LECTURES ON THE BLASCHKE CONJECTURE

NOTES BY WERNER BALLMANN

1. Introductory remarks

These are lecture notes of a course on differential geometry, taught jointly with Karsten Grove in the summer of 2014. The main topic is the generalized Blaschke conjecture, the main source for the lecture is [4].

The notes do not cover everything that occured in class; in particular, the part on the volumes of Blaschke manifolds is missing. In comparison to [4], our notation and definitions differ slightly. We also added details where we deemed it appropriate and changed the exposition and formulations in some places. Comments on the notes are welcome.

I would like to thank Karsten Grove for joining me in the endeavour to understand where we are in the Blaschke conjecture and for joining me in teaching the students a part of differential geometry which uses a number of important tools from different mathematical fields and offers a bounty of appealing results on the way.

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Date: March 10, 2016.

I would like to thank Simon Lohove and Saskia Voß for their careful reading of the notes and their helpful comments.

2. On the geometry of tangent and cotangent bundles

2.1. Connections and the splitting of TTM. Let M be a manifold with a connection ∇. Let $v \in TM$ and $Z \in T_vTM$ be a tangent vector of TM with foot point v. Represent $Z = \dot V(0)$, where $V = V(s)$ is a curve in TM through $v = V(0)$. Then V is a vector field along its curve $c = \pi \circ V$ of foot points. Define the *connection map*

$$C\colon TTM \to TM \quad \text{by} \quad C(Z) := V'(0), \tag{2.1}$$

where V' denotes the covariant derivative of V along c. In Proposition 2.2 below we will see that C is well defined and linear on the fibres T_vTM.

With respect to local coordinates x on M, we have associated local coordinates (x, y) of TM for tangent vectors $y^i\partial/\partial x^i(p)$, where $x = x(p)$. In terms of these, π is the projection onto the x-component. Moreover, tangent vectors of TM correspond to quadruples (x, y, ξ, η), where the first two components represent the foot point and the last two the principal part of the tangent vector.

Proposition 2.2. *With respect to coordinates (x, y) for TM as above,*

$$\pi_*(x, y, \xi, \eta) = (x, \xi) \quad \textit{and} \quad C(x, y, \xi, \eta) = (x, \eta^i + \xi^j y^k \Gamma^i_{jk}(x)).$$

Proof. With respect to coordinates of TM as above, a curve V in TM is given by a curve $(x, y) = (x(s), y(s))$. The first component $x = x(s)$ corresponds to the curve of foot points of V, the covariant derivative V' of V along the curve of foot points is given by

$$V' = (\dot y^i + \dot x^j y^k \Gamma^i_{jk}) \frac{\partial}{\partial x^i},$$

and $\dot V$ corresponds to $(x, y, \dot x, \dot y)$. Hence

$$\pi_* \circ \dot V = (x, \dot x) \quad \text{and} \quad C \circ \dot V = V' = (x, \dot y^i + \dot x^j y^k \Gamma^i_{jk}). \tag{2.3}$$ □

Corollary 2.4. *1) $\mathcal{H} := \ker C$ and $\mathcal{V} := \ker \pi_*$ are subbundles of TTM, called* horizontal *and* vertical distribution*, respectively, and $TTM = \mathcal{H} \oplus \mathcal{V}$.*
2) For each $v \in TM$, $\pi_*\colon \mathcal{H}_v \to T_pM$ and $C\colon \mathcal{V}_v \to T_pM$ are isomorphisms, where p is the foot point of v, $p = \pi(v)$. □

Henceforth, we will write tangent vectors of TM as pairs, $Z = (X, Y)$, where $Z^{\mathcal{H}} = X = \pi_*(Z)$ is the *horizontal* and $Z^{\mathcal{V}} = Y = C(Z)$ the *vertical component* of Z. Horizontal components do not depend on the connection, but vertical ones do. On the other hand, the vertical distribution does not depend on the connection, $\mathcal{V} = \ker \pi_*$, but the horizontal distribution does.

Corollary 2.5. *For $v \in TM$ with $p = \pi(v)$, we have:*
1) A parallel vector field $V = V(s)$ along a curve $c = c(s)$ through p with $\dot c(0) = X$ and $V(0) = v$ satisfies $\dot V(0) = (X, 0)$
2) The curve $V = V(s) = v + sY$ in T_pM satisfies $\dot V(0) = (0, Y)$. □

Corollary 2.6. *For any piecewise smooth curve $c\colon [a, b] \to M$, its horizontal lifts to TM induce parallel translation $h_c\colon T_{c(a)}M \to T_{c(b)}M$ along c.* □

For each $v \in TM$, denote by $\gamma_v\colon I_v \to M$ the *maximal geodesic* with initial velocity v, that is, $0 \in I_v$, $\dot\gamma_v(0) = v$, and I_v is the maximal domain of definition of a geodesic with initial velocity v. We say that ∇ is *complete* if $I_v = \mathbb{R}$ for all $v \in TM$.

The domain of definition $\mathcal{G}$ of the *geodesic flow* $g = (g_t)$ of ∇ is the open set of pairs $(t, v) \in \mathbb{R} \times TM$, where $t \in I_v$, and g is given by

$$g\colon \mathcal{G} \to TM, \quad g(t, v) = g_t(v) = \dot\gamma_v(t). \tag{2.7}$$

For any geodesic γ of ∇, $\dot\gamma$ is parallel along γ. Hence the vector field on TM associated to the geodesic flow is $X = X(v) = (v, 0)$.

Let $V = V(s)$ be a curve in TM through v and consider the associated geodesic variation $\gamma_s = \gamma_s(t) = \gamma_{V(s)}(t)$ of $\gamma = \gamma_v$ with Jacobi field $\partial\gamma_s/\partial s$. Note that the Jacobi equation contains a torsion term since we do not assume that ∇ is torsion free. For $\dot V(0) = (X, Y) \in T_vTM$ and $t \in I_v$, we get

$$g_{t*}(X, Y) = (J(t), (\nabla_s\dot\gamma_s)(0, t)) = (J, J' + T(J, \dot\gamma))(t), \tag{2.8}$$

where $J = J(t) = (\partial\gamma_s/\partial s)(0, t)$ and dot and prime indicate the derivative and covariant derivative in the t-direction, respectively. In the torsion free case, the main case further on, the T-term on the right hand vanishes.

2.2. **The cotangent bundle as a symplectic manifold.** We consider now the cotangent bundle $\bar\pi\colon T^*M \to M$. The *canonical* or *tautological one-form* λ on T^*M is defined by

$$\lambda_\alpha(v) = \alpha(\bar\pi_* v). \tag{2.9}$$

With respect to local coordinates x of M, we have associated local coordinates $(x, a) = (x^1, \dots, x^m, a_1, \dots, a_m)$ of T^*M for one-forms $a_i dx^i(p)$, where $x = x(p)$. In terms of these, $\bar\pi$ is the projection onto the x-component. Moreover, tangent vectors of T^*M correspond to quadruples (x, a, ξ, α), where the first two compnents represent the foot point and the last two the principal part of the tangent vector. We get

$$\lambda_{(x,a)}(x, a, \xi, \alpha) = (a_i dx^i)(\xi^j\partial/\partial x^j) = a_i\xi^i.$$

We conclude that $\lambda_{(x,a)} = a_i dx^i$, but now considered as a one-form on T^*M.

The differential $\omega := -d\lambda$ turns T^*M into an *exact symplectic manifold*; that is, the two-form ω is exact and non-degenerate. The latter is obvious from the expression for ω with respect to local coordinates (x, a) of T^*M as above, $\omega = dx^i \wedge da_i$.

If $f\colon M \to N$ is a local diffeomorphism, then $f^*\colon T^*N \to T^*M$ is a local diffeomorphism such that $f^{**}\lambda_M = \lambda_N$ and $f^{**}\omega_M = \omega_N$. In particular, f^* is a symplectic diffeomorphism of T^*M, for any diffeomorphism f of M.

We recall the notion of *Hamiltonian system* from symplectic geometry: Let N be a symplectic manifold with symplectic form ω. For a function $h \in \mathcal{F}(N)$ and a vector field X on N, h is called a *Hamiltonian potential* of X and X the *Hamiltonian vector field* associated to h, $X = X_h$, if

$$dh = i_{X_h}\omega \quad \text{that is,} \quad dh(Y) = (i_{X_h}\omega)(Y) = \omega(X_h, Y), \tag{2.10}$$

for all vector fields Y on N. The dynamical system associated to the Hamiltonian vector field X_h is called the *Hamiltonian system* associated to h and h the *Hamiltonian* of the system. Hamiltonian systems are *symplectic*; that is, they preserve the symplectic form.

By the non-degeneracy of ω, a function $h \in \mathcal{F}(N)$ determines a unique Hamiltonian vector field X_h. On the other hand, not any vector field X on N has a Hamiltonian potential. Furthermore, Hamiltonian potentials are only unique up to locally constant functions.

2.3. **The Legendre transform.** Suppose now that M is endowed with a semi-Riemannian metric, as usual denoted by $\langle .,. \rangle$. Since $\langle .,. \rangle$ is non-degenerate at each point of M,

$$\mathcal{L}\colon TM \to T^*M, \quad \mathcal{L}(v)(w) := \langle v, w \rangle, \tag{2.11}$$

is an isomorphism, called the *Legendre transform*. For any isometry f of M, we have

$$\mathcal{L}(f_*v)(w) = \langle f_*v, w \rangle = \langle v, f_*^{-1}w \rangle = (\mathcal{L}(v) \circ f_*^{-1})(w), \tag{2.12}$$

and hence the action of f_* on TM corresponds, under the Legendre transform, to the action of $(f^{-1})^* = (f^*)^{-1}$ on T^*M. The one-form λ on T^*M corresponds to the one-form on TM, also called λ, given by

$$\lambda_v(Z) = \langle v, \pi_*Z \rangle, \quad Z \in T_vTM. \tag{2.13}$$

Assume from now on that M is also endowed with a metric connection ∇. To compute $d\lambda$, we consider a map $V = V(s,t)$ to TM with $V(0,0) = v$, $(\partial V/\partial s)(0,0) = Z_1$, and $(\partial V/\partial t)(0,0) = Z_2$. At $s = t = 0$ (suppressed in the computation), we have

$$\begin{aligned} d\lambda(Z_1, Z_2) &= \frac{\partial}{\partial s}\lambda\big(\frac{\partial V}{\partial t}\big) - \frac{\partial}{\partial t}\lambda\big(\frac{\partial V}{\partial s}\big) \\ &= \frac{\partial}{\partial s}\big\langle V, \frac{\partial c}{\partial t}\big\rangle - \frac{\partial}{\partial t}\big\langle V, \frac{\partial c}{\partial s}\big\rangle \\ &= \big\langle \frac{\nabla V}{\partial s}, \frac{\partial c}{\partial t}\big\rangle + \big\langle V, \frac{\nabla}{\partial s}\frac{\partial c}{\partial t}\big\rangle - \big\langle \frac{\nabla V}{\partial t}, \frac{\partial c}{\partial s}\big\rangle - \big\langle V, \frac{\nabla}{\partial t}\frac{\partial c}{\partial s}\big\rangle \\ &= \big\langle \frac{\nabla V}{\partial s}, \frac{\partial c}{\partial t}\big\rangle - \big\langle \frac{\partial c}{\partial s}, \frac{\nabla V}{\partial t}\big\rangle + \big\langle V, T\big(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\big)\big\rangle, \end{aligned} \tag{2.14}$$

where T denotes the torsion tensor of ∇. We conclude that

$$\omega(Z_1, Z_2) = -d\lambda(Z_1, Z_2) = \langle X_1, Y_2 \rangle - \langle Y_1, X_2 \rangle - \langle v, T(X_1, X_2) \rangle, \tag{2.15}$$

where X_1 and X_2 denote the horizontal and Y_1 and Y_2 the vertical component of Z_1 and Z_2, respectively. This formula is well known in the case of the Levi-Civita connection, where the T-term on the right vanishes since the Levi-Civita connection is torsion free.

2.4. **The Sasaki metrics on TM and SM.** Let $\langle .,. \rangle$ be a semi-Riemannian metric and ∇ be a metric connection on M. Endow TM with the semi-Riemannian metric such that $\mathcal{H} \perp \mathcal{V}$ and such that $\pi_*\colon \mathcal{H}_v \to T_pM$ and $C\colon \mathcal{V}_v \to T_pM$ are orthogonal transformations, for all $p \in M$ and $v \in T_pM$.

Proposition 2.16. *With respect to the above metric, we have:*
1) $\pi\colon TM \to M$ is a Riemannian submersion.
2) For any piecewise smooth curve $c\colon [a,b] \to M$, its horizontal lifts induce an orthogonal transformation $h_c\colon T_{c(a)}M \to T_{c(b)}M$.
3) The fibers T_pM, $p \in M$, of π are totally geodesic with respect to the Levi-Civita connection on TM.

Proof. Assertion 1) is immediate from the definition of the metric and Assertion 2) since ∇ is metric and h_c is parallel translation along c, see Corollary 2.6. Assertion 3) follows since the fibers of a Riemannian submersion are totally geodesic with respect to the Levi-Civita connection if and only if the maps h_c are isometric (where defined). $\square$

Exercise 2.17. Prove the above assertion about Riemannian submersions: The second fundamental forms of the fibers $E_p := \pi^{-1}(p)$ of a Riemannian submersion $\pi : E \to M$ vanish if and only if the maps $h_c \colon \mathcal{D}_c \to E_{c(b)}$ are isometric, for all piecewise smooth curves $c\colon [a,b] \to M$, where $\mathcal{D}_c \subseteq E_{c(a)}$ denotes the domain of definition of h_c.

Before we proceed with the discussion of the tangent bundle, we introduce some notation concerning integration: For a semi-Riemannian manifold N, we denote the volume element of N by vol_N or simply by vol. Depending on readability, we write

$$\int_A f(p)\, d\mathrm{vol}_N(p) \ \text{ or } \int_A f(p) dp \ \text{ or } \int_A f, \tag{2.18}$$

respectively, for the integral of a function f over a measurable subset A of N against the volume element of N.

Exercise 2.19 (Fubini)**.** Let $\pi\colon P \to N$ be a Riemannian submersion of semi-Riemannian manifolds and $f\colon P \to \mathbb{R}$ be an integrable function. Then the restriction of f to almost any fiber P_p, $p \in N$, of π is integrable and

$$\int_P f(p)\, d\mathrm{vol}_P(p) = \int_N \int_{P_p} f(q)\, d\mathrm{vol}_{P_p}(q)\, d\mathrm{vol}_N(q). \tag{2.20}$$

Suppose now that ∇ is the Levi-Civita connection associated to the given semi-Riemannian metric on M. Then the metric on TM introduced further up is called the *Sasaki metric.*

Proposition 2.21. *With respect to the Sasaki metric,* $|\omega^m| = m!\,\mathrm{vol}$ *and the geodesic flow* (g_t) *is Hamiltonian with Hamiltonian function* $h(v) = \|v\|^2/2$.

Proof. Let $(e_1, \dots, e_m)$ be an orthonormal basis of T_pM. Let $v \in T_pM$ and set $Z_i = (e_i, 0)$ and $Z_{m+i} = (0, e_i)$, $1 \le i \le m$. Then $(Z_1, \dots, Z_{2m})$ is an orthonormal basis of T_vTM, and we have

$$\omega(Z_j, Z_k) = \begin{cases} 0 & \text{if } k \neq j \pm m; \\ \langle e_j, e_j\rangle & \text{if } k = j + m. \end{cases} \tag{2.22}$$

This shows the first claim. As for the second, note first that $\operatorname{grad} h$ is vertical. It follows that $\operatorname{grad} h(v) = (0, v)$ and, hence, that $X_h(v) = (v, 0)$, the vector field of the geodesic flow. $\square$

Exercise 2.23. It follows from Proposition 2.21 that the geodesic flow is symplectic. Prove this explicitly, using (2.15). Show also that the geodesic flow does not preserve λ.

Suppose from now on that $\langle .,. \rangle$ is Riemannian, and denote by SM the unit tangent bundle of M, $SM = \{v \in TM \mid \|v\| = 1\}$. The unit tangent bundle is a submanifold of TM of codimension 1, and SM is invariant under the geodesic flow. Throughout, the restrictions of λ and ω to SM will also be denoted by λ and ω, respectively.

Exercise 2.24. 1) For $p \in M$ and $v \in SM$,

$$T_v SM = \{(X, Y) \in T_p M \oplus T_p M \mid Y \perp v\}.$$

2) On SM, we have $|\lambda \wedge \omega^{m-1}| = (m-1)! \operatorname{vol}$.
3) The geodesic flow on SM preserves λ and ω.

Assume now that M is complete so that g_t has domain SM.

Lemma 2.25. *For any integrable function f on SM, we have*

$$\int_{SM} f(g_t v)\, dv = \int_{SM} f(v)\, dv.$$

Proof. By Exercise 2.24, the Jacobian of g_t is identically equal to 1. Hence the asserted equality follows from Exercise 2.34. □

Remark 2.26. A corresponding formula holds for TM in place of SM.

As an exercise, we recall a useful formula concerning the trace of self-adjoint endomorphisms on Euclidean vector spces.

Exercise 2.27. If V is a Euclidean vector space and $A \colon V \to V$ is a self-adjoint endomorphism, then

$$\frac{1}{k} \operatorname{tr} A = \frac{1}{\operatorname{vol}(S^{k-1})} \int_S \langle Av, v \rangle dv,$$

where $k = \dim V$ and $S \subseteq V$ denotes the unit sphere of V.

As a first application of our insights into the geometry of the unit tangent bundle, we prove Theorem 5.1 from L. Green's article [10]:

Theorem 2.28. *Let M be a closed Riemannian manifold and assume that the first conjugate point along any unit speed geodesic does not occur earlier than $a > 0$. Then*

$$\frac{a^2}{\pi^2} \int_M \operatorname{scal}(p)\, dp \le m(m-1) \operatorname{vol}(M)$$

with equality if and only if M has constant sectional curvature π^2/a^2.

Proof. Consider a unit speed geodesic $\gamma \colon [0, a] \to M$ and let $E = E(t)$ be a parallel unit field along γ and perpendicular to γ. Since there is no conjugate point of $c(0)$ along γ, the second variation of energy applied to the vector field $V = V(t) = \sin(\pi t/a) E(t)$ is non-negative, that is

$$\int_0^a \sin^2(\pi t/a) \langle R(E(t), \dot\gamma(t))\dot\gamma(t), E(t) \rangle\, dt \le \frac{\pi^2}{2a}.$$

Moreover, equality holds if and only if V is a Jacobi field, that is, if and only if the sectional curvature $K(E(t) \wedge \dot\gamma(t) \equiv \pi^2/a^2$. Integrating this inequality over all parallel orthonormal frame E along γ and perpendicular to γ as above and using Exercise 2.27, we get

$$\int_0^a \sin^2(\pi t/a) \operatorname{Ric}(\dot\gamma(t))\, dt \le (m-1) \frac{\pi^2}{2a},$$

where equality holds if and only if the sectional curvature of any tangential plane containing any $\dot\gamma(t)$ is equal to π^2/a^2. Now we integrate both sides over the unit tangent bundle. The term on the right integrates to

$$\operatorname{vol}(M)\operatorname{vol}(S^{m-1})(m-1)\pi^2/2a.$$

As for the term on the left, we have

$$\begin{aligned}\int_{SM}\int_0^a \sin^2(\pi t/a)\operatorname{Ric}(\dot\gamma_v(t))\,dt\,dv &= \int_0^a \sin^2(\pi t/a)\int_{SM}\operatorname{Ric}(g_t v)\,dvdt\\ &= \int_0^a \sin^2(\pi t/a)\int_{SM}\operatorname{Ric}(v)\,dvdt\\ &= \frac{a}{2}\int_M\int_{S_pM}\operatorname{Ric}(v)\,dvdp\\ &= \frac{a}{2m}\operatorname{vol}(S^{m-1})\int_M \operatorname{scal}(p)\,dp,\end{aligned}$$

where we use the definition of the geodesic flow g_t, Lemma 2.25, Exercise 2.19, and Exercise 2.27, respectively. □

Corollary 2.29. *If M is a closed Riemannian manifold without conjugate points, then $\int_M \operatorname{scal}(p)\,dp \le 0$.* □

Remark 2.30 (Recommended reading)**.** Two-dimensional tori without conjugate points are flat, by E. Hopf [13]. The arguments in [13] are short and brilliant and also use integration on the unit tangent bundle.

Corollary 2.31. *Let M be a closed Riemannian surface and assume that the first conjugate point along any unit speed geodesic in M does not occur earlier than $a > 0$. Then the area of M is at least $2a^2\chi(M)/\pi$, and equality holds if and only if the curvature of M is constant π^2/a^2.*

Reminder 2.32. An oriented Riemannian surface is a Riemann surface in a natural way. The converse is the content of the uniformization theorem.

We end this section with a useful formula concerning integration on SM. For any map $g\colon P\to N$ between semi-Riemannian manifolds of the same dimension, there is a function $|g_*|\colon P\to\mathbb{R}_+$, called the *Jacobian* of g, such that $g^*\operatorname{vol}_N = |g_*|\operatorname{vol}_P$. We also write $|g_{*p}|$ instead of $|g_*|(p)|$.

Exercise 2.33. For $g\colon P\to N$ as above, $p\in P$, and a basis $(b_1,\dots,b_n)$ of T_pP, we have $|g_{*p}| = |g_{*p}b_1\wedge\cdots\wedge g_{*p}b_n|/|b_1\wedge\cdots\wedge b_n|$, where the vertical bars indicate the volumes of the parallelepipeds spanned by the corresponding tuples of vectors; e.g., $|b_1\wedge\cdots\wedge b_n| = |\det((\langle b_i,b_j\rangle))|^{1/2}$.

Exercise 2.34 (Transformation rule)**.** If $g\colon P\to N$ is a diffeomorphism, then

$$\int_{g(A)} f(q)\,dq = \int_A (f\circ g)(p)|\,g_{*p}|dp,$$

for any measurable subset $A\subseteq P$ and integrable function f on $g(A)$.

Let Σ be a non-degenerate hypersurface in a semi-Riemannian manifold N and X be a vector field on N with flow $\Phi = (\Phi_t)$. Recall that $\operatorname{div} X$, the divergence of X, vanishes if and only if (Φ_t) preserves volume.

Let $U \subseteq \mathbb{R} \times \Sigma$ be the open subset of (t, x) which are in the domain of Φ and consider the map

$$F : U \to N, \quad F(t, x) = \Phi_t(x).$$

Endow Σ with the induced semi-Riemannian metric and $\mathbb{R} \times \Sigma$ with the product metric, where we view $\mathbb{R}$ as the standard Euclidean line.

Lemma 2.35. *If* $\operatorname{div} X$ *vanishes, then* $|F_*|$ *does not depend on* t,

$$|F_*|(t, x) = |\langle X^\perp(x), X^\perp(x)\rangle|^{1/2},$$

where $X^\perp$ *denotes the component of* X *perpendicular to* Σ.

Proof. Let (t, x) in U. Then there is an open neighborhood U' of (t, x) in U and an $\epsilon > 0$ such that $(s+t', x') \in U$ for all $(t', x') \in U'$ and $s \in (-\epsilon, \epsilon)$. For $(t', x') \in U'$, set $\Psi_s(t', x') = (s + t', x') \in U$. Then $\Psi_s^* \operatorname{vol}_{\mathbb{R}\times\Sigma} = \operatorname{vol}_{\mathbb{R}\times\Sigma}$ and $F \circ \Psi_s = \Phi_s \circ F$. Furthermore, since $\operatorname{div} X = 0$, we have $\Phi_s^* \operatorname{vol}_N = \operatorname{vol}_N$. We conclude that, on U',

$$\begin{aligned} |F_*| \operatorname{vol}_{\mathbb{R}\times\Sigma} &= F^* \operatorname{vol}_N = F^* \Phi_s^* \operatorname{vol}_N = \Psi_s^* F^* \operatorname{vol}_N \\ &= (|F_*| \circ \Psi_s) \Psi_s^* \operatorname{vol}_{\mathbb{R}\times\Sigma} = (|F_*| \circ \Psi_s) \operatorname{vol}_{\mathbb{R}\times\Sigma}, \end{aligned}$$

and therefore $|F_*|(t + s, x) = |F_*|(t, x)$. The first claim follows.

As for the second, the right hand side of the formula is equal to $|F_*|(0, x)$ since Σ is endowed with the induced semi-Riemannian metric and $\mathbb{R} \times \Sigma$ with the product metric; compare with Exercise 2.33. □

We apply Lemma 2.35 to the geodesic flow on the unit tangent bundle of a Riemannian manifold. It is common to refer to the following formula as *Santaló's formula*; see [20, pp. 336–338] and also [4, p. 147].

Proposition 2.36. *Let* H *be a hypersurface in a Riemannian manifold* M *and* $\Sigma = SM|_H$. *Let* X *be the vector field of the geodesic flow on* SM *and consider* U *and* F *as above. Then*

$$|F_*|(t, v) = \sin\theta(v),$$

where $\theta(v) \in [0, \pi/2]$ *is the angle between* v *and* T_pH, $p = \pi(v) \in H$.

Proof. Identifiying $T_vSM = T_pM \oplus v^\perp$ as usual, we have $T_v\Sigma = T_pH \oplus v^\perp$. Now $X(v) = (v, 0)$, and hence the assertion follows from Lemma 2.35. □

3. Wiedersehen manifolds

For $v \in SM$, denote by $\mathrm{con}(v) \in (0,\infty]$ the first $t > 0$ that is conjugate to 0 along γ_v. For $p \in M$, set

$$\begin{aligned}\mathrm{Con}_T(p) &= \{\mathrm{con}(v)v \mid v \in S_pM, \mathrm{con}(v) < \infty\} \subseteq T_pM,\\ \mathrm{Con}(p) &= \exp(\mathrm{Con}_T(p)) = \{\gamma_v(\mathrm{con}(v)) \mid v \in S_pM\} \subseteq M.\end{aligned}$$

the *tangential first conjugate locus* and *first conjugate locus* of p, respectively. Recall that $\mathrm{con}\colon SM \to (0,\infty]$ is continuous; cf. also Exercise A.1. It follows that $\mathrm{Con}_T(p)$ is a closed subset of T_pM.

Exercise 3.1. For all $v \in S_pM$, we have $\mathrm{con}(-\dot\gamma_v(\mathrm{con}(v))) = \mathrm{con}(v)$. (Hint: Recall the relation of conjugate points with the index form of geodesics.)

Following Blaschke, we say that a complete and connected Riemannian surface M is a *Wiedersehen surface* if $\mathrm{Con}(p)$ consists of one single point, for all $p \in M$. The original *Blaschke conjecture* says that a Wiedersehen surface has constant positive curvature, hence that it is a sphere or a projective plane with a standard Riemannian metric.

Theorem 3.2 (Green)**.** *The Blaschke conjecture is true.*

In the proof of Theorem 3.2, we follow the exposition of Green in [10]. The first part of the arguments does not involve that M is a surface. Until and including Exercise 3.9, we assume that M is a *Wiedersehen manifold*, that is, that M is a complete and connected Riemannian manifold such that $\mathrm{Con}(p)$ consists of a single point, for all $p \in M$. Then we may consider Con as a map $M \to M$; as such, Con is continuous and involutive, $\mathrm{Con}^2 = \mathrm{id}$, since con is continuous and $\mathrm{con}(-\dot\gamma_v(\mathrm{con}(v))) = \mathrm{con}(v)$.

Lemma 3.3. *For all $p \in M$, $\mathrm{con}\colon S_pM \to (0,\infty]$ is constant $=: a(p) < \infty$. In particular, the universal covering space of M is compact.*

Proof. Let $p \in M$. Since p does have a conjugate point, namely $q = \mathrm{Con}(p)$, there is a $v \in S_pM$ with $\mathrm{con}(v) < \infty$. Given any such v, there is an open spherical ball $B \subseteq S_pM$ about v such that $\mathrm{con}(w) < \infty$ for all $w \in B$, by the continuity of con.

Choose an $\varepsilon > 0$ smaller than the injectivity radius of $q = \mathrm{Con}(p)$ and let $S_\varepsilon(q)$ be the geodesic sphere of radius ε about q. Then the unit speed geodesics γ_w, $w \in B$, intersect $S_\varepsilon(q)$ transversally, in fact, perpendiculary, at time $\mathrm{con}(w) - \varepsilon$. It follows that $\mathrm{con} - \varepsilon$, hence also con, is smooth on B.

Let now $v = v(s)$ be a curve in B and consider the corresponding geodesic variation $\gamma = \gamma_s(t) = \gamma_{v(s)}(t)$. Then we have $\gamma_s(\mathrm{con}(v(s))) = q$, and hence

$$\begin{aligned}0 &= \frac{d}{ds}\gamma_s(\mathrm{con}(v(s)))\\ &= J(s,\mathrm{con}(v(s))) + \dot\gamma_s(\mathrm{con}(v(s))) \cdot \frac{d\,\mathrm{con}(v(s))}{ds},\end{aligned}$$

where $J = \partial\gamma/\partial s$ is the family of associated Jacobi fields. Since $J(s,0) = 0$ and $J'(s,0) = dv/ds$, $J(s,t)$ is perpendicular to $\dot\gamma_s(t)$, for all s,t. Hence the above calculation shows that $d\,\mathrm{con}(v(s))/ds$ vanishes. It follows that con is constant on B and therefore, by the continuity of con, on S_pM. ∎

Lemma 3.4. *The function* con *is constant* $=: a > 0$ *on* SM. *All geodesics of* M *are periodic with* $2a$ *as a common period.*

Proof. Let $p \in M$ and $q = \mathrm{Con}(p)$. Then $p = \mathrm{Con}(q)$, and hence any unit speed geodesic $\gamma\colon \mathbb{R} \to M$ starting at p comes back to p at time $2a(p)$. Choose $\epsilon > 0$ less than half the injectivity radius of M at p and so that $|2a(p) - 2a(p')|$ is less than half the injectivity radius of M at p for all p' of distance $< 2\epsilon$ to p. For $p' = \gamma(\epsilon)$, we have

$$q' = \mathrm{Con}(p') = \gamma(\epsilon + a(p')) \quad \text{and} \quad p' = \mathrm{Con}(q') = \gamma(\epsilon + 2a(p')).$$

Now $\gamma(2a(p)) = p$ and $|\epsilon + 2a(p') - 2a(p)|$ is less than the injectivity radius of M at p. Hence $\gamma|_{[2a(p),\epsilon+2a(p')]}$ is the unique minimal geodesic from p to p' and therefore is equal to $\gamma|_{[0,\epsilon]}$, up to the parameter. Therefore γ closes smoothly at p, $a(p') = a(p)$, and $\gamma(\epsilon) = \gamma(2a(p) + \epsilon)$. We also conclude that the function $a = a(p)$ is locally constant along unit speed geodesics. Since any point in M can be reached by a unit speed geodesic from p, we get that a is constant and that any unit speed geodesic on M is periodic with $2a$ as a period. □

Lemma 3.5. *The map* $\mathrm{Con}\colon M \to M$ *is an isometry.*

Proof. For any unit speed geodesic $\gamma\colon \mathbb{R} \to M$, $\mathrm{Con}(\gamma(t)) = \gamma(t + a)$. □

Exercise 3.6. Let $f\colon M \to N$ be a map between connected Riemannian manifolds of the same dimension, and suppose that f preserves distances. Show that f is a smooth local isometry. (Hint: Start with the case where M and N are equal to Euclidean space $\mathbb{R}^m$.)

Lemma 3.7. *For* $p \in M$, *consider the Euclidean sphere* $S^m(a/\pi)$ *of radius* a/π *as the quotient of the closed ball* $\bar{B}(0_p, a)$ *of radius* a *in* T_pM, *where the boundary* $S(0_p, a)$ *is identified to a point. Then* $\exp_p\colon \bar{B}(0_p, a) \to M$ *factors through a smooth covering map* $F\colon S^m \to M$.

Proof. Consider 0_p as the north pole N and the collapsed $S(0_p, a)$ as the south pole S of S^m. By definition, exp has maximal rank on the open ball $B(0_p, a)$. The only question is whether F is smooth and has maximal rank at the south pole. This follows easily, however, since all geodesics from p pass through $\mathrm{Con}(p)$ and intersect there with the same angle as in p, by Lemma 3.5. In other words, for all $v \in S_pM$ and $0 \le t \le a$, we have

$$\exp_{\mathrm{Con}\, p}(t\, \mathrm{Con}_{*p}\, v) = \mathrm{Con}(\exp_p(tv)) = \exp_p((t - a)v),$$

exactly as on $S^m(a/\pi)$ with the antipodal map in place of Con. □

The above discussion is summed up in the following

Theorem 3.8. *If* M *is a simply connected Wiedersehen manifold, then*
1) M *is diffeomeorphic to* S^m;
2) the injectivity radius of M *is equal to its diameter* $a := \operatorname{diam} M$;
3) $\gamma(a) = \mathrm{Con}(p)$, *for all* $p \in M$ *and unit speed geodesics* γ *through* p;
4) all unit speed geodesics of M *are periodic with period* $2a$;
5) Con *is an involutive isometry of* M *with* $d(p, \mathrm{Con}(p)) \equiv a$. □

Exercise 3.9. Show that the Wiedersehen property of a Riemannian manifold M passes to Riemannian covering and subcovering spaces of M. Show also that the isometry Con commutes with all isometries of M.

End of proof of Theorem 3.2. We may assume that M is simply conected, by Exercise 3.9. Then M is diffeomorphic to the sphere, by Theorem 3.8. By Corollary 2.31 it therefore suffices to show that the area of M is $4a^2/\pi$. To that end we choose $H \subseteq M$ to be one of the closed geodesics of M of length $2a$; see Theorem 3.8. Then $\Sigma = SM|_H$ is a hypersurface of SM. For U and F as in Proposition 2.36 and up to the set of measure zero of unit vectors tangent to H, SM is simply covered by the set of $F(t, v)$ with $0 \le t < a$ and $v \in \Sigma$, by Theorem 3.8. Therefore

$$\mathrm{vol}(SM) = \int_0^a \int_\Sigma \sin\theta(v)\, dvdt = 2a^2 \int_0^{2\pi} |\sin\theta|\, d\theta = 8a^2,$$

by Proposition 2.36. Hence $\mathrm{area}(M) = \mathrm{vol}(SM)/2\pi = 4a^2/\pi$ as desired. □

Exercise 3.10. Let M be a complete and connected Riemannian surface such that the function $\mathrm{con}\colon SM \to \mathbb{R}$ is constant and finite. Show that M is a Wiedersehensfläche. What about higher dimensions? Recall also that, in all dimensions, there are closed manifolds with $\mathrm{con} \equiv \infty$.

In a first and maybe erroneous version of the proof of Lemma 3.3, I used a somewhat carelessly formulated version of the following exercise. Since the statement of the exercise is useful, I decided to include it at the end of this section, although it is not required in our discussion anymore.

Exercise 3.11. Let $C = \{tw \mid w \in S_pM \text{ and } 0 < t < \mathrm{con}(w)\} \subseteq T_pM$. Let $c\colon [0,1] \to T_pM$ be a piecewise smooth curve with $c(0) = 0$ and $c((0,1)) \subseteq C$. Then

$$L(\exp \circ c) \ge \|c(1)\|.$$

Moreover, equality holds if and only if $c = c(t) = \phi(t)c(1)$, for all $0 \le t \le 1$, where ϕ is a monotonic surjection of $[0,1]$.

4. Zoll surfaces

Zoll surfaces are (Riemannian) metrics of revolution on the sphere S^2, such that all their geodesics are closed of the same length and without self-intersections. We discuss now the existence of Zoll surfaces, following the presentation in [4, Chapter 4]. Another reference–with a presentation close to Zoll's original one in [24]–is Section 8 of [3, Chapter IV].

Using polar coordinates at the north pole N of S^2, we view the sphere S^2 as a strip $[0,a] \times \mathbb{R}$, where the second parameter θ counts modulo 2π and where the boundary lines $s = 0$ and $s = a$ are collapsed to north and south pole N and S of the sphere, respectively:

$$(s,\theta) \longleftrightarrow (\sin(\pi s/a)\cos\theta, \sin(\pi s/a)\sin\theta, \cos(\pi s/a)). \tag{4.1}$$

We consider metrics on the open strip $(0,a) \times \mathbb{R}$ of the form

$$g = \mu^2(s)\, ds^2 + \lambda^2(s)\, d\theta^2, \tag{4.2}$$

where $\lambda, \mu\colon (0,a) \to \mathbb{R}$ are positive smooth functions. By substituting the s-parameter appropriately, we may also assume that $\mu = 1$, and then g has the simpler form

$$g = ds^2 + \lambda^2(s)\, d\theta^2. \tag{4.3}$$

The class of functions λ and μ, such that g extends to a smooth metric on S^2, will be identified in Exercise 4.5 below. To that end, the following characterization of the regularity of functions on $\mathbb{R}^2$ in terms of polar coordinates is useful.

Proposition 4.4. *Let $f = f(s,\theta)$ be a function on $\mathbb{R}^2$. Then there is a smooth function $g = g(x,y)$ on $\mathbb{R}^2$ with $f(s,\theta) = g(s\cos\theta, s\sin\theta)$ if and only if f is smooth and*
1) $f(-s,\theta) = f(s,\theta+\pi)$ for all $(s,\theta) \in (-\epsilon,\epsilon) \times \mathbb{R}$;
2) for all integers $j \geq 0$, $s^j(\partial^j f/\partial s^j)(0,\theta)$ is a homogeneous polynomial of degree j in $x = s\cos\theta$ and $y = s\sin\theta$.

Proposition 4.4 appears as Théorème IV in [9], p. 206, where it follows from a more general result, and Proposition 2.7 in [14], p. 208, where it comes with a direct and elementary proof.

Exercise 4.5. Let g be a metric of the form (4.2) with $\lambda, \mu > 0$ on $(0,a)$, and view S^2 as a quotient of $[0,a] \times \mathbb{R}$ as in (4.1). Show that g extends to a smooth metric on $S^2 \setminus \{S\}$ if and only if μ and $\nu = \lambda(s)/s$ extend to positive and even smooth functions on $(-a,a)$ such that $\mu(0) = \nu(0)$. Formulate also a corresponding criterion for $S^2 \setminus \{N\}$ and discuss under which conditions (s,θ) are (polar) normal coordinates about N and S.

Remark 4.6. A function $f\colon (-\epsilon,\epsilon) \to \mathbb{R}$ is smooth if and only if there is a smooth function $g\colon (-\epsilon,\epsilon) \to \mathbb{R}$ such that $f(s) - f(0) = sg(s)$.

Consider metrics on $(0,a) \times \mathbb{R}$ of the form (4.2). Then shifts $\theta \mapsto \theta + \alpha$ of the angle parameter θ are isometries, and hence $\partial/\partial\theta$ is a Killing field. The following assertion is immediate from Exercise A.2.1

Proposition 4.7 (Clairaut's theorem)**.** *If $\gamma = (s(t),\theta(t))\colon I \to (0,a) \times \mathbb{R}$ is a geodesic, then* $\langle\dot\gamma, \partial/\partial\theta\rangle = \lambda^2(\gamma(t))\dot\theta(t) = \text{constant}$.

Remark 4.8. Recall that, up to parametrization, all *meridians* $\theta = \theta_0$ are geodesics and that a *parallel* $s = s_0$ is a geodesic if and only if s_0 is a critical point of λ.

We consider now a smooth metric g on $(0, a) \times \mathbb{R}$ of the form (4.3) and assume that it extends to a smooth Riemannian metric on S^2, keeping the names λ, μ, and g for the corresponding extensions; compare with Exercise 4.5. Then, since $\lambda(0) = \lambda(a) = 0$ and $\lambda > 0$ on $(0, a)$, λ has a maximum λ_0 in $(0, a)$, say at a time $s = s_0$.

Proposition 4.9 (A necessary condition)**.** *If all unit speed geodesics of g are periodic with a common period, then s_0 is the only critical point of λ. Furthermore, s_0 is non-degenerate, that is, $\ddot{\lambda}(s_0) < 0$.*

Proof. The assertion of Proposition 4.9 follows if any critical point of λ in $(0, a)$ is a non-degenerate local maximum.

Suppose now that ℓ is a common period of all unit speed geodesics of g, and let $s_1 \in (0, a)$ be a critical point of λ. Then $\gamma = (s_1, t/\lambda(s_1))$ is a unit speed geodesic, and hence $\gamma(\ell) = \gamma(0)$. Let γ_α be the unit speed geodesic with $\gamma_\alpha(0) = (s_1, 0)$ and such that the oriented angle $\angle(\dot{\gamma}(0), \dot{\gamma}_\alpha(0)) = \alpha$. Then the γ_α constitute a geodesic variation of $\gamma = \gamma_0$ with fixed starting point $(s_1, 0) = \gamma_\alpha(0)$. By assumption $\gamma_\alpha(\ell) = \gamma_\alpha(0) = (s_1, 0)$ for all α. Hence the Jacobi field V associated to this geodesic variation satisfies $V(0) = V(\ell) = 0$. It follows that V is perpendicular to γ. Since the curvature K of g does not depend on θ, the oriented length v of V satisfies the scalar Jacobi equation $\ddot{v} + k_1 v = 0$, where $k_1 = K(s_1)$. Since v is non-trivial with $v(0) = v(\ell) = 0$, this is only possible if $k_1 > 0$.

Since all meridians are geodesics, $\partial/\partial\theta$ is a Jacobi field along each of them. Hence $\ddot{\lambda} = -K\lambda$. We conclude that $\ddot{\lambda}(s_1) = -k_1\lambda(s_1) < 0$, and hence λ achieves a non-degenerate local maximum at $s = s_1$. □

Remark 4.10. Suppose, more generally, that all unit speed geodesics of g are closed, but maybe not with a common period. Let s_1 be a critical point of λ. Then $K(s_1) < 0$ would imply that there are geodesics $\gamma = (s(t), \theta(t))$ of g which are asymptotic to the geodesic $s = s_1$ in the sense that $s(t)$ would converge strictly monotonically to s_1 as $t \to \infty$. Such geodesics would never close up, contradicting the hypothesis. The situation is maybe similar in the case $K(s_1) = 0$, and then Proposition 4.9 would follow under the given weaker assumption.

Metrics of the form (4.3) that satisfy the assertion of Proposition 4.9 have a unique parallel of maximal length, the *equator* $s = s_0$. Replacing g by $\lambda_0^{-2} g$ and substituting the parameter s by $\lambda_0 s$, we arrive at the first normalization $\lambda_0 = 1$. Then the equator is of length 2π. Our next aim is a normal form for such metrics.

Lemma 4.11. *Let $f\colon (-\epsilon, \epsilon) \to \mathbb{R}$ be a smooth function.*
1) If $s = 0$ is a critical point of f with $f(0) = 1$ and $\ddot{f}(0) < 0$, then there is a unique smooth function $r = r(s)$ about $s = 0$ with $r(0) = 0$ and $\dot{r} > 0$ such that $f(s) = \cos r(s)$.
2) If f is even, then there are smooth functions h_0 and h_1 about $s = 0$ and $s = 1$, respectively, such that $f(s) = h_0(s^2) = h_1(\cos s)$.

The existence of the function h_0 in the second assertion of Lemma 4.11 is Theorem 1 in [23]. The existence of h_1 is an immediate consequence.

Proof of the first assertion of Lemma 4.11. Diminishing ϵ if necessary, f is strictly monotonically increasing to 1 on $(-\epsilon, 0]$ and strictly monotonically decreasing on $[0, \epsilon)$. Hence there is a unique strictly increasing continuous function r on $(-\epsilon, \epsilon)$ such that $r(0) = 0$ and $f = \cos r$, namely

$$r = -\arccos f \text{ on } (-\epsilon, 0] \quad \text{and} \quad r = \arccos f \text{ on } [0, \epsilon),$$

respectively. Since cos is real analytic and diffeomorphic on $(-\pi/2, 0)$ and $(0, \pi/2)$, r is smooth on $(-\epsilon, 0)$ and $(0, \epsilon)$. Moreover,

$$\dot r = \dot f(1 - f^2)^{-1/2} \text{ on } (-\epsilon, 0) \quad \text{and} \quad \dot r = -\dot f(1 - f^2)^{-1/2} \text{ on } (0, \epsilon),$$

respectively. Now $\dot f$, $1 - f^2$ and the first derivative of $1 - f^2$ vanish at 0. Furthermore, the second derivative of $1 - f^2$ at 0 exists and equals $-2\ddot f(0)$. From Remark 4.6, we get that there are smooth functions $\phi, \psi\colon (-\epsilon, \epsilon) \to \mathbb{R}$ with $\phi(0) = \psi(0) = 1$ such that

$$\dot f(s) = \ddot f(0)\phi(s)s \quad \text{and} \quad 1 - f^2(s) = -\ddot f(0)\psi(s)s^2.$$

Since $f(s) < 1$ for $s \neq 0$, we have $\psi > 0$. Now $\sqrt{s^2} = \pm s$ with minus-sign on $(-\epsilon, 0)$ and plus sign on $(0, \epsilon)$, respectively. Since $\ddot f(0) < 0$, we obtain

$$\dot r = \sqrt{-\ddot f(0)}\phi\psi^{-1/2} \quad \text{on } (-\epsilon, 0) \cup (0, \epsilon).$$

Hence $\dot r$ extends to a smooth function on $(-\epsilon, \epsilon)$ with $\dot r(0) = \sqrt{-\ddot f(0)}$. It follows that r is smooth with $\dot r > 0$. $\square$

Proposition 4.12 (Normal form 1)**.** *Let g be a smooth metric of the form* (4.3) *which extends to a smooth metric on S^2. Assume that s_0 is the only critical point of λ with $\lambda(s_0) = 1$ and $\ddot\lambda(s_0) < 0$. Then there is a smooth parameter change $r\colon [0, a] \to [0, \pi]$ such that*

$$g = \mu^2(r)\, dr^2 + \sin^2(r)\, d\theta^2,$$

where $\mu\colon [0, \pi] \to \mathbb{R}$ is a positive smooth function which is even about 0 *and* π *with $\mu(0) = \mu(\pi) = 1$.*

Proof. Since λ is strictly monotonically increasing from 0 to 1 on $[0, s_0]$ and strictly monotonically decreasing from 1 to 0 on $[s_0, a]$, there is a unique increasing homeomorphism $r\colon [0, a] \to [0, \pi]$ such that $\lambda = \sin r$, namely

$$r = \arcsin\lambda \text{ on } [0, s_0] \quad \text{and} \quad r = \pi - \arcsin\lambda \text{ on } [s_0, a],$$

respectively. Since λ is diffeomorphic on $(0, s_0)$ and (s_0, a) and sin likewise on $[0, \pi/2)$ and $(\pi/2, \pi]$, r is diffeomorphic on $[0, s_0)$ and $(s_0, a]$. By Lemma 4.11.1, r is diffeomorphic about $s = s_0$. Hence $r\colon [0, a] \to [0, \pi]$ is a smooth parameter change.

As for the fundamental matrix of the metric in the (r, θ)-coordinates, we note that $\partial/\partial s = \dot r\,\partial/\partial r$. Hence $g = \dot r^{-2}dr^2 + \sin^2(r)\, d\theta^2$. Now λ is odd about 0 and a with $\dot\lambda(0) = 1$ and $\dot\lambda(a) = -1$. Hence r is odd about 0 and a with $\dot r(0) = \dot r(a) = 1$, and therefore $s = s(r)$ is a smooth function which is odd about 0 and π with $\dot s(0) = \dot s(\pi) = 1$. We conclude that

$\mu = \mu(r) = \dot{r}^{-1}(s(r))$ is a smooth function which is even about 0 and π with $\mu(0) = \mu(\pi) = 1$. $\square$

Example 4.13. On the standard sphere S^2 of radius one, we have

$$g = dr^2 + \sin^2(r)\, d\theta^2,$$

where r denotes the angle between north pole and foot point in question.

Proposition 4.14 (Normal form 2)**.** *Let $\mu\colon [0,\pi] \to \mathbb{R}$ be a positive smooth function, which is even about 0 and π with $\mu(0) = \mu(\pi) = 1$, and let g be the corresponding metric on S as in Proposition 4.12. Then there is a positive smooth function $f\colon [-1,1] \to \mathbb{R}$ with $f(-1) = f(1) = 1$ such that $\mu(r) = f(\cos r)$, that is, such that*

$$g = f^2(\cos r)\, dr^2 + \sin^2(r)\, d\theta^2. \tag{4.15}$$

Conversely, any such metric, where f is smooth with $f(-1) = f(1) = 1$, defines a smooth metric on S^2 with equator $r = \pi/2$ of length 2π.

Assume from now on that the Riemannian metric on S^2 is given in the (r,θ)-coordinates as in (4.15). By Proposition 4.7 and Proposition 4.9, for any unit speed geodesic $\gamma = (r(t), \theta(t))$ on M, which is not a meridian and not the equator, there is a number $r_0 = r_0(\gamma) \in (0, \pi/2)$ such that $\min_t r(t) = r_0$ and $\max_t r(t) = \pi - r_0$. We now compute the angle difference $\Delta\theta$ of γ between two events $r = r_0$ and $r = \pi - r_0$. Without loss of generality we may assume that $\dot{r} > 0$ in the corresponding time interval.

Lemma 4.16. *In the above situation, let*

$$r(t_0) = r_0,\ r(t_1) = \pi - r_0,\ \text{and } r_0 < r(t) < \pi - r_0 \text{ for all } t \in (t_0, t_1).$$

Then

$$\Delta\theta = \theta(t_1) - \theta(t_0) = \int_{r_0}^{\pi - r_0} \frac{f(\cos r)\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2} \sin r}\, dr.$$

Proof. Without loss of generality, we may assume that $\dot{\theta} > 0$. Clairaut's theorem says that $\langle \dot{\gamma}, \partial/\partial\theta \rangle = \dot{\theta}\sin^2 r = \text{constant}$. At time $t = t_0$, we have $\dot{r}(t_0) = 0$, therefore $\dot{\theta}(t_0)^2 \sin^2 r_0 = 1$, and hence $\dot{\theta}\sin^2 r = \sin r_0$. Since γ has unit speed, $\dot{r}^2 f^2(\cos r) + \dot{\theta}^2 \sin^2 r = 1$, we obtain

$$\dot{r}^2 = \frac{1 - \dot{\theta}^2 \sin^2 r}{f^2(\cos r)} = \frac{1 - \sin^2 r_0 / \sin^2 r}{f^2(\cos r)} = \frac{\sin^2 r - \sin^2 r_0}{f^2(\cos r)\sin^2 r}.$$

We conclude that

$$\frac{dt}{dr} = \frac{f(\cos r)\sin r}{(\sin^2 r - \sin^2 r_0)^{1/2}} \tag{4.17}$$

and therefore that

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{f(\cos r)\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2}\sin r}. \tag{4.18}$$

$\square$

Example 4.19. On the standard sphere of radius 1, we have $\Delta\theta = \pi$. In other words,

$$\int_{r_0}^{\pi - r_0} \frac{\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2}\sin r}\, dr = \Delta\theta = \pi,$$

for all $0 < r_0 < \pi/2$. This formula will be crucial in our discussion below.

Clearly, $\Delta\theta$ also gives half the θ-difference between two consecutive times that γ reaches $r = r_0$ or $r = \pi - r_0$, respectively, or that γ, after starting on the equator, comes back to the equator the next time. (Note that we do not, and should not, assume that the northern and southern hemissphere are isometric). We conclude that γ closes after $n > 0$ such events if and only if $2n\Delta\theta = 2m\pi$ for some integer $m > 0$, hence if and only if

$$\Delta\theta = \int_{r_0}^{\pi - r_0} \frac{f(\cos r)\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2}\sin r}\,dr = \frac{m}{n}\pi. \tag{4.20}$$

Assume now that all unit speed geodesics of M are periodic with a common period. Then, since $\Delta\theta$ is continuous in r_0 and rational, it is constant, say equal to $m\pi/n$. Choosing m and n divisor free, we conclude that all unit speed geodesics, except for the equator and the meridians, close after n events $r = r_0$. By continuity, the smallest common period of all unit speed geodesics, except for the meridians, is equal to $2m\pi$. Therefore the length of a segment of a geodesic between two consecutive events $r = r_0$ and $r = \pi - r_0$ as above is equal to $m\pi/n$. Again by continuity, the meridians also have length $m\pi/n$. Note that this discussion did not involve the regularity of the metric at the poles.

Theorem 4.21. *Up to the sign of $h = f - 1$, the isometry classes of smooth or real analytic metrics of revolution on S^2 with all geodesics closed and simple of length 2π are in one-one correspondence with metrics of the form*

$$g = (1 + h(\cos r))^2\,dr^2 + \sin^2 r\,d\theta^2, \tag{4.22}$$

where $h\colon [-1,1] \to (-1,1)$ is a smooth or real analytic function, respectively, which is odd about 0 and such that $h(-1) = h(1) = 0$.

Proof. The zeros of a non-trivial Killing field on a connected surface are isolated and of index one. Hence, by the Poincaré-Hopf index theorem, a non-trivial Killing field X of a metric g on S^2 has exactly two zeros; call them north and south pole N and S, respectively. Then N and S are fixed points of the flow (ϕ_t) of X. It follows easily that a minimizing unit speed geodesic $c\colon [0,a] \to S^2$ from N to S is perpendicular to X and that (ϕ_t) is periodic. Normalizing X so that the period of (ϕ_t) is 2π, we arrive at a correspondence $(s,\theta) \longleftrightarrow \phi_\theta(c(s))$ as in (4.1) such that $X = \partial/\partial\theta$ and g is of the form (4.3). Since g is smooth or real analytic, the change to the normal form (4.15) does not change regularity; that is, f is smooth or real analytic, respectively.

Now we consider geodesics γ as in Lemma 4.16. By Example 4.19 and the discussion further up, we must have

$$\Delta\theta = \int_{r_0}^{\pi - r_0} \frac{f(\cos r)\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2}\sin r}\,dr = \pi,$$

for any $r_0 \in (0, \pi/2)$. Writing $f = 1 + h$, this holds if and only if

$$\int_{r_0}^{\pi - r_0} \frac{h(\cos r)\sin r_0}{(\sin^2 r - \sin^2 r_0)^{1/2}\sin r}\,dr = 0,$$

for any $r_0 \in (0, \pi/2)$, by Example 4.19. It is easy to see that the latter condition holds if and only if h is odd with respect to $\pi/2$. From the oddness of h about $\pi/2$ and (4.17), we also get that the length of a geodesic between two consecutive events $r = r_0$ and $r = \pi - r_0$ as above is equal to π, as on the unit sphere.

Note that for any metric of the form in (4.22), except for the standard metric, the space of Killing fields is one-dimensional and generated by $\partial/\partial\theta$. It then follows that the parallels are invariantly defined as flow lines of $\partial/\partial\theta$. The parallels come in pairs of length $2\pi \sin r$, and the distance of them to the equator is an isometry invariant of the metric which determines h up to sign. Hence metrics g_1 and g_2 of the form (4.22) and with $h_2 \neq \pm h_1$ are not isometric. $\square$

Exercise 4.23. Let g be a metric on S^2 as in (4.22). Then, up to the parametrization, the meridians $\theta = \theta_0$ are geodesics and $\partial/\partial\theta$ is a Jacobi field along them. The speed of the meridians is $\mu(r) = f(\cos r) = 1+h(\cos r)$. Write $\partial/\partial\theta = \sin r\, E$, where E is of length one (and therefore parallel along the meridians) and show that the curvature of g is given by

$$K = K(r) = \frac{1 + h(\cos r) - \dot{h}(\cos r)\cos r}{(1 + h(\cos r))^3}.$$

Zoll's original surfaces were real analytic surfaces of revolution in $\mathbb{R}^3$. Following his Ansatz as in (4.24) below, we derive his main examples. By Proposition 4.9, we know that, up to a motion of the ambient Euclidean space, the surface should be of the form

$$(x, y, z) = (\sin r \cos\theta, \sin r \sin\theta, z_\pm(\sin r)), \quad 0 \le r \le \pi, \tag{4.24}$$

where we normalize the height functions $z_\pm$ by $z_\pm(1) = 0$. The height functions z_+ and z_- are responsible for the part of the surface above and below the (x, y)-plane, respectively. We also have

$$\cos^2(r)(1 + \dot{z}_\pm^2(\sin r)) = (1 + h(\cos r))^2. \tag{4.25}$$

Thus a necessary condition for the existence of the surface is

$$(1 + h(\cos r))^2 \ge \cos^2 r. \tag{4.26}$$

We set $\rho = \sin r$ and $\eta = \cos r$. Then $\eta^2 + \rho^2 = 1$ and

$$\sqrt{1 + \dot{z}_\pm^2(\rho)} = \frac{1}{\sqrt{1-\rho^2}} \pm \frac{h(\sqrt{1-\rho^2})}{\sqrt{1-\rho^2}} =: \frac{1}{\sqrt{1-\rho^2}} \pm \varphi(\rho) \ge 1, \tag{4.27}$$

where φ is real analytic and $\varphi \equiv 0$ corresponds to the standard sphere. For $0 \le \rho < 1$, the right hand side is given by

$$\frac{1}{\sqrt{1-\rho^2}} \pm \varphi(\rho) = 1 + \frac{1}{2}\rho^2 + \frac{1\cdot 3}{2\cdot 4}\rho^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\rho^6 + \cdots \pm \varphi(r), \tag{4.28}$$

The right hand side should be ≥ 1 and equal to 1 at $\rho = 0$. Zoll's main candidate is $\varphi_c(\rho) = c\rho^2$, where $0 \le c \le 1/2$. Then $h_c(\eta) = c\eta(1-\eta^2)$ is an odd real analytic function which vanishes in ± 1. It has maximum $2c/3\sqrt{3}$, hence it defines a real analytic metric of revolution as in Proposition 4.14 under the weaker restriction $0 \le c < 3\sqrt{3}/2$. However, in the following discussion, we require $0 \le c \le 1/2$.

To show the regularity of the surface S_c of revolution associated to φ_c, we consider

$$\zeta = \zeta(r) = \begin{cases} z_+(\sin r) & \text{for } 0 \le r \le \pi/2, \\ z_-(\sin r) & \text{for } \pi/2 \le r \le \pi. \end{cases} \tag{4.29}$$

Then $c = c(r) = (\sin r, \zeta(r))$ is a profile curve for S_c. We have

$$\begin{aligned} \dot\zeta(r) &= -\sqrt{(1 + h_c(\cos r))^2 - \cos^2 r} \\ &= -\sqrt{(1 + 2c\cos r)\sin^2 r + c^2\cos^2 r \sin^4 r} \end{aligned} \tag{4.30}$$

and hence $\dot\zeta$ is a real analytic function of r on $(0, \pi/2)$, therefore also ζ. We thus see that S_c, so far except for north and south pole, is real analytic.

Observe that S_c is real analytic at north and south pole if and only if the height functions $z_\pm$ are even real analytic functions of ρ about $\rho = 0$. This holds if and only if the derivatives $\dot z_\pm$ are odd real analytic functions of ρ about $\rho = 0$. Now we have

$$\begin{aligned} \dot z_\pm^2(\rho) &= \big(\frac{1}{\sqrt{1-\rho^2}} \pm c\rho^2\big)^2 - 1 \\ &= \frac{1}{1-\rho^2} \pm \frac{2c\rho^2}{\sqrt{1-\rho^2}} + c^2\rho^4 - 1 \\ &= \rho^2 + \rho^4 + \dots \pm 2c\rho^2\big(1 + \frac{1}{2}\rho^2 + \frac{1\cdot 3}{2\cdot 4}\rho^4 + \dots\big) + c^2\rho^4 \\ &= (1 \pm 2c)\rho^2 + (1 \pm c + c^2)\rho^4 + \text{higher order terms.} \end{aligned} \tag{4.31}$$

For the functions on the right hand side to be the squares of odd real analytic functions of ρ, we actually need $0 \le c < 1/2$, slightly stronger than Zoll's assumption $0 \le c \le 1/2$.

Exercise 4.32. 1) For $c \in [0, 1/2)$, show that the curvature of S_c is strictly positive. In other words, S_c is strictly convex in $\mathbb{R}^3$.
2) Discuss the surface S_d associated to the choice $\varphi_d = \varphi_d(\rho) = d\rho^4$. What are appropriate $d \ge 0$? For which d does S_d have regions with negative curvature? (Compare with the figures on page 111 in [4].)

5. Blaschke manifolds

Let M be a compact and connected Riemannian manifold. We say that M is a *Blaschke manifold with respect to* $p \in M$ if $\mathrm{inj}_M(p) = \mathrm{rad}_M(p)$. We say that M is a *Blaschke manifold* if $\mathrm{inj}(M) = \mathrm{diam}(M)$. It is easy to see that the latter holds if and only if M is a Blaschke manifold with respect to any $p \in M$. The only known examples of Blaschke manifolds are the compact symmetric spaces of rank one, that is, spheres and projective spaces together with their standard metrics. The *(generalized) Blaschke conjecture* says that these are in fact the only examples.

Let $k \in \{1, \dots, m-1\}$ and $l > 0$. Following [4], we say that M is an *Allamigeon-Warner manifold of type* (k, l) with respect to $p \in M$, if $\mathrm{con}(v) = l$ and if the multiplicity of $\gamma_v(l)$ as a conjugate point of p along γ_v is k, for all $v \in S_pM$. We say that M is an *Allamigeon-Warner manifold* if it is an Allamigeon-Warner manifold (of some type) at any $p \in M$. It is then easy to see that the type does not depend on p. Clearly, the Allamigeon-Warner property is stable under Riemannian covers and subcovers.

It is clear from our discussion in Subsection 3 that M is a Wiedersehen manifold if and only if M is an Allamigeon-Warner manifold of type $(m-1, l)$ for some $l > 0$. The Blaschke conjecture is known for these:

Theorem 5.1 (Berger-Kazdan [4])**.** *If M is a Wiedersehen manifold, then M has constant positive sectional curvature.*

The proof is involved and will not be presented in these lecture notes. Our aim is to develop the general picture around the Blaschke conjecture. We will see that Allamigeon-Warner and Blaschke manifolds describe the same class of manifolds, up to Riemannian covers or subcovers. We will also see that the pointed versions of both concepts are, more or less, completely understood, where we say that M is a *pointed Blaschke* or *Allamigeon-Warner manifold* if M is a Blaschke or Allamigeon-Warner manifold at some point $p \in M$.

Exercise 5.2. After studying Subsections 5.1 and 5.2, prove the two 'It is easy to see' statements from the beginning of this subsection.

5.1. **Pointed Allamigeon-Warner manifolds.**

In what follows, we assume that M is an Allamigeon-Warner manifold of type (k, l) with respect to a given point $p \in M$. Rescaling the metric, we may assume that $l = 1$. This will make the presentation easier.

Let f be the restriction of the exponential map to the unit sphere $S = S_pM$ in T_pM,

$$f = \exp_p|_S, \tag{5.3}$$

and define $F\colon S \to SM$ by

$$F(v) = \dot\gamma_v(1). \tag{5.4}$$

Note that $f = \pi \circ F$.

Proposition 5.5. *We have: 1) f has constant rank $m-k-1$.*
2) The connected components of the levels of f, that is, the leaves of the foliation Φ associated to the smooth distribution $\ker f_{*v}$*, $v \in S$, are diffeomorphic to the sphere S^k.*

3) For each $v \in S$, F maps the leave $L = L_v$ of Φ through v diffeomorphically to the unit sphere in $(\operatorname{im} f_{*v})^\perp$.
4) $F\colon S \to SM$ is an embedding.

Proof. The Gauss lemma implies that the kernel of $\exp_{p*v}$ is perpendicular to v, for all non-zero $v \in T_pM$. Hence the multiplicity of $\gamma_v(1)$ as a conjugate point of p along γ_v equals the dimension of $\ker f_{*v}$, and 1) follows.

A general version of the implicit function theorem maintains that, for each v in S, there exist coordinate charts $x = (x^1, \dots, x^{m-1})$ about v in S and $y = (y^1, \dots, y^m)$ about $f(v)$ in M such that

$$(y \circ f \circ x^{-1})(x^1, \dots, x^{m-1}) = (x^1, \dots, x^{m-k-1}, 0, \dots, 0). \tag{5.6}$$

It follows that the distribution $\ker f_*$ on S is smooth and integrable and that the image of the domain of x under f is a submanifold W of M of dimension $m - k - 1$. (In Theorem 5.8 below we will see that W is actually part of a compact immersed manifold N.) The non-explicit assertions of 2) follow. The remaining assertion of 2) is immediate from 3).

Since F is the restriction of the time one map of the geodesic flow to S, we get that $F\colon S \to SM$ is an embedding. By the definition of L, all the geodesics γ_w, $w \in L$, hit $q = f(v) = \gamma_v(1)$ at time 1. With respect to a coordinate chart x about v as above such that $x(v) = 0$, the vectors w with coordinates $(0, \dots, 0, x^{m-k}, \dots, x^{m-1})$ constitute a neighborhood of v in the leaf $L = L_v$ of Φ. It follows from (5.6) that $\operatorname{im} f_{*v} = \operatorname{im} f_{*w}$ for all such w. Since this argument applies to any $v \in L$ and L is connected, we conclude that $\operatorname{im} f_{*v}$ is independent of $v \in L$.

By the first variation formula, the geodesics γ_w, $w \in L$, hit the image of f perpendicularly at q, that is, $F(w) = \dot\gamma_w(1) \in (\operatorname{im} f_{*v})^\perp$ for all w. Now L is a compact and connected manifold of dimension k, and the restriction of F to L is an embedding of L. Furthermore, the unit sphere in $(\operatorname{im} f_{*v})^\perp$ is also a compact and connected manifold of dimension k and contains the image of L under the embedding F. It follows that F is a diffeomeorphism between L and the unit sphere in $(\operatorname{im} f_{*v})^\perp$. $\square$

Exercise 5.7. Let M be a foliated manifold, and let p, q be points in a leaf L of the foliation. Then there is a compactly supported (in particular, complete) vector field X on M such that
1) X is tangent to the foliation and
2) $\varphi_1(p) = q$, where (φ_t) denotes the flow of X.

Theorem 5.8. *Let N be the space of leaves of Φ and $\nu\colon S \to N$ be the projection. Then N is a smooth manifold of dimension $m - k - 1$ in a natural way such that ν is a submersion and such that the following holds:*
1) N is compact and simply connected.
2) f factors over an immersion $g\colon N \to M$, $f = g \circ \nu$.
3) F induces an isomorphism between $\nu\colon S \to N$ and the normal sphere bundle of the immersion g.

Proof. We show first that N has a natural atlas of coordinate charts such that the coordinate transitions are smooth. The heart of the matter are coordinates x and y as in (5.6). Let $U \subseteq S$ be the domain of x and assume without loss of generality that $x(U) = U' \times U''$, where $U' \subseteq \mathbb{R}^{m-k-1}$ and $U'' \subseteq \mathbb{R}^k$ are open and path connected. Split $x = (x', x'')$ with

$$x' = (x^1, \dots, x^{m-k-1}) \quad \text{and} \quad x'' = (x^{m-k}, \dots, x^{m-1}).$$

Then U intersects the levels of f in the subsets $x' = \text{constant}$. Since these are diffeomorphic to U'', they are path connected and, therefore, belong to pairwise different leaves of Φ. Thus, denoting $\nu(v)$ by $\bar{v}$, $v \in S$, and setting $\bar{U} = \nu(U)$, the map

$$\bar{x} \colon \bar{U} \to U', \quad \bar{x}(\bar{v}) = x'(v),$$

establishes a bijection between the leaves of Φ passing through U and U'.

We show next that the coordinate transformations between such coordinate charts are smooth. To that end, let v_0 and v_1 belong to the same leaf, L, of Φ and consider coordinate charts x_0 about v_0 and y_0 about $f(v_0)$ and, respectively, x_1 about v_1 and y_1 about $f(v_1)$ as above. Let X be a vector field on S tangent to Φ such that the time one map φ_1 of the flow (φ_t) of X maps v_0 to v_1; compare with Exercise 5.7. Then $\varphi_1 \circ x_0^{-1}$ maps a sufficiently small neighborhood $V = V' \times V''$ of $x_0(v_0)$ into a neighborhood of v_1 which is contained in the domain of x_1. Moreover, since X is tangent to Φ, the flow lines of X stay in their respective leaves of Φ. Therefore the smooth functions

$$\psi^j = x_1^j \circ \phi_1 \circ x_0^{-1}, \quad 1 \le j \le m-k-1,$$

only depend on the first $m-k-1$ coordinates in V. Thus we obtain an induced map $(\bar{\psi}^1, \dots, \bar{\psi}^{m-k-1})$ in a small neighborhood of $\bar{x}_0(\bar{v}_0)$ in U_0'. It is smooth and, locally about $\bar{v}_0$, it is equal to the coordinate transition $\bar{x}_1 \circ \bar{x}_0^{-1}$. It follows that coordinate transitions are smooth and, therefore, that they are diffeomorphism.

We now say that a subset V of N is open if $\bar{x}(V) \subseteq U'$ is open, for all charts $\bar{x}$ as above. Since coordinate transitions are homeomorphisms, we get that the coordinate charts $\bar{x}$ as above are homeomorphisms. It is not hard to see that any two different leaves L_0 and L_1 of Φ have neighborhoods U_0 and U_1 in S such that no leaf of Φ intersects U_0 and U_1 simultaneously. It follows that N with the above topology is a Hausdorff space. In conclusion, we obtain that N is a smooth manifold of dimension $m-k-1$ such that $\nu \colon S \to N$ is a submersion.

The remaining assertions are easy to see: Since N is the image of S, it is compact. If $0 < k = m-1$, then f is constant and N is a point. If $0 < k < m-1$, then S is simply connected and the fiber S^k of ν is path connected. Thus N is simply connected in either case, and 1) follows.

By the definition of N, there is a smooth map $g \colon N \to M$ with $f = g \circ \nu$. The lift to S of a vector u in the kernel of g_* belongs to the kernel of f_* and thus is tangent to the corresponding fiber of ν. Hence $u = 0$, and hence g is an immersion. Thus 2) follows. The remaining assertion 3) is an immediate consequence of Proposition 5.5.3. $\square$

Since geodesics are not minimal any more after the first conjugate point, M is the image of the closed unit ball $\bar{B} = \bar{B}(0_p, 1) \subseteq T_pM$ under the exponential map. Since M is an Allamigeon-Warner manifold with $l = 1$, the map

$$(0,1) \times S \to M, \quad (t, v) \mapsto \gamma_v(t),$$

is a local diffeomeorphism. We also have $\gamma_v(t) = \gamma_{-F(v)}(1-t)$, and this identity suggests a diffeomorphism G between $(0,1) \times S$, thought of as open unit ball $B = B(0_p, 1) \subseteq T_pM$ minus center 0_p, and the open unit disc bundle D in the normal bundle associated to the immersion g minus the zero section N,

$$G(t, v) = (t-1)F(v). \tag{5.9}$$

Thus we obtain a manifold $\tilde{M}$ by gluing B to D via the diffeomorphism G.

Theorem 5.10. *The exponential maps on B and D fit together and define a local diffeomorphism and covering map $\tilde{M} \to M$. Moreover,*
1) $\tilde{M}$ is compact and simply connected;
2) with respect to the induced metric, $\tilde{M}$ is a Blaschke manifold with respect to $\tilde{p} := 0_p$, that is, $\mathrm{inj}_{\tilde{M}}(\tilde{p}) = \mathrm{rad}_{\tilde{M}}(\tilde{p})$.

Proof. By the definition of G, the exponential maps on B and D fit together to a local diffeomorphism $\tilde{M} \to M$. Since $\tilde{M}$ and M are compact and connected, such a local diffeomorphism is a covering map. Now B and D are simply connected and intersect in a set diffeomorphic to $(0,1) \times S^{m-1}$. Hence $\tilde{M}$ is simply connected, by the Seifert-van Kampen theorem.

As for the last claim, note that the lines $(0,1) \times \{v\}$ in $(0,1) \times S \subseteq B$ correspond to the geodesics in $\tilde{M}$ issuing from $\tilde{p}$. Since they are pairwise disjoint, we conclude that $\mathrm{inj}_{\tilde{M}}(\tilde{p}) = 1 = \mathrm{rad}_{\tilde{M}}(\tilde{p})$. $\square$

Corollary 5.11. *If M is simply connected, then M is a Blaschke manifold with respect to p.* $\square$

5.2. **Characterizations of pointed Blaschke manifolds.** For a complete and connected Riemannian manifold M and points $p, q \in M$, denote by $\Sigma_{pq} \subseteq S_pM$ the closed subset of unit vectors which are tangent to minimal geodesic segments from p to q. For $q \in \mathrm{Cut}(p)$, denote by C_q the set of unit vectors $u \in S_qM$ such that there is a sequence (q_n) in $\mathrm{Cut}(p) \setminus \{q\}$ converging to q such that $u_n \to u$, where u_n is the unit tangent vector at q tangent to the minimal geodesic from q to q_n.

We assume throughout Lemmata 5.12–5.15 that M is a Blaschle manifold with respect to a given point $p \in M$ and let $l = \mathrm{rad}_M(p)$ and $q \in \mathrm{Cut}(p)$.

Lemma 5.12. *For all $v \in \Sigma_{qp}$ and $w \in C_q$, we have $\langle v, w\rangle \leq 0$.*

Proof. Choose a sequence (q_n) in $\mathrm{Cut}(p) \setminus \{q\}$ such that the sequence of unit tangent vectors w_n at q pointing to q_n converges to w. Then $\langle v, w\rangle > 0$ would imply that $\langle v, w_n\rangle > 0$ for all sufficiently large n. But then we would have $d(p, q_n) < d(p, q)$ for all sufficiently large n. On the other hand, we have $d(p, q_n) = d(p, q) = l = \mathrm{rad}_M(p)$ for all n since $q_n \in \mathrm{Cut}(p)$. $\square$

Lemma 5.13. *For all $u \in S_qM$ with $u \notin \Sigma_{qp} \cup C_q$, there exist $v \in \Sigma_{qp}$ and $w \in C_q$ with $\langle v, w\rangle = 0$ such that $u = av + bw$ with $a, b > 0$.*

Proof. Let σ be the unit speed geodesic with $\dot{\sigma}(0) = u$. Then, since $u \notin C_q$, we have $d(p, \sigma(\varepsilon)) < l = \mathrm{rad}_M(p)$ for all sufficiently small $\varepsilon > 0$. Therefore, for these there exists a unique minimal geodesic γ_ε from p to $p_\varepsilon = \sigma(\varepsilon)$. Then $q_\varepsilon = \gamma_\varepsilon(l) \in \mathrm{Cut}(p)$. Since $u \notin \Sigma_{qp}$, $q_\varepsilon \neq q$ for all sufficiently small $\varepsilon > 0$. Let w_ε be the unit tangent vector at q pointing at q_ε.

Since $d(p,\sigma(\varepsilon)) \to l$ as $\varepsilon \to 0$, we have $d(p_\varepsilon, q_\varepsilon) \to 0$ and therefore $q_\varepsilon \to q$ as $\varepsilon \to 0$. By compactness, there exists a sequence $\varepsilon_n \to 0$ such that the sequences $\gamma_n = \gamma_{\varepsilon_n}$ and $w_n = w_{\varepsilon_n}$ converge to a minimal unit speed geodesic γ from p to q and a unit tangent vector $w \in C_q$, respectively. By Lemma 5.12, we have $\angle(v,w) \geq \pi/2$, where $v = -\dot\gamma(l) \in \Sigma_{qp}$.

Thus we obtain a sequence of isosceles geodesic triangles $(\gamma|_{[0,l]}, \gamma_n|_{[0,l]}, \beta_n)$, where β_n is the shortest geodesic segment from q to $q_n = q_{\varepsilon_n}$, together with a sequence of points $p_n = \sigma(\varepsilon_n)$ on $\gamma_n|_{[0,l]}$ such that the length of β_n tends to 0 and $\gamma_n \to \gamma$. Using Fermi coordinates along γ, we see that, after rescaling the metric of M so that β_n has length one, we get a Euclidean half strip as a limit, with base β of length one in the direction of w, with sides in the direction of v, and with interior angles $\pi/2$ at the vertices q and q_∞ of the base. Moreover, the sequence p_n tends to the limit point p_∞ on the q_∞-side which lies in the direction of u as seen from q. Hence $\langle v,w\rangle = 0$ and u is a positive linear combination of v and w. □

Lemma 5.14. *For all $v_0, v_1 \in \Sigma_{qp}$ with $v_0 \neq \pm v_1$, the great circle arc between v_0 and v_1 is contained in Σ_{qp}. That is, Σ_{qp} is* π-convex.

Proof. Let u be on the great circle arc from v_0 to v_1 and assume that $u \neq \Sigma_{qp}$. Write $u = av_0 + bv_1$ with $a,b \geq 0$. Now $u \notin C_q$, by Lemma 5.12, since at least one of the angles between u and v_0 or u and v_1 is acute. Hence we can write $u = a'v + b'w$ with $v \in \Sigma_{qp}$ and $w \in C_q$ such that $\langle v,w\rangle = 0$ and $a', b' > 0$. By Lemma 5.12, the (spherical) distance between w and v_0 and between w and v_1 is at least $\pi/2$, hence also the distance to u. This contradict $\langle v,w\rangle = 0$ which says that the distance between v and w is $\pi/2$ and hence that the distance between w and u is less than $\pi/2$. □

Lemma 5.15. *For all $v \in \Sigma_{qp}$, there exists $v' \in \Sigma_{qp}$ with $\langle v, v'\rangle < 0$.*

Proof. In the decomposition of $\Sigma = \Sigma_{qp}$ as in Exercise 5.23, we cannot have $\Sigma = \Sigma_2$ since otherwise Σ would be contained in an open hemisphere. But that is not possible, by Berger's lemma, since q is at maximal distance from p. That is, q is a critical point of the distance function $d(p,.)$ in the sense of Gromov-Grove-Shiohama. It follows that Σ contains a unit vector v_0 such that $-v_0$ is also contained in Σ.

If v is not perpendicular to v_0, then $\langle v, v'\rangle < 0$ for $v' = v_0$ or $v' = -v_0$, and the assertion follows. Hence we may assume that $\langle v, v_0\rangle = 0$. Then the great circle arc from v_0 to $-v_0$ through v is contained in Σ, by Lemma 5.14. This great circle arc gives rise to a smooth one-parameter family γ_s, $0 \leq s \leq 1$, of minimal geodesics from p to q such that $\dot\gamma_0(l) = v_0$ and $\dot\gamma_1(l) = -v_0$. In particular, there are Jacobi fields J_0 and J_1 along γ_0 and γ_1, respectively, such that

$$J_0(0) = J_1(0) = 0, \quad J_0(l) = J_1(l) = 0, \quad J_0'(l) = J_1'(l) = v.$$

Let now $w \in T_qM$ with $\langle v,w\rangle < 0$. Assume first that $\langle v_0, w\rangle \leq 0$, and let W be a smooth vector field along γ_0, perpendicular to γ_0, such that $W(0) = 0$ and $W(l) = w$. For a constant $k > 0$, to be determined later, let $V = J_0 + kW$ and (c_s) be a variation of $c_0 = \gamma_0$ with $c_s(0) = p$ and variation field V such that $\gamma = \gamma(s) = c_s(l)$ is the geodesic with initial velocity kw.

Then we have, for the length $L(s)$ of c_s,

$$L(0) = l, \quad L'(0) = k\langle u_0, w\rangle \le 0, \quad L''(0) = I(V, V),$$

where

$$I(V, V) = \int_0^l \{\langle V', V'\rangle - R(V, \dot\gamma_0)\dot\gamma_0, V\rangle\}$$

denotes the index form of γ_0. Now $I(J_0, J_0) = 0$ since J_0 is a Jacobi field along γ_0 vanishing at $t = 0$ and $t = l$. Furthermore,

$$I(J_0, kW) = \langle J_0'(l), kW(l)\rangle = k\langle v, w\rangle < 0.$$

Hence we get, for $k > 0$ sufficiently small, that the first derivative of L is non-positive and the second is strictly negative. It follows that $d(p, \gamma(s)) < l$ for all sufficiently small $s > 0$. The same conclusion also follows if $\langle v_0, w\rangle \ge 0$, now using variations of γ_1. Thus all points close to q in a direction sufficiently close to $-v$ have distance $< l$ to p and, hence, do not belong to $\mathrm{Cut}(p)$. We conlude that $-v$ does not belong to C_q.

To arrive at the conclusion of the lemma, we may also assume that $-v$ is not contained in Σ_{qp}. By what we found above and Lemma 5.13, we may then write $-v = av' + bw$ with $v' \in \Sigma_{qp}$ and $w \in C_q$ such that $\langle v', w\rangle = 0$ and $a, b > 0$. Then

$$\langle v, v'\rangle = -\langle av' + bw, v'\rangle = -a < 0. \qquad \square$$

Theorem 5.16. *For any $p \in M$, the following conditions are equivalent:*
1) M is a Blaschke manifold with respect to p.
2) $\mathrm{Cut}_T(p) \subseteq T_pM$ is a round sphere about 0_p.
3) For any $v \in S_pM$, we have $\gamma_v(2\,\mathrm{cut}(v)) = p$.
4) For any $q \in \mathrm{Cut}(p)$, Σ_{qp} is a great sphere in S_qM.

Proof. 1) $\Rightarrow$ 4): This is the hardest step of the proof. Therefore we out-sourced the mayor part of the arguments to the above Lemmata 5.12–5.15. By Lemma 5.14, Σ_{qp} is a π-convex subset of S_qM. Hence Σ_{qp} is the spherical join of two π-convex subsets Σ_1 and Σ_2 as in Exercise 5.23. Now $\Sigma_2 = \emptyset$, by Lemma 5.15, and hence $\Sigma = \Sigma_1$ is a great sphere.

4) $\Rightarrow$ 3): Let $v \in S_pM$. Then $q = \gamma_v(\mathrm{cut}(v))$ is in the cut locus of p and $-\dot\gamma_v(\mathrm{cut}(v))$ is tangent to γ_v run backwards, a minimal geodesic from q to p. Since the set of unit tangent vectors in T_qM tangent to minimal geodesics from q to p is a great sphere in S_qM, we conclude that $\dot\gamma_v(\mathrm{cut}(v))$ is also tangent to a minimal geodesic from q to p. In other words, γ_v is back at p at time $2\,\mathrm{cut}(v)$.

3) $\Rightarrow$ 2): Following the argument in the proof of Lemma 3.3, we show first that $\mathrm{cut} = \mathrm{cut}(v)$ is not only continuous, but that it depends smoothly on $v \in S_pM$. To that end, we choose a sufficiently small $\varepsilon > 0$ and let $S(p, \varepsilon) = \{q \in M \mid d(p, q) = \varepsilon\}$, the geodesic sphere of radius ε about p. Then each γ_v, $v \in S_pM$, hits the hypersurface $S(p, \varepsilon)$ at time $t(v) = 2\,\mathrm{cut}(v) - \varepsilon$ and hits it transversally; in fact, perpendicularly. Hence $t = t(v)$ depends smoothly on $v \in S_pM$, by the implicit function theorem. It follows that $\mathrm{cut}(v)$ also depends smoothly on $v \in S_pM$.

Let now $v = v(s)$ be a curve in S_pM and consider the corresponding geodesic variation $\gamma = \gamma_s(t) = \gamma_{v(s)}(t)$. Then we have $\gamma_s(2\,\mathrm{cut}(v(s))) = p$, and hence

$$\begin{aligned}0 &= \frac{d}{ds}\gamma_s(2\operatorname{cut}(v(s)))\\ &= J(s, 2\operatorname{cut}(v(s))) + 2\dot\gamma_s(2\operatorname{cut}(v(s)))\cdot\frac{d\operatorname{cut}(v(s))}{ds},\end{aligned}$$

where $J = \partial\gamma/\partial s$ is the family of associated Jacobi fields. Since $J(s, 0) = 0$ and $J'(s, 0) = dv/ds$, $J(s, t)$ is perpendicular to $\dot\gamma_s(t)$, for all s, t. Hence the above calculation shows that $d\operatorname{cut}(v(s))/ds$ vanishes. It follows that cut is constant on S_pM.
2) $\Rightarrow$ 1): This is just a reformulation. □

Corollary 5.17. *If M is a Blaschke manifold, then Σ_{pq} is a great sphere in S_pM, for all $p \in M$ and $q \in \operatorname{Cut}(p)$. In particular, S_pM is fibered into the great spheres Σ_{pq}, where $q \in \operatorname{Cut}(p)$.*

Proof. This follows immediately from Theorem 5.16 by noting that q belongs to $\operatorname{Cut}(p)$ if and only if p belongs to $\operatorname{Cut}(q)$. □

Corollary 5.17 is the motivation for attacking the generalized Blaschke conjecture via great sphere fibrations of spheres, as has been done in many studies.

Proposition 5.18. *Let M be a Blaschke manifold at p and $l := \operatorname{rad}_M(p)$. Then 1) $\gamma_v|_{[0,2l]}$ is a simple geodesic loop at p, for all $v \in S_pM$;*
2) there is a $k \in \{0, \dots, m-1\}$ such that γ_v does not have conjugate points on $(0, l)$ and $(l, 2l)$ and such that $\gamma_v(l)$ is a conjugate point of p along γ_v of multiplicity k, for all $v \in S_pM$.

Proof. 1) is clear from Theorem 5.16. Since any γ_v is minmal up to l, there is no conjugate point to p along γ_v up to time l. Now $p = \gamma_v(2l)$ is conjugate to p along γ_v of multiplicity $m-1$ since all unit speed geodesics from p come back to p at time $2l$. If a non-trivial Jacobi field J along γ_v with $J(0) = 0$ would vanish at a time $t \in (l, 2l)$, then, since $J(2l) = 0$, $\gamma_v(t)$ would be a conjugate point to p along γ_w at time $2l - t \in (0, l)$, where $w = -\dot\gamma_v(2l) \in S_pM$. It follows that only $\gamma_v(l)$ can possibly be a conjugate point to p along γ_v. Since $\gamma_v(l + \varepsilon)$ is not a conjugate point of p along γ_v for any $v \in S_pM$, we get that the index of $\gamma_v|_{[0,l+\varepsilon]}$ does not depend on $v \in S_pM$, hence is equal to a constant $k \in \{0, \dots, m-1\}$. By what we said above, k is the multiplicity of $\gamma_v(l)$ as a conjugate point of p along γ_v. □

Theorem 5.19. *Let M be a Blaschke manifold at p with associated numbers k and l as in Proposition 5.18. Then we have:*
1) If $k \geq 1$, then M is a simply connected Allamigeon-Warner manifold of type (k, l).
2) If $k = 0$, then $\pi_1(M) = \mathbb{Z}/2$ and M is an Allamigeon-Warner manifold of type $(m-1, 2l)$.

Proof. The first claim is clear from Theorem 5.10. As for the second claim, we note that M is an Allamigeon-Warner manifold of type $(m-1, 2l)$ since all unit speed geodesics starting in P come back to p at time $2l$. It also follows that the covering $\tilde M \to M$ as in Theorem 5.10 is twofold. □

Example 5.20. Let $k \in \{0, \dots, m-1\}$ and $l > 0$. Let N be a compact manifold of dimension $m-k$ and $E \to N$ be a vector bundle of rank k. Let D_E and S_E be the disc and sphere bundle of E with respect to some given Riemannian metric h on E as a vector bundle; that is, h_x is an inner product on the fiber E_x, for each $x \in N$. Assume that S_E is diffeomorphic to the sphere S^{m-1}. Identify the zero section of E with N and $D_E \setminus N$ with $(0,l) \times S_E$. Let g_E be a Riemannian metric on D_E such that the restriction of g_E to $(0,l) \times S_E$ is of the form $ds^2 + g_{E,s}$ where $g_{E,s}$ is a smooth family of Riemannian metrics on S_E.

Let $D^m \subseteq \mathbb{R}^m$ be the ball of radius l. Identify $D^m \setminus \{0\}$ with $(0,l) \times S^{m-1}$. Let g_D be a Riemannian metric on D such that the restriction of g_D to $(0,l) \times S^{m-1}$ is of the form $dt^2 + g_{D,t}$, where $g_{D,t}$ is a smooth family of Riemannian metrics on S^{m-1}.

Let $F\colon S_E \to S^{m-1}$ be a diffeomorphism. Then we obtain a compact manifold $M = D^m \cup_F D_E$ by identifying

$$D^m \setminus \{0\} \ni (t,x) = (l-t, F(x)) \in D_E \setminus N.$$

The origin $0 \in D$ is a distinguished point of M and will be denoted by p.

Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a monotone smooth function such that $\chi(t) = 1$ for $t \le l/3$ and $\chi(t) = 0$ for $t \ge 2l/3$. Then we obtain a smooth Riemannian metric g on M by letting $g = g_D$ in a neighborhood of p, $g = g_E$ in a neighborhood of N, and

$$\begin{aligned} g &= \chi(t)(dt^2 + g_{D,t}) + (1-\chi(t))(ds^2 + g_{E,l-t}) \\ &= dt^2 + \chi(t) g_{D,t} + (1-\chi(t)) g_{E,1-t} \end{aligned}$$

on $M \setminus (\{p\} \cup N)$, where we observe that $ds = ds(t) = -dt$. It follows easily that the t-lines are unit speed geodesics. As a consequence, g turns M into a Blaschke manifold of type (k,l) with respect to p. Our previous discussion shows that any Riemannian manifold M, which is a Blaschke manifold of type (k,l) with respect to some point $p \in M$, arises in this way.

Exercise 5.21. Let N be a Riemannian manifold and $E \to N$ be a vector bundle of rank k. Let h be a Riemannian metric on E as a vector bundle over N, that is, h_x is an inner product on the fiber E_x, for all $x \in N$.
1) Show that there is a Riemannian metric g on E such that the restriction g_x of g to the fiber E_x coincides with h_x, for each $x \in N$, and such that $E \to N$ is a Riemannian submersion with totally geodesic fibers. Conclude that $g = g_E$ is a Riemannian metric on E as required in Example 5.20.
2) Let (ϕ_α) be a partition of unity subordinate to an open covering (U_α) of N such that there are trivializations $f_\alpha\colon E_\alpha \to U_\alpha \times \mathbb{R}^k$ of E over U_α which preserve inner products on the fibers. Let (g_α) be a family of Riemannian metrics on the restrictions D_α of the disc bundle D_E of E to U_α such that each g_α is of the form $ds^2 + g_{\alpha,s}$ with respect to the identification of $D_\alpha \setminus U_\alpha$ with $(0,l) \times U_\alpha \times S^{k-1}$ (induced by f_α), where $g_{\alpha,s}$ is a family of Riemannian metrics on $U_\alpha \times S^{k-1}$ which is smooth in s, for each α. Show that

$$g_E = \sum \phi_\alpha g_\alpha$$

is a Riemannian metric on D_E as required in Example 5.20 and that, conversely, any such metric can be obtained in the described way.

Exercise 5.22. Discuss restrictions on the topology of compact manifolds N which admit a vector bundle $E \to N$ whose sphere bundle is diffeomorphic to a sphere.

Exercise 5.23. Let S be the unit sphere in some Euclidean space E and Σ be a closed π-convex subset of S. Show that there is an orthogonal (and possibly trivial) decomposition $E = E_1 \oplus E_2$ such that $\Sigma_1 = \Sigma \cap E_1$ is the unit sphere in E_1 and such that $\Sigma_2 = \Sigma \cap E_2$ has circumradius $< \pi/2$. Conclude also that Σ is the *spherical join* of Σ_1 with Σ_2 if Σ_1 and Σ_2 are non-empty; that is, Σ then consists of all points which lie on great circle arcs of length $\pi/2$ from Σ_1 to Σ_2.

5.3. **The cohomology of pointed Blaschke manifolds.** Let M be a Blaschke manifold with respect to some point $p \in M$. Assume that M is simply connected and of type (k, l), where $k \in \{1, \dots, m-1\}$.

Fix a point $q \in M \setminus \mathrm{Cut}(p)$; that is, $d(p, q) = l - \varepsilon$ for some $\varepsilon \in (0, l)$. Then there is precisely one geodesic loop γ at p of length $2l$ which passes through q, and q divides γ into two segments γ_1 of length $l - \varepsilon$ and γ_2 of length $l+\varepsilon$ from p to q. All other geodesic segments from p to q are obtained by concatenation of geodesic loops at p with γ_1 and γ_2. Geodesic loops at p have lengths $2il$, $i \geq 0$, and thus the set of geodesic segments from p to q is an infinite family (γ_n), $n \geq 1$, where the γ_{2i+1} and γ_{2i+2} are obtained by the concatenation of a geodesic loop of length $2il$ with γ_1 and γ_2, respectively. Thus the set of geodesic segments from p to q is strictly ordered by length,

$$L(\gamma_{2i+1}) = (2i+1)l - \varepsilon \quad \text{and} \quad L(\gamma_{2i+2}) = (2i+1)l + \varepsilon.$$

Note that q is not a conjugate point along any geodesic segment from p to q and that the indices of the geodesic segments are given by

$$\operatorname{ind} \gamma_{2i+1} = i(k+m-1) \quad \text{and} \quad \operatorname{ind} \gamma_{2i+2} = i(k+m-1) + k.$$

Thus the indices are strictly increasing with the lengths of the segments.

We now invoke the critical point theory of the length functional L on the space Ω_{pq} of continuous paths $c\colon [0,1] \to M$ from p to q. There are several ways of making critical point theory precise in this context. We assume that the reader is familiar with at least one of them.

By the first variation of the length, the critical points of L are the geodesic segments from p to q. Moreover, L is a Morse function since q is not a conjugate point of p along any geodesic segment from p to q.

Theorem 5.24. *Let M be a simply connected Blaschke manifold with respect to a point $p \in M$. Let M be of type (k, l), and let q belong to $M \setminus \mathrm{Cut}(p)$. Then the length functional L on Ω_{pq} is a perfect Morse function. In particular, we have*

$$H_j(\Omega_{pq}) = \begin{cases} \mathbb{Z} & \text{if } j = i(k+m-1) \text{ for some } i \geq 0, \\ \mathbb{Z} & \text{if } j = i(k+m-1) + k \text{ for some } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We enumerate the geodesic segments from p to q as above (and parameterize them on $[0,1]$). Then their lengths and indices increase strictly. Thus it suffices to show now that, through each geodesic segment $\gamma_n \in \Omega_{pq}$, there is an embedded closed and orientable manifold P_n in Ω_{pq} of dimension equal to $\operatorname{ind}\gamma_n$ which consists of paths with lengths at most $L(\gamma_n)$. To that end, consider the foliation Φ of S_pM into k-dimensional spheres as in Proposition 5.5. For $n = 2i+1$, let P_n be the manifold of $2i$-tuples $(v_1, w_1, \dots, v_i, w_i)$ of vectors in S_pM, where each w_j belongs to the same leaf of Φ as the corresponding v_j. Then P_n is closed and orientable of dimension $i(k+m-1)$ as required. For each (v_j, w_j), let $\sigma_j\colon [0, 2l] \to M$ be the geodesic loop at p with initial velocity v_j and $\sigma_j'\colon [0, l] \to M$ be the geodesic segment with initial velocity w_j. Let σ_j'' be the restriction of σ_j to its second half $[l, 2l]$. Finally, associate to the tupel $(v_1, w_1, \dots, v_n, w_n)$ the broken geodesic segment

$$\sigma_1' * \sigma_1'' * \dots \sigma_i' * \sigma_i'' * \gamma_1 \in \Omega_{pq}.$$

Thus we have a natural embedding $P_n \to \Omega_{pq}$ as required.

For $n = 2i+2$, the construction is similar. In this case, we let P_n be the manifold of $2i+1$-tuples $(v_1, w_1, \dots, v_i, w_i, w)$ of vectors in S_pM, where each w_j belongs to the same leaf of Φ as the corresponding v_j as above and where w belongs to the same leaf of Φ as $\dot\gamma_2(0)$. To any such $(2i+1)$-tuple we associate the broken geodesic segment

$$\sigma_1' * \sigma_1'' * \dots \sigma_i' * \sigma_i'' * \sigma' * \gamma_2'' \in \Omega_{pq},$$

where the first part is as above, $\sigma'\colon [0, l] \to M$ is the geodesic segment with initial velocity w, and γ_2'' is the restriction of γ_2 to $[l, l+\varepsilon]$. $\square$

Theorem 5.25. *Let M be a simply connected Blaschke manifold with respect to a point $p \in M$ and of type (k, l). Then the cohomology of M is generated by an element in degree $k+1$. In particular, the dimension of M is a multiple of $k+1$.*

Sketch of proof. Let Ω_p be the space of paths $c\colon [0,1] \to M$ with $c(0) = p$. Recall that the end point map

$$\pi\colon \Omega_p \to M, \quad \pi(c) = c(1),$$

is a Serre fibration. Since M is simply connected, the Serre spectral sequence for cohomology applies,

$$E_2^{i,j} = H^i(M, H^j(\Omega_{pq})) \Longrightarrow H^{i+j}(\Omega_p).$$

Since Ω_p is contractible, the assertion is now a rather straightforward consequence of diagram chasing, using Theorem 5.24 and the multiplicative properties of the Serre spectral sequence. $\square$

Results from topology imply that there are only the following possibilities:

$$\begin{aligned} k &= 1, & m &= 2n, \\ k &= 3, & m &= 4n, \\ k &= 7, & m &= 16, \\ k &= m-1, & m &\ge 2. \end{aligned}$$

These are realized by complex projective spaces, quaternionic projective spaces, the Cayley projective plane, and spheres. However, the construction in Example 5.20 shows that there exist further pointed Blaschke manifolds.

6. Harmonic spaces

In Euclidean space $\mathbb{R}^m$, the Laplace equation $\Delta f = 0$ has the solution $||x||^{2-m}$ for $m > 2$ and $\ln ||x||$ for $m = 2$, respectively. It is defined on $\mathbb{R}^m \setminus \{0\}$ and depends only on the distance of x to 0. We read in the literature, e.g., [4, Chapter 6], that, in 1930, Ruse tried to establish the existence of non-trivial solutions of the Laplace equation on a general Riemannian manifold M, depending only on the distance to a point p in M. The attempt was doomed to fail, as is clear from today's perspective, but it was a productive failure.

Following Copson and Ruse [5], we say that M is *harmonic about a point* p *in* M if, on some punctured neighborhood of p, $\Delta r = \phi(r)$ for some function ϕ, where r denotes the distance to p. We say that M is a *harmonic space* if it is harmonic about each of its points. In [5], Copson and Ruse showed that geodesic spheres of sufficiently small radius in harmonic spaces are of constant mean curvature and that harmonic spaces are Einstein spaces. In particular, if the dimension of M is 2 or 3 and M is harmonic, then the curvature of M is constant. This led Copson and Ruse to conjecture that harmonic spaces are of constant curvature, another productive error.

In [16], Lichnerowicz showed that non-flat harmonic spaces are irreducible and reproved the result of Copson and Ruse that harmonic spaces are Einstein spaces. He also oracled whether harmonic spaces, say complete and simply connected, are either flat or symmetric spaces of rank one. This became known as the *Lichnerowicz conjecture* and was studied in many articles. The conjecture holds true in dimension 4, as shown by Lichnerowicz and Walker. The conjecture had, and still has, its fascination and is the source of many insights and results in differential geometry. Today there are counterexamples to the conjecture, see below, but it holds true in the case where M is compact and simply:

Theorem 6.1 (Szabó [22]). *Let M be a compact and simply connected harmonic space. Then M is a compact rank one symmetric space.*

Szabó's proof relies on important previous work, as exposited and developed by Besse and his friends, see [4]. In particular, his proof uses Besse's immersion as discussed on pages 174–178 in [4]. In these notes, we do not prove Theorem 6.1, but discuss Besse's immersion.

Given a compact and simply connected harmonic space of diameter d and an eigenvalue $\lambda > 0$ of the Laplacian Δ of M, there is a smooth $2d$-periodic and even function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 1$ such that, for any point $p \in M$, the function $f = f(q) = \phi(d(p,q))$ is a λ-eigenfunction of Δ, see Subsection 6.3 below.

Theorem 6.2 (Besse's immersion). *Let λ, ϕ be as above and $(f_1, \dots, f_l)$ be an orthonormal basis of the λ-eigenspace of Δ in $L^2(M)$. Then the mapping*

$$F : M \to \mathbb{R}^l, \quad F(p) := C \cdot (f_1(p), \dots, f_l(p)),$$

is a minimal isometric immersion into the sphere $S^{l-1}(R)$ of radius R in $\mathbb{R}^l$, where $C^2 = \operatorname{vol} M/\lambda$ and $R^2 = m/\lambda$. For all points p, q in M,

$$||F(p) - F(q)|| = R\sqrt{2 - 2\phi(d(p,q))}.$$

The map F as in Theorem 6.2 had been considered before [4], not in the context of harmonic spaces, however, but for compact homogeneous spaces. It occurs, for example, in the work [8] of do Carmo and Wallach on minimal isometric immersions of spheres into spheres. That the map F works well for harmonic spaces is a nontrivial insight, and this development was a milestone in the theory of harmonic spaces. Below, we present a proof of Theorem 6.2 which avoids a somewhat unpleasant technical detail in the argument of Besse, concerned with Brownian motion or the fundamental solution of the heat equation and their properties, respectively, see Theorem 6.17 in [4].

A simplification of Szabó's original proof of Theorem 6.1, which avoids Besse's immersion, appeared in [18]. A short survey on recent results on harmonic spaces is contained in [15].

6.1. **Preliminary remarks.** Let M be a complete and connected Riemannian manifold of dimension m, and let $p \in M$ be a point. For $v \in T_pM$, let P_v be parallel translation along the geodesic γ_v from $p = \gamma_v(0)$ to $\gamma_v(1)$. Then $P_v^{-1} \circ d\exp_p|_v$ is an endomorphism of T_pM, hence

$$\omega_p(v) := \det(P_v^{-1} \circ d\exp_p|_v) \tag{6.3}$$

is well defined on all of T_pM and depends smoothly on p and v.

Choose $r_0 > 0$ such that $\exp_p : B_0(r_0) \to B_p(r_0)$ is a diffeomorphism, where $0 = 0_p$. If g denotes the fundamental matrix of the Riemannian metric with respect to $\exp_p$ on $B(r_0)$, then we have, for all $v \in B(r_0)$,

$$\omega_p(v) = \sqrt{\det g}(v). \tag{6.4}$$

We recall that, for all $v, w \in T_pM$ and $t \in \mathbb{R}$,

$$d\exp_p|_{tv}(tw) = J(t), \tag{6.5}$$

where J is the Jacobi field along γ_v with initial conditions $J(0) = 0$ and $J'(0) = w$. Hence, if $E_1, \dots, E_m$ is a parallel frame along γ_v, then

$$\omega_p(tv) = t^{-m} \det(J_1(t), \dots, J_m(t)), \tag{6.6}$$

where J_i is the Jacobi field along γ_v with initial conditions $J_i(0) = 0$ and $J_i'(0) = E_i(0)$, expressed as linear combination in the frame $E_1, \dots, E_m$. If v is a unit vector and $E_1, \dots, E_m$ is orthonormal with $E_1 = \dot\gamma_v$, we have $J_1(t) = t\gamma_v(t)$ and then, by the Gauss Lemma,

$$\omega_p(tv) = t^{1-m} \det(J_2(t), \dots, J_m(t)). \tag{6.7}$$

The matrix $J(t) = (J_2(t), \dots, J_m(t))$ satisfies the *Jacobi equation*

$$J'' + RJ = 0 \tag{6.8}$$

with initial condition $J(0) = 0$ and $J'(0) = I$, where $R(t)$ is the matrix with entries

$$R_{ij}(t) := \langle R(E_i(t), \dot\gamma_v(t))\dot\gamma_v(t), E_j(t)\rangle, \quad 2 \le i, j \le m. \tag{6.9}$$

Lemma 6.10. *Let $\gamma = \gamma_v$, where $v \in T_pM$ is a unit vector. Let $t_0 > 0$ and $q := \gamma_w(t_0)$, where $w := -\dot\gamma(t_0)$. Then $\omega_p(t_0v) = \omega_q(t_0w)$.*

Proof. Let $E_1 = \dot\gamma, E_2, \dots, E_m$ be an orthonormal frame along γ. Let $J = (J_2, \dots, J_m)$ be as above and $K := (K_2, \dots, K_m)$, where K_i is the Jacobi field along γ with $K_i(t_0) = 0$ and $K_i'(t_0) = E_i(t_0)$, expressed as a linear combination in the frame $E_2, \dots, E_m$, $2 \le i \le m$. Then

$$\omega_q(t_0 w) = (-t)^{1-m} \det K(0).$$

Now K satisfies the Jacobi equation (6.9) as well, and hence

$$J^* K' - (J^*)' K$$

does not depend on t, by the symmetry of R. Therefore

$$\begin{aligned}\det J(t_0) &= \det(J^*(t_0)K'(t_0)) = \det(J^*K' - (J^*)'K) \\ &= \det(-J^*(0)'K(0)) = (-1)^{m-1} \det K'(0),\end{aligned}$$

where we use $J(0) = K(t_0) = 0$ and $J'(0) = K'(t_0) = I$. □

It is immediate from (6.9) that $W := J'J^{-1}$ satisfies the *Riccati equation*

$$W' + W^2 + R = 0. \tag{6.11}$$

Exercise 6.12. Show that, along the given unit speed geodesic $\gamma = \gamma(r)$, $W = W(r)$ is equal to the corresponding *Weingarten map* of the geodesic sphere $S_p(r)$ of radius $r \in (0, r_0)$ about p (with respect to the inner normal and written as a matrix with respect to the chosen frame along γ). Use (6.7) to conclude that the *mean curvature* $h(r) := \operatorname{tr} W(r)$ of $S_p(r)$ is given by

$$h := \frac{\omega'}{\omega} + \frac{m-1}{r}. \tag{6.13}$$

Conclude also that $rh(r)$ extends naturally and smoothly to $\mathbb{R}$.

Lemma 6.14. *The matrix function $tW = tW(t)$ extends smoothly to a neighborhood of $t = 0$ and*

$$(tW)''|_{t=0} = -\frac{2}{3} R(0).$$

Proof. Since $J(0) = 0$, we may write $J(t) = tV(t)$ in a neighborhood of $t = 0$, where V is smooth in t. We get

$$J' = V + tV', \quad J'' = 2V' + tV'', \quad J''' = 3V'' + tV'''.$$

Hence

$$V(0) = I, \quad V'(0) = 0, \quad 3V''(0) = J'''(0) = -R(0),$$

where we recall (6.9). Now $tW = J'V^{-1}$, hence tW extends smoothly to $t = 0$ with $(tW)' = J''V^{-1} - J'V^{-1}V'V^{-1}$. Differentiating once more, the claimed equality follows by substituting the above values for V, V', and V'' at $t = 0$. □

Remark 6.15. Note that $R(0)$ represents the curvature tensor $R(., v)v$, written as a matrix with respect to the orthonormal basis $E_2, \dots, E_m$ of $v^\perp$. The point of Lemma 6.14 is that we obtain $R(., v)v$ from W. Clearly, using higher derivatives of tW at $t = 0$, we get more information on the curvature tensor and its covariant derivatives in the direction of v.

6.2. **Harmonic spaces.** We say that M is *harmonic about* p if $\omega_p(v)$ depends only on $||v||$, for all $v \in T_pM$ with sufficiently small norm. We say that M is *harmonic* if it is harmonic about any $p \in M$.

It is clear from (6.13) that M is harmonic iff the mean curvature of geodesic spheres with sufficiently small radius is constant. The latter is mentioned as a consequence of harmonicity as defined in the introduction. However, from our discussion below it will become clear that the two definitions are equivalent.

Theorem 6.16. *If M is harmonic, then M is an Einstein space and hence an analytic Riemannian manifold. Moreover, there is an even analytic function $\omega : \mathbb{R} \to \mathbb{R}$, the* associated volume density*, such that $\omega_p(tv) = \omega(t)$ for any $p \in M$, unit vector $v \in T_pM$, and $t \in \mathbb{R}$.*

Proof. For any $p \in M$, unit tangent vector $v \in T_pM$, and $\gamma = \gamma_v$ as above,

$$-\frac{2}{3}\operatorname{Ric}(v,v) = (t \operatorname{tr} W)''|_{t=0} = (th(t))''|_{t=0},$$

by Exercise 6.12 and Lemma 6.14. Hence harmonic manifolds are Einstein spaces. It follows that they are analytic Riemannian manifolds, see Theorem 5.2 in [7].

For analytic Riemannian manifolds, the functions ω_p are analytic. Hence, for any given $p \in M$, unit vectors $v_1, v_2 \in T_pM$, and $t \in \mathbb{R}$, we have $\omega_p(tv_1) = \omega_p(tv_2)$, by the unique continuation property of analytic functions. In particular, $\omega_p(-tv) = \omega_p(tv)$, for any unit vector $v \in T_pM$ and $t \in \mathbb{R}$, and hence there exists an even analytic function $\omega_p : \mathbb{R} \to \mathbb{R}$ such that $\omega_p(tv) = \omega_p(t)$ for any unit vector $v \in T_pM$ and $t \in \mathbb{R}$. Moreover, $\omega(p,t) = \omega_p(t)$ is smooth in p and t.

To show that ω_p does not depend on p, let $p \in M$ and $u \in T_pM$, and assume $\omega_t(p) := \omega_p(t) \neq 0$. Let $v \in T_pM$ be a unit vector perpendicular to u and $q = \exp_p(v)$. By Lemma 6.10, we have $\omega_q(tw) = \omega_p(tv)$, where $w = -\dot\gamma_v(t)$, and hence $\omega_t(q) \neq 0$. In particular, p is not conjugate to q along γ_w. Hence there is a smooth curve w_s, $-\epsilon < s < \epsilon$, of unit tangent vectors in T_qM with $w(0) = w$ such that the curve $c(s) := \exp_q(tw(s))$ has derivative u in $s = 0$. Then $\omega_t(c(s)) = \omega_t(q) = \omega_t(p)$, again by Lemma 6.10. Hence the derivative of ω_t in the direction of u vanishes. Since ω is smooth in p and t, we get that the derivative of ω_t vanishes for each t, and hence ω_t is constant. $\square$

Suppose from now on that M is harmonic with associated volume density ω. Then there are two cases, by Theorem 6.16: Either ω does not have (real) zeroes, then M does not have conjugate points, and then $\exp_p : T_pM \to M$ is the universal covering, for any $p \in M$. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^m$. Or, else, ω does have a first positive zero l (where we recall that $\omega(0) = 1$). Then M does have conjugate points, and, in fact, the first conjugate point along each unit speed geodesic in M occurs precisely at time l and has multiplicity $k \in \{1, \dots, m-1\}$ equal to the order of vanishing of ω at $t = l$. Thus we arrive at the theorem of Allamigeon, see [1, 2] and also Theorem 6.82 in [4]:

Theorem 6.17. *Let M be a compact harmonic manifold with associated volume density ω and first positive zero l of ω. Then M is an Allamigeon-Warner manifold of type (k, l) with k and l as above.* □

Assume from now on that M is a compact and simply connected harmonic manifold with associated volume density ω and first positive zero d of ω. Let $p \in M$. Then the cut locus $C(p) \subseteq M$ is a submanifold of dimension n independent of p. Unit speed geodesics starting from p hit $C(p)$ perpendicularly at time d and, vice versa, for any unit normal vector v of $C(p)$, the geodesic γ_v hits p at time d. Moreover, such geodesics from a point $q \in C(p)$ correspond precisely to a great sphere of dimension $m - n - 1$ in the unit sphere S_pM in T_pM, thus giving rise to a smooth foliation of S_pM by great $(m - n - 1)$-spheres.

Lemma 6.18. *Let $p \in M$. For a smooth function $f : M \to \mathbb{R}$, let $\bar{f} : M \to \mathbb{R}$ be the function which associates to $q \in M$ the average of f over the geodesic sphere $S_p(r)$ if $r := d(p, q) < d$ and over $N := C(p)$ if $d(p, q) = r$. Then $\bar{f}$ is p-radial about p and smooth.*

Proof. With $N = \{p\}$, smoothness of $\bar{f}$ in $M \setminus C(p)$ is immediate from Lemma A.3. Now with $N = C(p)$, we have $S_p(r) = S_N(d - r)$. Hence smoothness of $\bar{f}$ in $M \setminus \{p\}$ is also immediate from Lemma A.3. □

Lemma 6.19. *Let $p \in M$. Then, via $f(q) = \phi(d(p, q))$, p-radial smooth functions $f : M \to \mathbb{R}$ are in one to one correspondence with $2d$-periodic even smooth functions $\phi : \mathbb{R} \to \mathbb{R}$.*

Proof. Define $\phi : [0, d] \to \mathbb{R}$ by $\phi(r) := \bar{f}(q)$, where $d(p, q) = r$. Since $\bar{f}$ is smooth, all derivatives of odd order of ϕ at $r = 0$ and $r = d$ vanish. Hence the extension of ϕ to an even and $2d$-periodic function on $\mathbb{R}$ is smooth. The converse is clear: any such function gives rise to a smooth p-radial function. □

As for the L^2-norms of a given radial function $f : M \to \mathbb{R}$ and the related $2d$-periodic even function $\phi : \mathbb{R} \to \mathbb{R}$, we have

$$||f||^2_{L^2(M)} = \operatorname{vol} S^{m-1} \int_{[0,d]} \phi^2 r^{m-1} \omega(r) dr =: ||\phi||^2_\omega. \tag{6.20}$$

6.3. **Besse's immersion.** Suppose again that M is simpy connected, compact, and harmonic with diameter d. Let $p \in M$. For a smooth p-radial function f on a normal geodesic ball $B_p(d)$ about p,

$$\Delta f = -f'' - hf', \tag{6.21}$$

where h denotes the mean curvature of the geodesic spheres about p as in (6.13). We keep in mind that h is an analytic function on $(0, d)$ which only depends on r.

We are interested in radially symmetric λ-eigenfunctions of Δ, where $\lambda > 0$ is given: On $[0, d)$, we want to solve

$$\phi'' + h\phi' + \lambda\phi = 0. \tag{6.22}$$

By (6.13), $r = 0$ is a regular singular point of this linear ordinary differential equation, and, therefore, it has precisely one solution $\phi : [0, d) \to \mathbb{R}$ with

$\phi(0) = 1$, and ϕ is analytic on $[0, d)$. It follows easily that the p-radial funtion f as in Lemma 6.19 is a λ-eigenfunction of Δ on $B_p(d)$ and, hence, that ϕ extends to an even analytic function on $(-d, d)$, also denoted by ϕ; that is, $\phi : (-d, d) \to \mathbb{R}$ is an even analytic function. This holds for any $\lambda > 0$.

Now we start with a λ-eigenfunction $f : M \to \mathbb{R}$ of Δ. Then the average $\bar{f}$ of f as in Lemma 6.18 is again a λ-eigenfunction of Δ, by harmonicity, and hence the associated $2d$-periodic even analytic function ϕ as in Lemma 6.19 satisfies (6.22) on $(0, d)$ with initial condition $\phi(0) = f(p)$. Conversely, any such function ϕ satisfying (6.22) on $(0, d)$ corresponds to an eigenfunction of the Laplacian for the eigenvalue λ.

Lemma 6.23. *If $f : M \to \mathbb{R}$ is a p-radial λ-eigenfunction with $f(p) = 0$, then $f = 0$.*

Proof. This is a restatement of the fact that a solution ϕ of (6.22) with $\phi(0) = 0$ vanishes. □

Let $V_\lambda \subseteq L^2(M)$, $\lambda > 0$, be an eigenspace of Δ. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the corresponding $2d$-harmonic even analytic function ϕ satisfying (6.22) on $(0, d)$ with initial condition $\phi(0) = 1$.

Lemma 6.24. *For any be an orthonormal basis $B = (f_1, \dots, f_l)$ of V_λ and pair of points p, q in M,*

$$\sum\nolimits_{1 \le i \le m} f_i(p) f_i(q) = \frac{m}{\operatorname{vol} M} \cdot \phi(d(p, q)).$$

Proof. It is clear that the left hand side does not depend on the choice of B, which gives us the freedom of making a convenient choice, for any given $p \in M$.

Let $f : V_\lambda \to \mathbb{R}$ be a λ-eigenfunction of Δ of L^2-norm 1, and choose $q \in M$ with $f(q) \neq 0$. Then the average $\bar{f}$ of f as in Lemma 6.18, but now with averages centered at q, is a q-radial λ-harmonic function of L^2-norm 1 with $\bar{f}(q) = f(q) \neq 0$. Hence the $2d$-harmonic even analytic function $\phi_1 : \mathbb{R} \to \mathbb{R}$ associated to $\bar{f}$ as in Lemma 6.19 has norm $||\phi_1||_\omega = 1$ and satisfies (6.22) on $(0, d)$ with initial condition $\phi_1(0) = f(p) \neq 0$. Hence the corresponding p-radial function f_1 is a λ-harmonic function of L^2-norm 1 with $f_1(p) \neq 0$.

The kernel of the linear map $\epsilon : V_\lambda \to \mathbb{R}$, $\epsilon(f) := f(p)$, consists of the space of λ-harmonic functions vanishing at p and is a compliment of the line $\mathbb{R} f_1$. Let $f \in \ker \epsilon$. Then the average $\bar{f}$ of f as in Lemma 6.18 is a p-radial λ-eigenfunction of Δ with $\bar{f}(p) = 0$, and hence $\bar{f} = 0$, by Lemma 6.23. Since f_1 is p-radial, we get

$$\langle f_1, f \rangle_{L^2(M)} = \langle f_1, \bar{f} \rangle_{L^2(M)} = 0,$$

and therefore $\ker \epsilon$ is the orthogonal complement of $\mathbb{R} f_1$ in V_λ. Hence, for any orthonormal basis $(f_1, f_2, \dots, f_l)$ of V_λ starting with the given f_1, the remaining functions $f_2, \dots, f_l$ belong to $\ker \epsilon$. But then

$$\sum\nolimits_{1 \le i \le m} f_i(p) f_i(q) = f_1(p) f_1(q) = \phi_1(0) \phi_1(d(p, q)).$$

In particular, $\sum f_i(p)^2 = \phi_1(0)^2$ does not depend on p and hence

$$\phi_1(0)^2 \operatorname{vol} M = \int_M \sum f_i(p)^2 = \sum ||f_i||^2_{L^2(M)} = m.$$

Now $\phi_1 = \phi_1(0)\phi$ and the asserted equality follows. $\square$

Proof of Theorem 6.2. We use Lemma 6.18 a couple of times, firstly by observing that the image of F is contained in $S^{l-1}(R)$. For $p \in M$ and $v \in T_pM$ a unit vector, we have

$$||dF(v)||^2 = C^2 \sum |df_i(v)|^2.$$

Differentiating $\sum f_i^2 = \text{const}$ twice, we conclude that

$$\begin{aligned} \sum |df_i(v)|^2 &= -\sum f_i(p) \operatorname{Hess} f_i(v,v) \\ &= -\sum f_i(p) f_i(\gamma_v(t))''|_{t=0} = -m\phi''(0)/\operatorname{vol} M, \end{aligned}$$

hence the left hand side does not depend on v. Therefore

$$\begin{aligned} m \sum |df_i(v)|^2 = \sum_{i,j} |df_i(v_j)|^2 &= -\sum_{i,j} f_i(p) \operatorname{Hess} f_i(v_j, v_j) \\ &= \sum f_i(p)\Delta f_i(p) = \lambda \sum f_i(p)^2 = \lambda m/\operatorname{vol} M, \end{aligned}$$

where $(v_1, \dots, v_m)$ is an orthonormal basis of T_pM. Hence F is an isometric imersion. Since $\Delta F = \lambda F$ is a multiple of F, it is perpendicular to $S^{l-1}(R)$, and hence F is a minimal immersion into $S^{l-1}(R)$. The remaining assertions follow since $\langle F(p), F(q)\rangle = C^2 \sum f_i(p) f_i(q)$. $\square$

Appendix A. Some background material

Let M be an affine manifold, i.e., a manifold together with a connection. For a non-trivial geodesic $\gamma\colon I \to M$ and $t_0 \neq t_1$ in I, we say that t_0 and t_1 are *conjugate along* γ and that $\gamma(t_0)$ and $\gamma(t_1)$ are *conjugate points along* γ if there is a non-trivial Jacobi field J along γ which vanishes at t_0 and t_1. Such Jacobi fields are perpendicular to γ.

Exercise A.1. Recall the two different ways in which conjugate points play a role, namely as critical values of the exponential map and, in the Riemannian case, for determining index and nullity of geodesics (Morse index theorem).

Exercise A.2. Let M be a semi-Riemannian manifold and (ϕ_t) be the flow of a vector field X on M. Show that the flow maps ϕ_t are isometric if and only if ∇X is a field of skew-symmetric endomorphisms. If this holds, then X is called a *Killing field*. Show that
1) for any geodesic $\gamma\colon I \to M$, $\langle \dot\gamma, X\rangle$ is constant along γ;
2) for any geodesic $\gamma\colon I \to M$, $X \circ \gamma$ is a Jacobi field;
3) X is complete if M is a complete Riemannian manifold.

A.1. **Mean averages.** Let $N \subseteq M$ be a compact submanifold of dimension n. Fix an $r_0 > 0$ such that $d_N : B(r_0) \setminus N \to \mathbb{R}$ is smooth. Then $S(r)$ is a smooth submanifold of M, for all $0 \le r < r_0$.

Lemma A.3. *Let $f : B(r_0) \to \mathbb{R}$ be a smooth function. For $0 \le r < r_0$ and $p \in S(r)$, let $\bar f(p)$ be the mean of f over $S(r)$ with respect to the induced volume element. Then $\bar f : B(r_0) \to \mathbb{R}$ is smooth.*

Proof. Let $\phi : U \to V$ be a local parameterization of N, where $U \subseteq \mathbb{R}^n$ and $V \subseteq N$ are open subsets and $n = \dim N$. Let $E_1, \dots, E_k$ be an orthonormal frame of the normal bundle of N over V, where $k = m - n$. Let D be the open disc of radius r_0 about 0 in $\mathbb{R}^k$. Then

$$\psi : U \times D \to B(r_0), \quad \psi(x,y) := \exp\big(\sum y^i E_i(\phi(x))\big),$$

is a parameterization of its image, W, in $B(r_0)$. Without loss of generality we may assume that the support of f is contained in W.

Let g be the fundamental matrix of the Riemannian metric of M in the parameterization ψ. Since $\operatorname{grad} d_N$ is a smooth vector field of norm one perpendicular to the hypersurfaces $S(r)$, $0 < r < r_0$, the induced volume element along these is given by the square root of $\det g$. Since the frame $E_1, \dots, E_k$ is orthonormal, the induced volume element along $N = S(0)$ is also given by the square root of $\det g$. Now $\mathrm{SO}(k)$ acts transitively on spheres about 0 in $\mathbb{R}^k$ and hence

$$\bar f(\psi(x,y)) = \frac{\int_U \int_{\mathrm{SO}(k)} f(\psi(x,Ay))\sqrt{\det g(x,Ay)}\,d\mu(A)dx}{\int_U \int_{\mathrm{SO}(k)} \sqrt{\det g(x,Ay)}\,d\mu(A)dx},$$

where μ denotes normalized Haar measure of $\mathrm{SO}(k)$. □

A.2. **Higher helices.** Let I be an open interval and $c : I \to \mathbb{R}^n$ be a smooth curve. We say that c is a *helix* if the inner products $\langle c^{(i)}(t), c^{(j)}(t)\rangle$, $i, j \geq 1$, do not depend on t. Suppose that c is a helix. Then, for all $i, j \geq 1$,

$$0 = \langle c^{(i)}, c^{(j)}\rangle' = \langle c^{(i+1)}, c^{(j)}\rangle + \langle c^{(i)}, c^{(j+1)}\rangle. \tag{A.4}$$

We conclude that

$$\langle c^{(i-k)}, c^{(j+k)}\rangle = -\langle c^{(i)}, c^{(j)}\rangle \tag{A.5}$$

for all $i, j \geq 1$ and k with $i-k, j+k \geq 1$. For $i, j, l \geq 1$ with $i + j = 2l + 1$ we get

$$\langle c^{(i)}, c^{(j)}\rangle = \pm\langle c^{(l)}, c^{(l+1)}\rangle = \pm\frac{1}{2}\langle c^{(l)}, c^{(l)}\rangle' = 0. \tag{A.6}$$

Let $t \in I$. Then there is a first $k \leq n$ such that $c^{(k+1)}(t)$ is linearly dependent on $c'(t), \ldots, c^{(k)}(t)$,

$$c^{(k+1)}(t) = \sum\nolimits_{1\leq i\leq k} a_i c^{(i)}(t). \tag{A.7}$$

Since the inner products $\langle c^{(i)}, c^{(j)}\rangle$ are constant, this equation holds for all $t \in I$, and thus (A.7) is a linear ordinary differential equation for c of order $k+1$ with constant coefficients $a_0 = 0, a_1, \ldots, a_k$, the *associated differential equation.* By the choice of k, $c'(t), \ldots, c^{(k)}(t)$ are linearly independent for all $t \in I$. Note that not any linear ordinary differential equation is associated to a helix. For example, the equation $c'' = c'$ is not such an associated equation.

A solution of the associated differential equation (A.7) is uniquely determined by its initial conditions $c(t_0), c'(t_0), \ldots, c^{(k)}(t_0)$ at any given time $t_0 \in I$. Moreover, for any affine function F of $\mathbb{R}^n$, $F \circ c$ is also a solution of (A.7). Note also that the maximal domain of definition of a solution of (A.7) is the real line $\mathbb{R}$. From what we just said it follows easily that the maximal solution containing c is a helix. In other words, we can assume without loss of generality that $I = \mathbb{R}$.

Proposition A.8. *The following are equivalent:*

1) c is a helix.

2) The function $||c(t+s) - c(t)||$ depends only on s.

Proof. Let $t_0, t_1 \in I$. Since the inner products $\langle c^{(i)}, c^{(j)}\rangle$ are constant, there is an orthogonal transformation A of $\mathbb{R}^n$ with $A(c^{(i)}(t_1)) = c^{(i)}(t_0)$, for $1 \leq i \leq k$. Let F be the affine map of $\mathbb{R}^n$ with $F(x) = Ax + b$, where $b = c(t_0) - Ac(t_1)$. Then $F \circ c$ solves (A.7) and has the same initial conditions at $t = t_1$ as c at $t = t_0$. Hence $(F \circ c)(t_1 + s) = c(t_0 + s)$, and therefore, since F is a Euclidean motion,

$$\begin{aligned} ||c(t_1+s) - c(t_1)|| &= ||F(c(t_1+s)) - F(c(t_1))|| \\ &= ||c(t_0+s) - c(t_0)||. \end{aligned}$$

Now assume conversely that the function $||c(t+s) - c(t)||$ depends only on s and set

$$f(s) := ||c(t+s) - c(t)||^2 = \langle c(t+s) - c(t), c(t+s) - c(t)\rangle.$$

By the product rule for higher derivatives,

$$f^{(k)}(0) = \sum_{1 \le i \le k-1} \binom{k}{i} \langle c^{(i)}(t), c^{(k-i)}(t) \rangle, \tag{A.9}$$

where we recall that the expression does not depend on t. In particular, $\langle c', c' \rangle$ is constant, and, therefore, $\langle c'', c' \rangle = 0$. Let $k \ge 4$ be even and assume inductively that $\langle c^{(i)}, c^{(j)} \rangle$ is constant, for all $i, j \ge 1$ with $i + j \le k - 1$. Assume that $k \ge 4$ is even, $k = 2l$. Then, by (A.5) and (A.9),

$$f^{(k)}(0) = \sum_{1 \le i \le k-1} \binom{k}{i} (-1)^{l+i} \langle c^{(l)}, c^{(l)} \rangle = 2(-1)^{l+1} \langle c^{(l)}, c^{(l)} \rangle,$$

where we recall that

$$\sum_{0 \le i \le k} (-1)^i \binom{k}{i} = 0.$$

We conclude that $\langle c^{(l)}, c^{(l)} \rangle$ is constant and, therefore, that $\langle c^{(i)}, c^{(j)} \rangle$ is constant for all $i, j \ge 1$ with $i + j = 2l = k$. Hence $\langle c^{(l+1)}, c^{(l)} \rangle = 0$, and hence also $\langle c^{(i)}, c^{(j)} \rangle = 0$ for all $i, j \ge 1$ with $i + j = 2l + 1 = k + 1$. Hence c is a helix. □

Remark A.10. Using the associated differential equation (A.7) we see that, for any $t_0 \in I$, c is contained in the k-dimensional affine subspace through $c(t_0)$ spanned by $c'(t_0), \dots, c^{(k)}(t_0)$. Normalizing c by $c(t_0) = 0$, we get that c is contained in the linear subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$ spanned by $c'(t_0), \dots, c^{(k)}(t_0)$. In this $\mathbb{R}^k$, c possesses a Frenet frame and is a curve of constant curvatures with respect to it. Conversely, any curve with a Frenet frame and constant curvatures with respect to it is a helix.

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