AUTOMORPHISM GROUPS

WERNER BALLMANN

In these lecture notes I discuss groups of automorphisms of certain natural structures which occur in differential geometry. My aim is to show that such groups of automorphisms, when endowed with the compact–open topology, are Lie transformation groups.

Many of the automorphism groups from differential geometry can be viewed as closed subgroups of automorphism groups of parallelizations. For example, this is the case for the group of isometries of a Riemannian or pseudo-Riemannian manifold, see Example 3.2. In the first part of these notes I therefore treat the case of parallelized manifolds.

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1. The Main Result

Let M be a smooth manifold of dimension n. A *parallelization* or *global frame* of M is a diffeomorphism $\Phi : M \times \mathbb{R}^n \to TM$ such that

$$\Phi(p,.) : \mathbb{R}^n \to T_pM \tag{1.1}$$

is an isomorphism for all $p \in M$. Sometimes it is convenient to replace $\mathbb{R}^n$ by some other real vector space of dimension n.

The existence of a parallelization imposes strong topological restrictions on M. For example, if $M = S^n$, the sphere of dimension n, then M has a parallelization iff $n \in \{1, 3, 7\}$, see [Ad].

Let Φ be a parallelization of M. For each vector $z \in \mathbb{R}^n$ we obtain a smooth vector field Z on M by setting

$$Z(p) = \Phi(p, z)\,. \tag{1.2}$$

We call Z a *constant field (with respect to* Φ*)*. The vector space of constant fields has dimension n. In general, it is not a subalgebra of the Lie algebra of all smooth vector fields on M.

The *automorphism group* $\mathrm{Aut}(\Phi)$ of Φ is the group of all diffeomorphisms f of M such that

$$f_* \circ Z = Z \circ f \tag{1.3}$$

for all constant fields Z. We endow $\mathrm{Aut}(\Phi)$ with the compact–open topology.

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1.4. *Example.* Let G be a Lie group and $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G. Define $\Phi : G \times \mathfrak{g} \to TG$ by $\Phi(p, Z) = Z(p)$. We call Φ the *left invariant parallelization* of G. The automorphisms of Φ are precisely the left translations by elements of G.

1.5. Proposition. *The action of* $\mathrm{Aut}(\Phi)$ *on* M *is free. For each* $p \in M$*, the orbit map* $\mathrm{Aut}(\Phi) \ni g \mapsto g(p) \in M$ *is proper and embeds* $\mathrm{Aut}(\Phi)$ *as a closed subset of* M*.*

The proof is easy and will be presented below.

1.6. Corollary. *A subgroup* G *of* $\mathrm{Aut}(\Phi)$ *is closed iff* $G(p)$ *is closed for some* $p \in M$ *or, equivalently, iff* $G(p)$ *is closed for any* $p \in M$*.*

Our main result is the following.

1.7. Theorem. *Let* G *be a closed subgroup of* $\mathrm{Aut}(\Phi)$*. Then*

(1) *for each* $p \in M$*, the orbit* $G(p)$ *is a submanifold of* M*;*
(2) *the smooth structure on* G *induced by the orbit map* $g \mapsto g(p)$ *is independent of* $p \in M$ *and turns* G *into a Lie transformation group of* M*;*
(3) *the orbit space* $G\backslash M$ *has a unique smooth structure for which the projection* $M \to G\backslash M$ *is a smooth left principal bundle with structure group* G*.*

In the beginning I follow arguments in Kobayashi's book [Ko] on transformation groups. The main argument is similar to the one which is used in the proof of the fact that a closed subgroup of a Lie group is a Lie subgroup. Note that this latter result is a consequence of Theorem 1.7, when applied to Example 1.4.

1.8. Corollary. *Let* L *be a Lie group and* $G \subset L$ *a closed subgroup. Then* G *is a Lie subgroup of* L*. Furthermore,* $G\backslash L$ *has a unique smooth structure such that* $L \to G\backslash L$ *is a left principle bundle with structure group* G*.*

By interchanging left and right we get the analogous conclusions for the more standard homogeneous space L/G, namely that it has a unique smooth structure such that $L \to L/G$ is a principle bundle with structure group G.

2. Proof of Proposition 1.5 and Theorem 1.7

Without loss of generality we may and do assume that M is connected.

Denote by $\varphi(p, z, t)$ the maximal integral curve of the constant field $Z = \Phi(\,.\,, z)$ with initial condition $\varphi(p, z, 0) = p$. The domain $\mathcal{O}$ of definition of φ is open in $M \times \mathbb{R}^n \times \mathbb{R}$ and contains the closed subset $M \times \mathbb{R}^n \times \{0\}$. Moreover, φ is smooth on $\mathcal{O}$ and

$$
\begin{aligned}
\varphi(p, \alpha z, t) &= \varphi(p, z, \alpha t) && \text{for all } \alpha \in \mathbb{R} \text{ and} \\
f(\varphi(p, z, t)) &= \varphi(f(p), z, t) && \text{for all } f \in \mathrm{Aut}(\Phi)\,.
\end{aligned}
\tag{2.1}
$$

For $z \in \mathbb{R}^n$ let $\varphi_z(p) = \varphi_p(z) = \varphi(p, z, 1)$ and denote by $D_z \subset M$ and $D_p \subset \mathbb{R}^n$ their (maximal) domains of definition. We may have $D_z = \emptyset$ for some $z \in \mathbb{R}^n$. By definition, $\varphi_{-z} = \varphi_z^{-1}$. We consider the family Γ of all local diffeomorphisms

$$\gamma = \varphi_{z_1} \circ \cdots \circ \varphi_{z_m}\,. \tag{2.2}$$

For any $\gamma \in \Gamma$, the inverse γ^{-1} also belongs to Γ. The domain of $\gamma \in \Gamma$ is denoted D_γ. We note the following obvious assertion.

2.3. Lemma. *For any $f \in \mathrm{Aut}(\Phi)$ and $\gamma \in \Gamma$ we have $f \circ \gamma = \gamma \circ f$. In particular, D_γ is invariant under* $\mathrm{Aut}(\Phi)$.

Proof. For any vector field Z and diffeomorphism f of M, the flow of $f_* \circ Z \circ f^{-1}$ is $f \circ \varphi \circ f^{-1}$, where φ denotes the flow of Z. □

2.4. Lemma. *For all $p \in M$ there are open neighborhoods U of p in M and V of 0 in $\mathbb{R}^n$ such that for each $q \in U$, $\varphi_q : V \to M$ is a diffeomorphism onto an open neighborhood of q in M.*

Proof. The map $(p, z) \mapsto (p, \varphi_z(p))$ is defined in an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}^n$. By the definition of φ, its differential in $(p, 0)$ is $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$. This matrix is invertible, hence the assertion follows from the inverse function theorem. □

2.5. Lemma. *For all points $p, q \in M$ there is $\gamma \in \Gamma$ with $\gamma(p) = q$.*

Proof. It follows from Lemma 2.4 that for all points $p \in M$ the orbit $\Gamma(p) = \{\gamma(p) \mid \gamma \in \Gamma\}$ of p under Γ is open. Now M is connected and $\gamma^{-1} \in \Gamma$ for all $\gamma \in \Gamma$, hence $\Gamma(p) = M$ for all $p \in M$. □

2.6. Lemma. *If $f : M \to M$ is a map with $f \circ \gamma = \gamma \circ f$ for all $\gamma \in \Gamma$, then f is smooth and $f_* \circ Z = Z \circ f$ for all constant fields Z. In particular, f is of maximal rank.*

Similarly, if X is a vector field on M such that $X \circ \gamma = \gamma_* \circ X$ for all $\gamma \in \Gamma$, then X is smooth and $[X, Z] = 0$ for all constant vector fields Z. Moreover, if X is complete, then the 1–parameter group generated by X belongs to $\mathrm{Aut}(\Phi)$.

Proof. Let $p \in M$ and choose V as in Lemma 2.4. Then for any $z \in V$, $f(\varphi_p(z)) = \varphi_{f(p)}(z)$ and hence f is smooth. We clearly have $f_* \circ Z = Z \circ f$ for all constant fields Z. This proves the assertions about f, the proof of the assertions about X is similar. □

2.7. Lemma. *Suppose (f_n) is a sequence in $\mathrm{Aut}(\Phi)$ such that $(f_n(p))$ converges for some $p \in M$. Then $(f_n(p))$ converges for all $p \in M$. The pointwise limit f defined by $f(p) := \lim f_n(p)$ belongs to $\mathrm{Aut}(\Phi)$ and (f_n) converges to f in the compact–open topology.*

Proof. Let $q = \lim f_n(p)$. Then $f_n(p) = \varphi_q(z_n)$ for all n sufficiently large, where $\mathbb{R}^n \ni z_n \to 0$. Hence $q = \varphi(f_n(p), -z_n, 1)$ and therefore by (2.1),

$$f_n^{-1}(q) = f_n^{-1}(\varphi(f_n(p), -z_n, 1)) = \varphi_p(-z_n) \to p\,.$$

Now let $p' = \gamma(p)$, $\gamma \in \Gamma$. Since the domain D_γ of γ is open, $f_n^{-1}(q) \in D_\gamma$ for all n sufficiently large. Since D_γ is invariant under $\mathrm{Aut}(\Phi)$ we conclude that $q = f_n(f_n^{-1}(q)) \in D_\gamma$. Hence by Lemma 2.3,

$$f_n(p') = f_n(\gamma(p)) = \gamma(f_n(p)) \to \gamma(q)\,.$$

Therefore by Lemma 2.5, $(f_n(p))$ converges for all $p \in M$. Clearly the limiting map f commutes with the action of Γ, hence f is smooth and $f_* \circ Z = Z \circ f$ for any constant field Z. In particular, f is of maximal rank.

Now by the first equation above we have $(f_n^{-1}(q)) \to p$, hence the pointwise limit of (f_n^{-1}) is an inverse to f. Hence f is a diffeomorphism and therefore $f \in \mathrm{Aut}(\Phi)$. The convergence $f_n \to f$ is locally uniform since $f_n(\varphi_p(z)) = \varphi(f_n(p), z, 1)$ for all $p \in M$. □

Proof of Proposition 1.5. Suppose $f \in \mathrm{Aut}(\Phi)$ fixes a point $p \in M$. Let $q \in M$. Then by Lemma 2.5, there is a $\gamma \in \Gamma$ with $\gamma(p) = q$. From Lemma 2.3 we infer

$$f(q) = f(\gamma(p)) = \gamma(f(p)) = \gamma(p) = q$$

and hence $f = \mathrm{id}$. Hence $\mathrm{Aut}(\Phi)$ acts freely. The second assertion of the proposition follows immediately from Lemma 2.7. □

Denote by ℓ the space of all smooth vector fields X on M such that $[X, Z] = 0$ for all constant fields Z. Then by the Jacobi identity, ℓ is a Lie algebra.

2.8. Lemma. *For any $p \in M$, the evaluation map $\ell \to T_pM$, $X \mapsto X(p)$, is injective. In particular, $\dim \ell \le \dim M$.*

Proof. Let $X \in \ell$ and let (ψ^s) be the flow of X. By the definition of ℓ,

$$\psi^s(\varphi(p, z, t)) = \varphi(\psi^s(p), z, t)$$

whenever both sides of the equation are defined. Hence if $X(p) = 0$, that is, if $\psi^s(p) = p$ for all $s \in \mathbb{R}$, then

$$\psi^s(\varphi(p, z, t)) \equiv \varphi(p, z, t)\,,$$

and hence $X(q) = 0$ for all q in a neighborhood of p. Therefore the set of zeroes of X is open and closed. Now M is connected, hence the set of zeroes of X is empty if $X \neq 0$. □

Now fix a closed subgroup $G \subset \mathrm{Aut}(\Phi)$. Let $\mathcal{C} \subset \ell$ be the cone of all $X \in \ell$ such that X is complete and $\exp(sX) \in G$ for all $s \in \mathbb{R}$, where $\exp(sX)(p)$, $s \in \mathbb{R}$ and $p \in M$, denotes the flow of X.

2.9. Lemma. *Let $g \in G$ and $X \in \mathcal{C}$. Then $g_*X \in \mathcal{C}$.*

Proof. The flow of g_*X is $g \circ \exp(sX) \circ g^{-1}$. □

Let $\mathfrak{g} \subset \ell$ be the linear hull of $\mathcal{C}$ and define a distribution Δ on M by

$$\Delta_p = \{X(p) \mid X \in \mathfrak{g}\}\,. \tag{2.10}$$

If $X_1, \dots, X_k$ is a basis of $\mathfrak{g}$, then by Lemma 2.8, $X_1(p), \dots, X_k(p)$ is a basis of Δ_p for all $p \in M$. In particular, Δ is smooth of rank k. We choose a basis $X_1, \dots, X_k$ of $\mathfrak{g}$ such that $X_i \in \mathcal{C}$, $1 \le i \le k$.

Let $p \in M$ and choose a smooth map $h : U \to M$, where $U \subset \mathbb{R}^{n-k}$ is a small open set, such that $h(u_0) = p$ for some $u_0 \in U$ and such that the image of h_{*u} is a linear complement of $\Delta_{h(u)}$ for all $u \in U$. For $v = (v_1, \dots, v_k) \in \mathbb{R}^k$ and $u \in U$ set

$$H(u, v) = \exp(v_1 X_1) \cdots \exp(v_k X_k)(h(u))\,. \tag{2.11}$$

Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then

$$H_{*(u,0)}(e_i) = \begin{cases} h_{*u}(e_i) & \text{if } 1 \le i \le n-k, \\ X_{i-n+k}(h(u)) & \text{if } n-k < i \le n. \end{cases} \tag{2.12}$$

We conclude that H has maximal rank equal to n in $(u, 0)$. Hence if $V \subset \mathbb{R}^k$ is a sufficiently small neighborhood of 0 and if we replace U by a smaller neighborhood of u_0 if necessary, then $H : U \times V \to W = H(U \times V)$ is a diffeomorphism.

2.13. Lemma. *For all $(u, v) \in U \times V$, the vectors $H_{*(u,v)}(e_{n-k+i})$, $1 \le i \le k$, form a basis of $\Delta_{H(u,v)}$. In particular, Δ is integrable and $H(\{u\} \times V)$ is the local leaf of Δ through $h(u)$.*

Proof. Let $v \in V$ and $i \in \{1, \dots, k\}$. Then

$$f = \exp(v_1 X_1) \cdots \exp(v_i X_i) \quad \text{and} \quad g = \exp(v_{i+1} X_{i+1}) \cdots \exp(v_k X_k)$$

are both in G. Now

$$\begin{aligned} H_{*(u,v)}(e_{n-k+i}) &= \partial_s (f \circ \exp(sX_i) \circ g(h(u)))|_{s=0} \\ &= \partial_s ((f \circ \exp(sX_i) \circ f^{-1})(fg(h(u))))|_{s=0} = (f_* X_i)(H(u, v))\,. \end{aligned}$$

Now Lemma 2.9 applies and shows $H_{*(u,v)}(e_{n-k+i}) \in \Delta_{H(u,v)}$. This concludes the proof since H has maximal rank. □

For example, if $X \in \mathcal{C}$, then X is tangent to Δ. Hence if $\exp(sX)(p) \in W$ for $0 \le s \le 1$, then $\exp(X)(p) \in H(\{u_0\} \times V)$.

Now let G_0 be the subgroup of G generated by the flows $\exp(sX)$, $X \in \mathcal{C}$. By Lemma 2.13, the orbit $G_0(p)$ is the set of all $q \in M$ such that there exists a piecewise smooth curve $c : [0, 1] \to M$ with $c(0) = p$, $c(1) = q$ and c' tangential to Δ.

2.14. Lemma. *The orbit $G_0(p)$ of p is closed in M. If $W = H(U \times V)$ is as above, where U and V are sufficiently small, then*

$$H(\{u_0\} \times V) = W \cap G_0(p) = W \cap G(p)\,.$$

Proof. The inclusions from left to right are obvious from the definitions. Hence it suffices to show that $W \cap G(p) \subset H(\{u_0\} \times V)$ if U and V are sufficiently small.

We use the following terminology. Let (p_n) be a sequence in M converging to p and let (r_n) be a sequence of positive numbers. We say that *the sequence (of formal expressions)* $r_n(p_n - p)$ *converges to* $X \in T_pM$ if

$$r_n(x(p_n) - x(p)) \to x_{*p}(X) \tag{2.15}$$

with respect to some local coordinate chart x of M about p. This is independent of the choice of x. Moreover, if $f : M \to N$ is smooth and $r_n(p_n - p) \to X$, then $r_n(f(p_n) - f(p)) \to f_*(X)$.

Suppose now that $W \cap G(p) \not\subset H(\{u_0\} \times V)$ even if U and V are as small as possible. Then there exists a sequence (g_n) in G such that $g_n(p) \to p$ with $g_n(p) = H(u_n, v_n)$ and $u_n \neq u_0$. By the definition of H we have $H(u_n, v_n) = h_n(H(u_n, 0))$ for appropriate $h_n \in G$, hence replacing g_n by $h_n^{-1}g_n$ if necessary we can assume $v_n = 0$ for all n. Set $r_n = 1/|u_n - u_0|$ and note that $r_n \to \infty$. Now H^{-1} is a coordinate chart, hence

$$r_n(g_n(p) - p) \to X_p \in T_pM\,, \quad X_p \neq 0\,,$$

after passing to a subsequence if necessary. Moreover, $X_p \notin \Delta_p$ since Δ_p is spanned by the v–directions.

Recall from Lemma 2.3 that g_n commutes with any $\gamma \in \Gamma$. Hence by the above,

$$r_n(g_n(\gamma(p)) - \gamma(p)) = r_n(\gamma(g_n(p)) - \gamma(p)) \to \gamma_* X_p\,.$$

Since any $q \in M$ can be written as $\gamma(p)$ for some appropriate $\gamma \in \Gamma$, we obtain a vector field X by setting

$$X(q) = \lim r_n(g_n(q) - q)\,.$$

Note that $X \circ \gamma = \gamma_* \circ X$ for all $\gamma \in \Gamma$. Hence by Lemma 2.6, X is smooth and belongs to ℓ. We now show that X is complete and $\exp(sX) \in G$ for all s, hence $X \in \mathfrak{g}$. This is a contradiction since $X(p) = X_p \notin \Delta_p$.

We choose small neighborhoods $W_0 \subset W$ of p in M and V_0 of 0 in $\mathbb{R}^n$ such that $\varphi(q, z, 1) \in W$ for all $q \in W_0$ and $z \in V_0$ and such that the restriction of φ_p to V_0 is a diffeomorphism onto a neighborhood of p in M. Fix a small $s \neq 0$ and suppose that the domain of the flow ψ of X contains the set $W \times (-2|s|, 2|s|)$. We will show that ψ^s is defined on all of M.

In what follows, differences and estimates are to be understood to be taken with respect to the coordinate chart H^{-1} on $W_0 \subset W$. For example, the first and second derivatives of the maps φ_z, $z \in V_0$, have a uniform bound C on W_0.

Hence for $q = \varphi_z(p)$ and n sufficiently large,

$$\begin{aligned}\|r_n(g_n(q) - q) - X_q\| &= \|r_n(g_n(\varphi_z(p)) - \varphi_z(p)) - \varphi_{z*p} X_p\| \\ &= \|r_n(\varphi_z(g_n(p)) - \varphi_z(p)) - \varphi_{z*p} X_p\| \\ &\le C \cdot \|r_n(g_n(p) - p) - X_p\| + r_n C \cdot \|g_n(p) - p\|^2 \\ &\le C \cdot \|r_n(g_n(p) - p) - X_p\| + C/r_n\,,\end{aligned}$$

where we use that $\|g_n(p) - p\| = 1/r_n$. Now choose $k_n \in \mathbb{Z}$ and $\rho_n \in (0,1)$ with

$$r_n s = k_n + \rho_n\,.$$

Since s is fixed,

$$\frac{\rho_n}{s}\,\|g_n(q) - q\|$$

can be estimated by C/r_n for some appropriate constant C, uniformly for all q sufficiently close to p. Hence we may replace r_n by k_n/s in the above estimate.

We now want to show that $g_n^{k_n}(p) \to \psi^s(p)$. To that end we choose coordinates about p such that X is the coordinate vector field in the direction of the first coordinate, $X = e_1$. Then $\psi^s(q) = q + se_1$ locally about p.

The above estimates also hold in the new coordinates since locally about p, there are uniform bounds on the derivatives of the coordinate transition map.

Without loss of generality we assume $s > 0$. Then $k_n \to \infty$ and by the above estimate

$$\begin{aligned}&\|g_n^{k_n}(p) - p - se_1\| \\ &\quad\le \Big\|g_n(p) - p - \frac{s}{k_n}e_1\Big\| + \Big\|g_n^2(p) - g_n(p) - \frac{s}{k_n}e_1\Big\| + \cdots \\ &\quad\le \frac{s}{k_n}\Big\|\frac{k_n}{s}(g_n(p) - p) - e_1\Big\| + \frac{s}{k_n}\Big\|\frac{k_n}{s}(g_n^2(p) - g_n(p)) - e_1\Big\| + \cdots \\ &\quad\le C\,s\,\Big\|\frac{k_n}{s}(g_n(p) - p) - e_1\Big\| + C\,\frac{s^2}{k_n}\,.\end{aligned}$$

It follows that $g_n^{k_n}(p) \to \psi^s(p)$ as claimed. By Proposition 1.5 there is a unique element $g^s \in \mathrm{Aut}(\Phi)$ such that $g^s(p) = \psi^s(p)$. Now G is closed and $g_n^{k_n}(p) \to g^s(p)$, hence $g^s \in G$.

We have $g^s g^t(p) = g^{s+t}(p)$ for all s, t small. Hence by Proposition 1.5, $g^s g^t = g^{s+t}$ for all small s, t. It follows that the family of g^s generates a 1–parameter group of diffeomorphisms of M. Since $\gamma_* \circ X = X \circ \gamma$ and $g^s \circ \gamma = \gamma \circ g^s$ for all $\gamma \in \Gamma$ and $s \in \mathbb{R}$, we conclude by Lemma 2.5 that (g^s) is the flow of X. Hence X is complete and $\exp(sX) \in G$ for all s, the desired contradiction. □

Proof of Theorem 1.7. The first assertion of Theorem 1.7 is an immediate consequence of Lemma 2.14. As for the smoothness of the group structure of G, it suffices to show that multiplication and inversion is smooth in a neighborhood of the identity. We use that G acts by diffeomorphisms on the orbit $G(p)$: Choose a basis $X_1, \dots, X_k$ of $\mathfrak{g}$ as above. Then by Lemma 2.14, the map

$$v = (v_1, \dots, v_k) \mapsto \exp(v_1 X_1) \cdots \exp(v_k X_k)(p) = H(u_0, v)$$

is a diffeomorphism of a small neighborhood V of 0 in $\mathbb{R}^k$ onto a neighborhood of p in $G(p)$. Now multiplication and inversion in a neighborhood of the identity in G correspond to composition and inversion of the local flows of the X_i close to p. Since $H(u_0, .)$ is a diffeomorphism, the output of such operations depends smoothly on the input. This proves the second assertion of Theorem 1.7, where the independence of p follows from Lemmas 2.3 and 2.5. It also follows that $\mathfrak{g}$ represents the Lie algebra of G.

As for the proof of the third assertion, consider the diffeomorphism

$$H : U \times V \mapsto H(U \times V) = W$$

as above and set $h(u) = H(u, 0)$. Consider the map

$$F : G \times U \to M\,, \quad F(g, u) = g(h(u))\,.$$

By making U smaller if necessary, F is injective. Hence by choosing U small enough, we get that F is a diffeomorphism onto a G–invariant tubular neighborhood of $G(p)$ in M. Then $\pi \circ h : U \to G\backslash M$ is a local homeomorphism onto its image, where $\pi : M \to G\backslash M$ denotes the natural projection. The claim follows. □

3. Applications

We discuss two examples from geometry, where the main result applies. The first concerns the group of affine transformations of an affine manifold, the second the group of isometries of a Riemannian manifold.

3.1. *Example.* Let M be a smooth manifold and $\pi : \mathrm{Gl}(M) \to M$ be the principal bundle of frames of M. Let $\mathcal{V} = \ker \pi_*$ be the vertical distribution of $\mathrm{Gl}(M)$. Let $\phi \in \mathrm{Gl}(M)$ and $x \in \mathfrak{g}l(n)$. Using the right action of $\mathrm{Gl}(n)$ on $\mathrm{Gl}(M)$ we define

$$\Phi_v(\phi, x) = \partial_t(\phi \exp(tx))|_{t=0}\,.$$

Then

$$\Phi_v : \mathrm{Gl}(M) \times \mathfrak{g}l(m) \to \mathcal{V}$$

is a trivialization of the vertical distribution.

Let D be a connection on M. The pair (M, D) is called an *affine manifold*, a diffeomorphism of M preserving D is called an *affine transformation*.

Now D induces a horizontal distribution $\mathcal{H}$ of $\mathrm{Gl}(M)$; that is, $\mathcal{H}$ is a distribution of $\mathrm{Gl}(M)$ such that

$$\pi_* : \mathcal{H}_\phi \to T_pM\,, \quad p = \pi(\phi)\,,$$

is an isomorphism for all $\phi \in \mathrm{Gl}(M)$. To define $\mathcal{H}$, let $\phi \in \mathrm{Gl}(M)$ and $v = \phi(z) \in T_pM$, where $p = \pi(\phi)$ and $z \in \mathbb{R}^n$. Choose a curve $c = c(t)$ through p

with $c'(0) = v$ and let $\psi = \psi(t)$ be the unique parallel frame along c such that $\psi(0) = \phi$. Set

$$\Phi_h(\phi, z) = \psi'(0)\,.$$

Using local coordinates, it is easy to see that $\Phi_h = \Phi_h(\phi, z)$ depends smoothly on ϕ and z and that $\Phi_h(\phi, .\,)$ is linear for each frame ϕ. Note that $\pi_*(\Phi_h(\phi, z)) = v = \phi(z)$. Hence for each $\phi \in \mathrm{Gl}(M)$, $\Phi_h(\phi, .\,) : \mathbb{R}^n \to T_\phi \mathrm{Gl}(M)$ is an injection. It follows that the images $\mathcal{H}_\phi$ define a smooth horizontal distribution of $\mathrm{Gl}(M)$. This is the horizontal distribution we are after. Note that $\mathcal{H}$ comes with a trivialization, namely our map

$$\Phi_h : \mathrm{Gl}(M) \times \mathbb{R}^n \to \mathcal{H}\,.$$

Now the pair $\Phi = (\Phi_v, \Phi_h)$ is a parallelization of $Gl(M)$.

A diffeomorphism f acts on $\mathrm{Gl}(M)$ by sending a frame ϕ to $f_* \circ \phi$, where f_* denotes the differential of f. By the definition of Φ, a diffeomorphism f is affine if this action preserves Φ. It follows easily that the group $\mathrm{Aut}(M, D)$ of affine transformations of (M, D), endowed with the compact–open topology, is a closed subgroup of $\mathrm{Aut}(\Phi)$ and hence is a Lie group.

We emphasize that for each frame ϕ, the orbit map $\mathrm{Aut}(M, D) \to \mathrm{Gl}(M)$, $f \mapsto f_* \circ \phi$, is an embedding.

3.2. *Example.* Let M be a smooth manifold and g be a Riemannian metric on M. Then the group $\mathrm{Aut}(M, g)$ of isometries of (M, g), endowed with the compact–open topology, is a Lie group such that the action of $\mathrm{Aut}(M, g)$ on M is smooth.

This follows either from the previous example by observing that $\mathrm{Aut}(M, g)$ is a closed subgroup of the group of affine diffeomorphisms of M with respect to the Levi–Civita connection D of (M, g). Or else one can replace $Gl(M)$ in the previous discussion by the bundle $\pi : O(M) \to M$ of orthonormal frames. Now D induces a parallelization $\Phi = (\Phi_v, \Phi_h)$ of $O(M)$ and $\mathrm{Aut}(M, g)$ is a closed subgroup of $\mathrm{Aut}(\Phi)$ as in the previous example.

Other applications include the group of conformal transformations of a conformal structure or of all projective transformations of a projective structure. For this, see [Ko] or [Ba].

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