above lemma occurs in the case where c is proper, that is, $c(s,a) = p$ and $c(s,b) = q$ for some points $p,q \in M$ and all $s \in I$. Then $S(s,a) \in T_p M$ and $S(s,b) \in T_q M$ for all s. Under the assumptions of Lemma 3.1.11, $S(\,.\,,a)$ is constant. The assertion of the lemma then says that

$$\partial_s S(s,b) = \Big(\int_a^b R_{s,t} dt\Big) \cdot S(s,b)\,, \tag{3.1.12}$$

a first order linear ordinary differential equation for $S(\,.\,,b)$. If $P_s = P_{s,a}$ denotes parallel translation along c_s from $p = c_s(a)$ to $q = c_s(b)$, then $S(s,b) = P_s S(s,a)$ by our assumptions on S. Hence we can reformulate (3.1.12) as

$$\partial_s P_s = \Big(\int_a^b R_{s,t} dt\Big) \cdot P_s\,. \tag{3.1.13}$$

In this precise sense, curvature tells us the dependence of parallel translation on the path.

Let $p \in M$ and $u, v \in T_pM$. Let $f : U \to M$ be a smooth map with

$$f(0) = p\,, \quad f_x(0) = u\,, \quad f_y(0) = v\,,$$

where $U \subset \mathbb{R}^2$ is an open neighborhood of 0. Define a smooth family $c_s(t) = c(s,t)$, $0 \le s, t \le 1$, of piecewise smooth curves by

$$c_s(t) = \begin{cases} f(4st, 0) & \text{for } 0 \le t \le 1/4, \\ f(s, s(4t-1)) & \text{for } 1/4 \le t \le 1/2, \\ f(s(3-4t), s) & \text{for } 1/2 \le t \le 3/4, \\ f(0, 4s(1-t)) & \text{for } 3/4 \le t \le 1\,. \end{cases} \tag{3.1.14}$$

The infinitesimal dependence of parallel translation on curvature is now captured by the following result.

3.1.15. Theorem. *Let $p \in M$, $u, v \in T_pM$ and $f : U \to M$ be a map as above. Let P_s be parallel translation along the curve c_s as in (3.1.14) from $c_s(0)$ to $c_s(1)$. Then*

$$\partial_s P_s(0) = 0 \quad \textit{and} \quad \partial_s \partial_s P_s(0) = 2R(v,u)\,.$$

Proof. By the definition of c and the skew symmetry of R,

$$R(\partial_t c, \partial_s c) = \begin{cases} 0 & \text{for } t \le 1/4 \text{ or } t \ge 3/4\,, \\ 4sR(f_y, f_x) & \text{for } 1/4 \le t \le 3/4\,. \end{cases}$$

Let $P_{s,t}$ be parallel translation along the curve c_s as (3.1.14) from $c_s(t)$ to $c_s(1)$ and define $R_{s,t}$ as in (3.1.10), where $a = 0$ and $b = 1$. Then $P_s = P_{s,0}$ and

$$\partial_s P_s = \Big(\int_{1/4}^{3/4} R_{s,t} dt \Big) P_s$$

by (3.1.13). Now parallel translation depends continuously on the path, hence

$$\frac{1}{4s} R_{s,t} = P_{s,t} \circ R(f_y, f_x) \circ P_{s,t}^{-1} \to R(v,u)$$

uniformly in t for $s \to 0$. □

Let $c : [0,1] \to M$ be a curve. Then the *inverse curve* c^{-1} of c is defined by

$$c^{-1}(t) := c(1-t) \quad 0 \le t \le 1\,. \tag{3.1.16}$$

Let $c_1, c_2 : [0,1] \to M$ be curves with $c_2(0) = c_1(1)$. Then the *composition* $c = c_1 * c_2$ of c_1 and c_2 is defined by concatenation,

$$(c_1 * c_2)(t) = \begin{cases} c_1(2t) & \text{if } 0 \le t \le 1/2\,, \\ c_2(2t-1) & \text{if } 1/2 \le t \le 1\,. \end{cases} \tag{3.1.17}$$

If c and c_1, c_2 are piecewise smooth, then both c^{-1} and $c_1 * c_2$ are piecewise smooth — this is one of the reasons for considering piecewise smooth curves. Note that the composition of smooth curves need not be smooth.

We denote by $\Pi = \Pi(M)$ the space of all piecewise smooth curves $c : [0,1] \to M$. For $p, q \in M$ we let $\Pi_p = \Pi_p(M)$ respectively $\Pi_{pq} = \Pi_{pq}(M)$ be the subspace of curves with initial point p respectively initial point p and end point q. Elements of Π_{pp} are also called *loops* at p.

For $c \in \Pi$ we denote by P_c parallel translation along c. We have

$$P_{c^{-1}} = P_c^{-1} \quad \text{and} \quad P_{c_1 * c_2} = P_{c_2} P_{c_1}\,. \tag{3.1.18}$$

We conclude that for any point $p \in M$, the set $\mathrm{Hol}(p)$ of parallel translations P_c, $c \in \Pi_{pp}$, is a group of linear automorphisms of E_p. We call $\mathrm{Hol}(p)$ the *holonomy group* of the connection D at p. The subset $\mathrm{Hol}_0(p)$ of parallel translations P_c, where c is a piecewise smooth contractible loop at p, is a normal subgroup, the *restricted holonomy group* of D at p.

Let $p, q \in M$ and $\gamma \in \Pi_{pq}$. Then the map

$$\Pi_{pp} \to \Pi_{qq}\,, \quad c \mapsto \gamma * c * \gamma^{-1}\,, \tag{3.1.19}$$

induces an isomorphism

$$\mathrm{Hol}(p) \to \mathrm{Hol}(q)\,, \quad P_c \mapsto P_\gamma P_c P_\gamma^{-1}\,, \tag{3.1.20}$$

which sends $\mathrm{Hol}_0(p)$ to $\mathrm{Hol}_0(q)$.

We now discuss the holomony theorem of Ambrose–Singer [AS] and Nijenhuis [Ni]. For $c \in \Pi_p$ and $u, v \in T_pM$ set

$$R_c(u,v) = P_c^{-1} \circ R(P_c u, P_c v) \circ P_c\,, \tag{3.1.21}$$

an endomorphism of E_p.

3.1.22. Theorem. *Let $p \in M$. In the Lie algebra of endomorphisms of E_p, let $\mathfrak{g}$ be the subalgebra generated by the endomorphisms $R_c(u,v)$ for all $c \in \Pi_p$ and $u, v \in T_pM$. Then* $\mathrm{Hol}_0(p)$ *is the connected Lie subgroup of $Gl(E_p)$ with Lie algebra $\mathfrak{g}$.*

In particular, $\mathrm{Hol}(p)$ *is a Lie group with* $\mathrm{Hol}_0(p)$ *as component of the identity.*

Proof. Let $c : [a, b] \to M$ be a piecewise smooth loop at p and assume that c is contractible. Choose a piecewise smooth proper homotopy c_s, $0 \le s \le 1$, from the point curve $c_0 = p$ to $c = c_1$. By (3.1.13), parallel translation $P_s = P_{c_s}$ along c_s satisfies the differential equation

$$\partial_s P_s = \Big(\int_0^1 R_{s,t} dt \Big) \cdot P_s \,.$$

Now the integrand on the right hand side is contained in $\mathfrak{g}$. It follows that the curve P_s, $0 \le s \le 1$, is contained in G. Hence parallel translation $P = P_1$ along $c = c_1$ belongs to G, hence $\mathrm{Hol}_0(p) \subset G$.

By the first part of the latter argument, $H = \mathrm{Hol}_0(p)$ is a subgroup of G such that for each element $g \in H$, there is a continuously differentiable curve from the neutral element of G to g which is contained in H. Hence H is a Lie subgroup of G, see Appendix 4 in Volume I of [KN].

Now let $c \in \Pi_p$ and $u, v \in T_pM$. Let $U \subset \mathbb{R}^2$ be an open neighborhood of 0 and $f : U \to M$ be smooth with $f(0) = q = c(1)$, $f_x(0) = P_c u$ and $f_y(0) = P_c v$. Define loops c_s, $0 \le s \le 1$, at q as in (3.1.14). Then the piecewise smooth curves $\gamma_s = c * c_s * c^{-1}$ are contractible loops at our point p.

For $0 \le \tau \le 1$, let $g(\tau)$ be parallel translation along γ_s, $s = \tau^2$. Then by (3.1.13) and Theorem 3.1.15, $g : [0, 1] \to Gl(E_p)$ is a continuously differentiable curve contained in $H = \mathrm{Hol}_0(p)$ and

$$\partial_\tau g(0) = P_c^{-1} \circ R(P_c v, P_c u) \circ P_c = R_c(v, u) \,.$$

Hence the Lie algebra $\mathfrak{h}$ of H contains the generators of the Lie algebra $\mathfrak{g}$. Therefore $\mathfrak{h} = \mathfrak{g}$ and hence $\mathrm{Hol}_0(p) = H = G$. □

4. **Miscellanea**

4.1. **Metrics.** For many vector bundles, the fibers are equipped with inner products which depend smoothly on the base points.

4.1.1. Definition. Let $\pi : E \to M$ be a $\mathbb{K}$–vector bundle over M. A *Riemannian metric* (if $\mathbb{K} = \mathbb{R}$) respectively *Hermitian metric* (if $\mathbb{K} = \mathbb{C}$) is a smooth map $\langle . , . \rangle : E \oplus E \to \mathbb{K}$ such that $\langle . , . \rangle : E_p \oplus E_p \to \mathbb{K}$ is an inner respectively Hermitian product on E_p for all $p \in M$.

We also speak of Riemannian or Hermitian bundles respectively.

4.1.2. Definition. Let E be a Riemannian or Hermitian bundle and D be a connection on E. We say that D is *Riemannian* or *Hermitian* respectively if D satisfies the product rule

$$X \langle S, T \rangle = \langle D_X S, T \rangle + \langle S, D_X T \rangle$$

for all vector fields X on M and sections S, T of E. We also simply say that D is *compatible* with $\langle . , . \rangle$.

If D is compatible with $\langle . , . \rangle$, then $R(X,Y) \in \mathrm{Hom}_{\mathbb{K}}(E,E)$ is skew symmetric for all vector fields X and Y on M.

4.2. **Cocycles and Bundles.** Let $\pi : E \to M$ be a $\mathbb{K}$–vector bundle of rank k over M. A family $\Phi_\alpha : U_\alpha \times \mathbb{K}^k \to E|U_\alpha$ of trivializations of π is called a *bundle atlas* if $\cup_\alpha U_\alpha = M$. For such an atlas and a point $p \in U_\alpha \cap U_\beta$, we have

$$(\Phi_\alpha^{-1}\Phi_\beta)(p,\xi) = (p, g_{\alpha\beta}(p)\cdot\xi)\,, \tag{4.2.1}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(k,\mathbb{K})$ is a smooth map. Obviously the family of maps $g_{\alpha\beta}$ satisfies the following *cocycle relations*,

$$\begin{aligned} g_{\alpha\alpha} &= 1 \quad \text{on } U_\alpha\,, \\ g_{\alpha\beta}g_{\beta\alpha} &= 1 \quad \text{on } U_\alpha \cap U_\beta\,, \\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= 1 \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma\,. \end{aligned} \tag{4.2.2}$$

An open cover $M = \cup_\alpha U_\alpha$ of M together with a family of smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(k,\mathbb{K})$ satisfying (4.2.2) is called a *cocyle with values in* $Gl(k,\mathbb{K})$.

Suppose now we are given such a cocycle. For $(p,\xi) \in U_\alpha \times \mathbb{K}^k$ and $(q,\eta) \in U_\beta \times \mathbb{K}^k$ define $(p,\xi) \sim (q,\xi)$ iff $p = q$ and $g_{\alpha\beta}(p)\cdot\xi = \eta$. Then "$\sim$" defines an equivalence relation on the disjoint union $\dot\cup(U_\alpha \times \mathbb{K}^k)$. The equivalence class of (p,ξ) is denoted $[p,\xi]$, the set of equivalence classes is denoted E. By the definiton of our relation, the map

$$\pi : E \to M\,, \quad \pi([p,\xi]) = p\,,$$

is well defined. For $p \in U_\alpha$, $\kappa \in \mathbb{K}$ and $x, y \in E_p$ let $(p,\xi), (p,\eta) \in U_\alpha \times \mathbb{K}^k$ be representatives, $x = [p,\xi]$, $y = [p,\eta]$, and define

$$x + y := [p, \xi + \eta]\,, \quad \kappa \cdot x := [p, \kappa\cdot\xi]\,,$$

where we consider $(p, \xi+\eta)$ and $(p, \kappa\cdot\xi)$ as elements of $U_\alpha \times \mathbb{K}^k$. It follows from (4.2.1) that $x + y$ and κx are well defined. In this way each fiber E_p of π carries a natural structure as a $\mathbb{K}$–vector space of dimension k. Furthermore, for each α and point $p \in U_\alpha$,

$$\Phi_{\alpha,p} : \mathbb{K}^k \to E_p\,, \quad \Phi_{\alpha,p}(\xi) = [p,\xi]\,,$$

is a $\mathbb{K}$–linear isomorphism. The following lemma is now immediate from Lemma 1.3.4.

4.2.3. Lemma. *The space E of equivalence classes as above carries a unique structure as a smooth manifold such that $\pi : E \to M$ is a $\mathbb{K}$–vector bundle for which the bijections Φ_α are local trivializations.* $\square$

4.2.4. Exercise. Starting from a bundle atlas (U_α, Φ_α) for a vector bundle $\pi : E \to M$, we obtain a cocycle $(g_{\alpha\beta})$ as in (4.2.1). The above construction associates to this cocycle a new vector bundle, now denoted $\pi' : E' \to M$. Show that the map on the disjoint union of the $U_\alpha \times \mathbb{K}^k$, defined by

$$U_\alpha \times \mathbb{K}^k \ni (p, \xi) \mapsto \Phi_\alpha(p, \xi) \in E$$

on the different parts of the disjoint union, induces an isomorphism $E' \to E$.

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