HOMOGENEOUS STRUCTURES

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In these lecture notes I discuss the theorem of Ambrose and Hicks on parallel translation of torsion and curvature [Am56], [Hi59] and the Lie theoretic description of affine manifolds with parallel torsion and curvature by Nomizu [No54]. The work of Ambrose, Hicks and Nomizu is based on earlier work of E. Cartan on the corresponding problems for locally symmetric Riemannian spaces. Some immediate applications concern results of Ambrose and Singer [AS58] and of Kostant [Ko60]. A related discussion and more references can be found in [He62], [KN63], [KN69], [TV83], and [NT90].

In Section 1 I discuss some preliminaries about affine manifolds, in Section 2 parallel translation of torsion and curvature after Ambrose and Hicks. Up to an immediate application of the result of Ambrose and Hicks, Theorem 3.2, the results in Sections 3–5 are more or less due to Nomizu. In Section 6 I discuss applications to Riemannian geometry.

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Contents

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1. Preliminaries about Affine Manifolds

We assume some familiarity with connections on manifolds. Recall that an *affine manifold* is a manifold together with a connection.

For the convenience of the reader we recall some definitions and general facts. Let M be an affine manifold with connection D, torsion T and curvature R. We will need the *first* and *second Bianchi identity*,

$$\mathcal{S}\{D_X T(Y,Z)\} = \mathcal{S}\{T(X,T(Y,Z))\} + \mathcal{S}\{R(X,Y)Z\} \tag{1.1}$$

$$\mathcal{S}\{D_X R(Y,Z)\} = \mathcal{S}\{R(X,T(Y,Z))\}\,, \tag{1.2}$$

where $\mathcal{S}$ denotes the sum over all cyclic permutations of X, Y and Z. Another formula we need is the *Jacobi equation*,

$$J'' + R(J,\dot c)\dot c = T(\dot c, J') + T'(\dot c, J)\,, \tag{1.3}$$

where c is a geodesic and the prime denotes covariant differentiation along c.

A continuous curve $c : I \to M$ is called a *geodesic polygon* if there is a subdivision $\dots t_{i-1} < t_i < t_{i+1} \dots$ of I such that $c|[t_{i-1}, t_i]$ is a geodesic for all i. For a point $p \in M$, we denote by Π_p the space of all geodesic polygons $c : [0,1] \to M$ with $c(0) = p$, endowed with the compact–open topology.

Let $p \in M$ and $c \in \Pi_p$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a subdivision of $[0,1]$ such that $c|[t_{i-1}, t_i]$ is a geodesic, $1 \le i \le k$. Let $v_i \in T_pM$ be the parallel translate of $\dot c(t_i + 0)$ along $c|[t_0, t_i]$, $0 \le i \le k-1$. Then c is completely determined by the data

$$(v_0, t_1), \dots, (v_{k-1}, t_k) \in T_pM \times [0,1]\,.$$

If M is geodesically complete, any such family of pairs in $T_pM \times [0,1]$ determines a geodesic polygon in Π_p. Since subdivision points may also occur in smooth points of c, the correspondence between c and the data is not one to one. The data correspond to geodesic polygons together with a subdivision of $[0,1]$.

1.4. Lemma. *Let M be a connected affine manifold. Then for any two points $p, q \in M$, there is a geodesic polygon $c : [0,1] \to M$ from p to q.*

Proof. Say that two points $p, q \in M$ are equivalent if there is a geodesic polygon connecting p and q. This defines an equivalence relation on M. The equivalence classes are open and non-empty. The claim now follows since M is connected. ☐

Let M and $\tilde M$ be manifolds of the same dimension and let $f : M \to \tilde M$ be a smooth map. Assume first that f is a diffeomorphism. For a vector field X on M, denote by f_*X the push forward of X by f,

$$f_*X(f(p)) = f_{*p}(X(p))\,. \tag{1.5}$$

Then for a connection $\tilde{D}$ on $\tilde{M}$, we obtain a connection $D = f^*\tilde{D}$ on M by setting

$$f_*(D_XY) = D_{f_*X}f_*Y\,, \tag{1.6}$$

for vector fields X and Y on M. Since the covariant derivative only involves local information about X and Y, the pull back connection $f^*\tilde{D}$ is also defined when f is a local diffeomorphism. With respect to local coordinate systems (x, U) on M and $(\tilde{x}, \tilde{U})$ on $\tilde{M}$, the Christoffel symbols Γ^k_{ij} of the pull back connection $D = f^*\tilde{D}$ and $\tilde{\Gamma}^k_{ij}$ of $\tilde{D}$ are related by

$$\Gamma^k_{ij}\partial_k f^l = \tilde{\Gamma}^l_{mn}\cdot\partial_i f^m\cdot\partial_j f^n + \partial^2_{ij}f^l\,, \tag{1.7}$$

where we evaluate in points which correspond to each other under f.

Now let M and $\tilde{M}$ be affine manifolds of the same dimension with connections D and $\tilde{D}$ respectively and let $f : M \to \tilde{M}$ be a smooth map. Then we say that f is an *affine map* if f is a local diffeomorphism and $f^*\tilde{D} = D$. In terms of local coordinates as above this means that the Christoffel symbols satisfy (1.7). In particular, if $x = \tilde{x}\circ f$ — since f is a local diffeomorphism such a choice of coordinates is always possible —, then

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij}. \tag{1.8}$$

The identity, the composition of affine maps and the inverse to an affine diffeomorphism are affine. In particular, the space of affine diffeomorphisms of M is a group, denoted here by $\mathfrak{A}(M)$. We also call elements of $\mathfrak{A}(M)$ *affine transformations* of the manifold M.

The following assertions about affine maps are immediate from the definition but (1.8) makes them even more obvious.

1.9. Proposition. *A local diffeomorphism $f : M \to \tilde{M}$ is affine if and only if for all smooth curves c in M and parallel vector fields X along c, the vector field $\tilde{X} = f_*X$ is parallel along $\tilde{c} = f\circ c$. That is, if P denotes parallel translation, then*

$$f_*\circ P_c = P_{\tilde{c}}\circ f_*\,,$$

1.10. Corollary. *If $f : M \to \tilde{M}$ is affine and c is a geodesic in M, then $f\circ c$ is a geodesic in $\tilde{M}$. That is,*

$$\widetilde{\exp}\circ f_* = f\circ\exp\,.$$

1.11. Remark. 1) It is immediate from Lemma 1.4 and Corollary 1.10 that an affine map $f : M \to \tilde{M}$ between affine manifolds in uniquely determined by its value and differential at a point of M if M is connected.

2) Let $f : M \to \tilde{M}$ be an affine map, $p \in M$ and $\tilde{p} = f(p) \in \tilde{M}$. Let $c \in \Pi_p$. Then by Corollary 1.10, $\tilde{c} = f \circ c \in \Pi_{\tilde{p}}$. If $(v_0, t_1), \dots, (v_{k-1}, t_k)$ is data for c as above (with $t_k = 1$) and $\tilde{v}_i := f_{*p} v_i$, then by Proposition 1.9, $(\tilde{v}_0, t_1), \dots, (\tilde{v}_{k-1}, t_k)$ is data for $\tilde{c}$. This will be one of the key observations in the discussion of affine maps further on.

For a local diffeomorphism $f : M \to \tilde{M}$ between smooth manifolds, and a tensor A on $\tilde{M}$ denote by f^*A the pull back of A to M. We now discuss the pull back of tensor fields with respect to affine maps. We start with the most important tensor fields in affine geometry, namely torsion and curvature. In local coordinates, their coefficients are expressions involving the Christoffel symbols and their first derivatives. Hence (1.8) gives the following result.

1.12. Proposition. *Suppose $f : M \to \tilde{M}$ is affine. Then $f^*\tilde{T} = T$ and $f^*\tilde{R} = R$.*

The following assertion is immediate from Proposition 1.9.

1.13. Proposition. *Suppose M is connected and $f : M \to \tilde{M}$ is affine. Let A be a parallel tensor field on M, $\tilde{A}$ be a parallel tensor field on $\tilde{M}$. Then if $f_p^* \tilde{A}_{f(p)} = A_p$ for one $p \in M$, then $f^*\tilde{A} = A$.*

For us, the most important example is the case where A and $\tilde{A}$ are Riemannian metrics. Recall that a connection D on a Riemannian manifold is metric if and only if the metric is parallel with respect to D. Hence Proposition 1.13 has the following immediate application.

1.14. Corollary. *Let M and $\tilde{M}$ be Riemannian manifolds, endowed with metric connections D and $\tilde{D}$. Assume that M is connected and that $f : M \to \tilde{M}$ is affine. Then if f_{*p} is orthogonal for one $p \in M$, then f is a local isometry.*

In fact, the following result of Kobayashi shows that it is quite hard for a diffeomorphism of a Riemannian manifold to be affine with respect to the Levi–Civita connection, but not to be isometric.

1.15. Theorem. *Let M be a complete and irreducible Riemannian manifold with Levi–Civita connection D. If $f : M \to M$ is an affine transformation of M and $\dim M \geq 2$, then f is an isometry.*

Proof. Denote by g the metric on M. Since f is affine, $h := f^*g$ is a parallel tensor field on M. Now the holonomy of M is not reducible, hence at each point $p \in M$, h_p is a multiple of g_p. Since h is parallel, h is a constant multiple of g on all of M, hence f is a homothety.

To finish the proof we show that a complete Riemannian manifold is flat if it admits a homothety f with $f^*g = c^2 g$ and $c \neq \pm 1$. By passing to f^{-1} if necessary, we can assume that $0 < c < 1$. Then f is a contraction and hence has a fixed point p. Furthermore, for any $q \in M$, $f^n(q) \to p$ as $n \to \infty$.

The curvature tensor R of M is preserved by f, hence for any point $q \in M$ and pair $u, v \in T_qM$ of orthonormal vectors we have

$$R(f_*^n u, f_*^n v) f_*^n v = f_*^n R(u,v)v\,.$$

The vectors $f_*^n u$ and $f_*^n v$ are perpendicular with norm c^n. Hence if K denotes the sectional curvature of M, then

$$\begin{aligned} K(f_*^n u \wedge f_*^n v) &= c^{-4n} \langle R(f_*^n u, f_*^n v) f_*^n v, f_*^n u \rangle \\ &= c^{-4n} \langle f_*^n R(u,v)v, f_*^n u \rangle \\ &= c^{-2n} \langle R(u,v)v, u \rangle = c^{-2n} K(u \wedge v)\,. \end{aligned}$$

Since $f^n(q) \to p$, the left hand side remains bounded. We conclude that $K(u \wedge v) = 0$. Hence M is flat. □

We now turn to the infinitesimal version of affine transformations. Let X be a vector field on M and let $f = f_t$ be the flow of X. We say that X is an *affine vector field* if for every t, f_t is an affine map on its domain of definition. To characterize affine vector fields by a differential equation we introduce the Lie derivative of a connection D,

$$\begin{aligned} (\mathcal{L}_X D)_Y Z &= \lim_{t \to 0} \frac{1}{t}((f_t^* D)_Y Z - D_Y Z) \\ &= [X, D_Y Z] - D_{[X,Y]} Z - D_Y [X, Z]\,. \end{aligned} \tag{1.16}$$

Note that $\mathcal{L}_X D$ is tensorial in Y and Z.

1.17. Proposition. *A vector field X on M is affine if and only if $\mathcal{L}_X D = 0$. The space of affine vector fields on M is a Lie subalgebra of the space of all vector fields.*

Proof. Since $f = f_t$ is a flow,

$$\begin{aligned} f_t^*(\mathcal{L}_X D) &= f_t^*\big(\lim_{\tau \to 0} \frac{1}{\tau}(f_\tau^*(D) - D)\big) \\ &= \lim_{\tau \to 0} \frac{1}{\tau} f_t^*\big(f_\tau^*(D) - D\big) \\ &= \lim_{\tau \to 0} \frac{1}{\tau}\big(f_{t+\tau}^*(D) - f_t^*(D)\big)\,. \end{aligned}$$

Therefore

$$f_t^*(D) - D = \int_0^t f_\tau^*(\mathcal{L}_X D) d\tau \,,$$

hence the first claim. The second claim is an immediate consequence. □

We now derive a characterization of affine vector fields by another differential equation, a global form of the Jacobi equation. Note that the flow of an affine vector field X maps geodesics to geodesics, hence X restricted to a geodesic c is a Jacobi field along c.

1.18. PROPOSITION. *A vector field X on M is affine if and only if for all vector fields Y and Z on M*

$$D^2X(Y,Z) + R(X,Y)Z = T(Z, D_Y X) + (D_Y T)(Z, X) \,,$$

where D^2X denotes the second covariant derivative of X.

Proof. Direct calculation shows that the right hand side is equal to

$$D^2X(Y,Z) + R(X,Y)Z - (\mathcal{L}_X D)_Y Z \,.$$

Now the claim is immediate from Proposition 1.17. □

We say that M or D respectively is *complete* if the maximal domain of definition of geodesics is $\mathbb{R}$. In difference to complete and connected Riemannian manifolds (with the Levi–Civita connection), the exponential map for complete and connected affine manifolds need not be surjective. An example is the Lie group $G = Sl_2(\mathbb{R})$ with the Levi–Civita connection D for the bi–invariant pseudo–Riemannian metric induced by the Killing form: Since the metric is bi–invariant, the 1–parameter subgroups and their left translates are the geodesics of D, hence D is complete. The exponential map is not surjective however.

1.19. PROPOSITION. *If M is complete, then affine vector fields on M are complete.*

Proof. We can assume that M is connected. Let X be an affine field on M and f_t be the flow of X. Let $p \in M$ and assume that $f_t(p)$ is defined on $(-\varepsilon, \varepsilon)$. Let $q \in M$ and $c = c(s)$, $0 \le s \le 1$, be a geodesic polygon from p to q. Let

$$(v_0, s_1), \dots, (v_{k-1}, s_k) \in T_pM \times [0,1]$$

be data for c (with $s_k = 1$). For $t \in (-\varepsilon, \varepsilon)$, set $v_i(t) = f_{t*}(v_i)$. Since M is complete, there exists a geodesic polygon c_t starting from $f_t(p)$ and with data

$$(v_0(t), s_1), \dots, (v_{k-1}(t), s_k) \,.$$

For each $t \in (-\varepsilon, \varepsilon)$, there is a $\delta > 0$ such that $f_\tau(c_t(s))$ is defined for all $s \in [0,1]$ and $\tau \in (-\delta, \delta)$.

By Proposition 1.10, $f_\tau(c_t(1)) = c_{t+\tau}(1)$. Since f is a flow, $f_t(q) = f_t(c(1))$ exists for all $t \in (-\varepsilon, \varepsilon)$ and is equal to $c_t(1)$. Since q is arbitrary, X is complete. □

2. Parallel Translation of Torsion and Curvature

Let M and $\tilde{M}$ be affine manifolds. Let $p \in M$, $\tilde{p} \in \tilde{M}$ and $L : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isomorphism. Suppose that U_0 is an open neighborhood of 0 in T_pM, star shaped with respect to 0, such that

$$\exp : U_0 \to U := \exp(U_0) \quad \text{and} \quad \widetilde{\exp} : LU_0 \to \tilde{U} := \widetilde{\exp}(LU_0)$$

are diffeomorphisms. Then $f = \widetilde{\exp} \circ L \circ \exp^{-1}$ is a diffeomorphism from U to $\tilde{U}$ with $f(p) = \tilde{p}$ and $df(p) = L$.

For $q \in U$, denote by $c_q : [0,1] \to U$ the radial geodesic from p to q and set $\tilde{q} = f(q) \in \tilde{U}$ and $\tilde{c}_q = f \circ c_q$. Furthermore, denote by P_q parallel translation along c_q from T_pM to T_qM and by $\tilde{P}_q$ the corresponding parallel translation along $\tilde{c}_q$.

2.1. Theorem. *The map $f = \widetilde{\exp} \circ L \circ \exp^{-1}$ above is affine if and only if $L^*\tilde{P}_q^*\tilde{T}_{\tilde{q}} = P_q^*T_q$ and $L^*\tilde{P}_q^*\tilde{R}_{\tilde{q}} = P_q^*R_q$ for all $q \in U$.*

Proof. Necessity of the condition on the pull back of torsion and curvature follows from Proposition 1.12. Assume now that f satisfies these conditions.

Choose a basis $b_1, \dots, b_n$ to identify $T_pM = \mathbb{R}^n$. Let $x = \exp^{-1} : U \to U_0$ and $\tilde{x} = L^{-1} \circ \widetilde{\exp}^{-1} : \tilde{U} \to U_0$. In these coordinates, f is represented by the identity, that is, $\tilde{x} \circ f \circ x^{-1} = \text{id}$. Hence using the chosen coordinates, we may think of the connections D of M and $\tilde{D}$ of $\tilde{M}$ as connections on $U_0 \subset \mathbb{R}^n$. Our aim is to show that D and $\tilde{D}$ are equal. Note that with respect to both connections, lines through 0 are geodesics. We will call these lines *radial geodesics*.

Let $E_1, \dots, E_n$ be the frame on U_0 which is D–parallel along radial geodesics and such that $E_m(0) = b_m$. Similarly, let $\tilde{E}_1, \dots, \tilde{E}_n$ be the frame on U_0 which is $\tilde{D}$–parallel along radial geodesics and with $\tilde{E}_m(0) = b_m$. Let T_{ij}^l and R_{ijk}^l be the coefficients of T and R with respect to the frame $E_1, \dots, E_n$ and $\tilde{T}_{ij}^l$ and $\tilde{R}_{ijk}^l$ be the coefficients of $\tilde{T}$ and $\tilde{R}$ with respect to the frame $\tilde{E}_1, \dots, \tilde{E}_n$. Now $\tilde{E}_m(0) = E_m(0)$, hence our assumption on the pull backs of torsion and curvature implies that $T_{ij}^l = \tilde{T}_{ij}^l$ and $R_{ijk}^l = \tilde{R}_{ijk}^l$.

Let $c(t) = tv$, $0 \le t \le 1$, be a radial geodesic in U_0. Then for any fixed m, $J_m(t) = tb_m$, $0 \le t \le 1$, is a Jacobi field along c — with respect to both connections — corresponding to the variation by the radial geodesics $c_s(t) = t(v + sb_m)$. Hence $J = J_m$ satisfies the Jacobi equation

$$J'' + R(J, \dot{c})\dot{c} = T(\dot{c}, J') + T'(\dot{c}, J)$$

with respect to D and the corresponding Jacobi equation with respect to $\tilde{D}$. Therefore the coefficients α^l_m of J_m with respect to the frame $E_1, \dots, E_n$ satisfy the system of differential equations

$$\ddot{\alpha}^l_m + R^l_{ijk}\alpha^i_m v^j v^k = T^l_{ij} v^i \dot{\alpha}^j_m + \dot{T}^l_{ij} v^i \alpha^j_m \,,$$

where $\dot{c} = v^m E_m$. The initial condition is

$$\alpha^l_m(0) = 0 \,, \quad \dot{\alpha}^l_m(0) = \delta^l_m \,.$$

The corresponding statement holds for the coefficients $\tilde{\alpha}^l_m$ of J_m with respect to the frame $\tilde{E}_1, \dots, \tilde{E}_n$. Now $v^m = \tilde{v}^m$ since c is a geodesic with respect to both connections, hence $\tilde{\alpha}^l_m = \alpha^l_m$. The vector fields $tb_1, \dots, tb_n$ are a frame along $c|(0,1]$ and therefore $E_m = \tilde{E}_m$, $1 \le m \le n$, along c. Since c was arbitrary we conclude that $E_m = \tilde{E}_m$, $1 \le m \le n$, on all of U_0.

Next we consider the Christoffel symbols Γ^k_{ij} of D and $\tilde{\Gamma}^k_{ij}$ of $\tilde{D}$ with respect to the frame $E_1, \dots, E_n$ on U_0,

$$D_{E_i} E_j = \Gamma^k_{ij} E_k \quad \text{and} \quad \tilde{D}_{E_i} E_j = \tilde{\Gamma}^k_{ij} E_k \,.$$

Now if c is a curve in an open subset U_0 of an affine manifold and X, Y are vector fields on U_0 such that Y is parallel along c, then $D_t D_X Y = R(\dot{c}, X)Y$. This we apply to the vector fields E_m along a radial geodesic c as above to get a system of differential equations for the Christoffel symbols as functions of the parameter t of c,

$$\dot{\Gamma}^k_{ij} E_k = D_t(\Gamma^k_{ij} E_k) = D_t D_{E_i} E_j = R(\dot{c}, E_i)E_j = v^l R^k_{lij} E_k \,,$$

where $\dot{c} = v^m E_m$. Now the Christoffel symbols $\tilde{\Gamma}^k_{ij}$ satisfy the same system of differential equations. Since the covariant derivatives of $E_1, \dots, E_n$ with respect to D and $\tilde{D}$ vanish at 0,

$$\Gamma^k_{ij}(0) = \tilde{\Gamma}^k_{ij}(0) = 0 \,,$$

we conclude that the Christoffel symbols of the two connections are equal. Hence $D = \tilde{D}$. □

2.2. Lemma. *Let M be an affine manifold. Then for any two points $p, q \in M$,*
1) if $c : [0,1] \to M$ is a continuous curve from p to q, then c is homotopic to a geodesic polygon from p to q;

2) if two geodesic polygons $c_0, c_1 : [0,1] \to M$ from p to q are homotopic as continuous curves, then there is a homotopy $c_s : [0,1] \to M$, $0 \le s \le 1$, from c_0 to c_1 admitting a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $c_s|[t_{i-1}, t_i]$ is a geodesic for all $s \in [0,1]$, $1 \le i \le k$.

We refrain from discussing the straightforward but tedious proof of these rather obvious assertions.

Let M and $\tilde{M}$ be affine manifolds and suppose in addition that $\tilde{M}$ is complete. Let $p \in M$, $\tilde{p} \in \tilde{M}$ be points and $L : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isomorphism. For a geodesic polygon $c \in \Pi_p$, let $(v_0, t_1), \dots, (v_{k-1}, t_k)$ be data in $T_pM \times [0,1]$ (with $t_k = 1$). Since $\tilde{M}$ is complete, there is a (unique) polygon $\tilde{c} \in \Pi_{\tilde{p}}$ with data $(Lv_0, t_1), \dots, (Lv_{k-1}, t_k)$ and $\tilde{c}$ is independent of the choice of data for c. For $c \in \Pi_p$ and the corresponding $\tilde{c} \in \Pi_{\tilde{p}}$, we denote by P_c parallel translation along c and $\tilde{P}_c$ parallel translation along $\tilde{c}$.

2.3. Theorem. *Suppose that M is simply connected and $\tilde{M}$ is complete and that $L^*\tilde{P}_c^*\tilde{T}_{\tilde{q}} = P_c^*T_q$ and $L^*\tilde{P}_c^*\tilde{R}_{\tilde{q}} = P_c^*R_q$ for all $c \in \Pi_p$, where $q = c(1)$ and $\tilde{q} = \tilde{c}(1)$. Then there is a unique affine map $f : M \to \tilde{M}$ with $f(p) = \tilde{p}$ and $f_{*p} = L$. Moreover, if M is complete and $\tilde{M}$ is connected, then f is a covering map.*

Proof. For $q \in M$, denote by $\Pi_{pq} \subset \Pi_p$ the space of geodesic polygons $c : [0,1] \to M$ from p to q, endowed with the compact–open topology. By Lemma 1.4, Π_{pq} is not empty. For $c \in \Pi_{pq}$ and the corresponding $\tilde{c} \in \Pi_{\tilde{p}}$ as above set $\tilde{q} = f(q) = \tilde{c}(1)$. The main point of the proof is to show that f is in fact well defined, that is, $\tilde{q}$ is independent of the choice of c.

Now M is simply connected and hence by Lemma 2.2, Π_{pq} is connected. Hence it suffices to show that for any geodesic polygon $c_0 \in \Pi_{pq}$, $\tilde{c}_1(1) = \tilde{c}_0(1)$ for any geodesic polygon $c_1 \in \Pi_{pq}$ sufficiently close to c_0.

For a given subdivision $0 = t_0 < t_1 < \cdots < t_k$ of $[0,1]$, denote by P_i and $\tilde{P}_i$ parallel translation along $c_0|[t_{i-1}, t_i]$ and $\tilde{c}_0|[t_{i-1}, t_i]$ respectively and set

$$L_i = \tilde{P}_i \circ L_{i-1} \circ P_i^{-1},$$

where $L_0 = L$. Then our assumption on the parallel translation of torsion and curvature implies that the triples $(p_i = c_0(t_i), \tilde{p}_i = \tilde{c}_0(t_i), L_i)$ satisfy the hypothesis of Theorem 2.1. Hence if the subdivision is chosen fine enough, then for any $i \le k-2$ there are open neighborhoods U_i of $c_0([t_i, t_{i+2}])$, $\tilde{U}_i$ of $\tilde{c}_0([t_i, t_{i+2}])$ and affine diffeomorphisms $f_i : U_i \to \tilde{U}_i$ with $f_i(c_0(t_i)) = \tilde{c}_0(t_i)$ and $df_i(c_0(t_i)) = L_i$. For $i \ge 2$ we choose neighborhoods V_i of $c_0(t_i)$ such that for any point $p \in V_i$, there is a piecewise smooth curve from $c_0(t_{i-1})$ to p which is contained in the intersection $U_{i-2} \cap U_{i-1}$.

Now let $c_1 \in \Pi_{pq}$ be a polygon with $c_1([t_i, t_{i+2}]) \subset U_i$ and $c_1(t_i) \in V_i$. (It is not required that $c_1|[t_{i-1}, t_i]$ be a geodesic, c_1 might have a different set of breaks.) We show that $\tilde{c}_1(1) = \tilde{c}_0(1)$.

For $i \geq 1$, let $\sigma_i : [t_i, t_{i+1}] \to U_i \cap U_{i-1}$ be a piecewise smooth curve from $c_0(t_i)$ to $c_1(t_{i+1})$, where we choose $\sigma_{k-1} := c_0|[t_{k-1}, 1]$. Let $\tilde{\sigma}_i = f_{i-1} \circ \sigma_i$. Denote by Q_i, $\tilde{Q}_i$, R_i and $\tilde{R}_i$ parallel translation along $c_1|[t_{i-1}, t_i]$, $\tilde{c}_1|[t_{i-1}, t_i]$, σ_i and $\tilde{\sigma}_i$ respectively. Set $K_i = \tilde{Q}_i \circ K_{i-1} \circ Q_i^{-1}$, where $K_0 = L$.

Since the curves $c_0|[0, t_1]$, $c_1|[0, t_2]$ and σ_1 are contained in U_0 and f_0 is affine with $f_0(p) = \tilde{p}$ and $df_0(p) = L$, we have

$$\tilde{c}_1(t_2) = f_0(c_1(t_2)) = f_0(\sigma_1(t_2)) = \tilde{\sigma}_1(t_2)\,,$$

$$L_1 = \tilde{P}_1 \circ df_0(p) \circ P_1^{-1} = df_0(c_0(t_1))$$

and similarly $K_1 = df_0(c_1(t_1))$ and $K_2 = df_0(c_1(t_2)) = \tilde{R}_1 \circ L_1 \circ R_1^{-1}$. Inductively we assume for some $i \leq k-2$ that

$$\tilde{\sigma}_i(t_{i+1}) = \tilde{c}_i(t_{i+1}) \quad \text{and} \quad K_{i+1} = \tilde{R}_i \circ L_i \circ R_i^{-1}.$$

Since the four curves $c_0|[t_i, t_{i+1}]$, $c_1|[t_{i+1}, t_{i+2}]$, σ_i and σ_{i+1} are in U_i and since f_i is affine with $f_i(c_0(t_i)) = \tilde{c}_0(t_i)$ and $df_i(c_0(t_i)) = L_i$, we have $K_{i+1} = df_i(c_1(t_{i+1}))$ and hence $\tilde{c}_0 = f_i \circ c_0$ on $[t_i, t_{i+1}]$ and $\tilde{c}_1 = f_i \circ c_1$ on $[t_{i+1}, t_{i+2}]$. In particular, $\tilde{\sigma}_{i+1}(t_{i+2}) = \tilde{c}_1(t_{i+2})$. Again since f_i is affine we have $L_{i+1} = df_i(c_0(t_{i+1}))$ and

$$K_{i+2} = df_i(c_1(t_{i+2})) = \tilde{R}_{i+1} \circ L_{i+1} \circ R_{i+1}^{-1}\,.$$

We conclude that induction applies and hence that $\tilde{c}_1(1) = \tilde{c}_0(1)$. Therefore f is well defined.

Locally about the point q as above we have $f = f_{k-2}$ in the above notation. Hence f is affine.

To prove the last assertion in the theorem, we assume first that $\tilde{M}$ is simply connected and complete. Then the roles of M and $\tilde{M}$ can be reversed: The first part of the theorem applies to L^{-1} and we obtain an affine map $\tilde{f} : \tilde{M} \to M$ with $\tilde{f}(\tilde{p}) = p$, $d\tilde{f}(\tilde{p}) = L^{-1}$. But then $\tilde{f} \circ f$ is an affine map from M to M with $f(p) = p$ and $df(p) = \mathrm{id}$. By uniqueness, $\tilde{f} \circ f$ is the identity of M. Similarly, $f \circ \tilde{f}$ is the identity of $\tilde{M}$ and hence f is a diffeomorphism. Now the general case follows by passing to the universal covering space of $\tilde{M}$ with the induced connection. □

Among the various immediate applications of Theorem 2.3 the following one is quite useful.

2.4. Corollary. *Let M and $\tilde{M}$ be connected affine manifolds and $f : M \to \tilde{M}$ be an affine map. If M is complete, then $\tilde{M}$ is complete and f is a covering map.*

3. Homogeneous Structures: Uniqueness

Let M be an affine manifold with parallel torsion and curvature, that is, $DT = 0$ and $DR = 0$. Then for any two points $p, q \in M$ and piecewise smooth curve c connecting p to q we have $P_c^* T_q = T_p$ and $P_c^* R_q = R_p$. This weakens the conditions needed on parallel translation of torsion and curvature considered in Theorem 2.1 and Theorem 2.3.

3.1. Theorem. *Let M, $\tilde{M}$ be affine manifolds with parallel torsion and curvature. Let $p \in M$, $\tilde{p} \in \tilde{M}$ and $L : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isomorphism with $L^*(\tilde{T}_{\tilde{p}}) = T_p$ and $L^*(\tilde{R}_{\tilde{p}}) = R_p$. Then there are open neighborhoods U of p in M, $\tilde{U}$ of $\tilde{p}$ in $\tilde{M}$ and an affine diffeomorphism $f : U \to \tilde{U}$ with $f(p) = \tilde{p}$ and $df(p) = L$.*

Proof. Since M and $\tilde{M}$ have parallel torsion and curvature, the assumption on the pull back of torsion and curvature in p implies that the hypothesis of Theorem 2.1 is satisfied. □

3.2. Theorem. *Let M, $\tilde{M}$ be affine manifolds with parallel torsion and curvature. Assume that M is simply connected and $\tilde{M}$ is complete. Let $p \in M$, $\tilde{p} \in \tilde{M}$ and $L : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isomorphism with $L^*(\tilde{T}_{\tilde{p}}) = T_p$ and $L^*(\tilde{R}_{\tilde{p}}) = R_p$. Then there is an affine map $f : M \to \tilde{M}$ with $f(p) = \tilde{p}$ and $df(p) = L$. Moreover, if M is complete and $\tilde{M}$ is connected, then f is a covering map.*

Proof. Since M and $\tilde{M}$ have parallel torsion and curvature, the assumption on the pull back of torsion and curvature in p implies that the hypothesis of Theorem 2.3 is satisfied. □

3.3. Examples. We now shortly discuss these results in the case where $M = \tilde{M}$. We assume that M is a connected affine manifold with parallel torsion and curvature.

1) For any two points $p, q \in M$ and piecewise smooth curve c from p to q, $L = P_c$ satisfies the assumption of Theorem 3.1, where $M = \tilde{M}$ and $q = \tilde{p}$. Hence M is locally affinely homogeneous. If M is simply connected and complete, L satisfies the assumption of Theorem 3.2. It follows that M is affinely homogeneous.

2) Assume in addition that the torsion is not only parallel, but vanishes identically, so $T = 0$ and $DR = 0$. Then for each point $p \in M$, $L = -\mathrm{id} : T_pM \to T_pM$ satisfies the assumption of Theorem 3.1,

where $M = \tilde{M}$ and $p = \tilde{p}$: For the torsion this is clear since it vanishes identically. For the curvature tensor we have

$$L(R_p(u,v)w) = -R_p(u,v)w = R_p(-u,-v)(-w) = R_p(Lu,Lv)Lw$$

since R has an odd number of arguments. If M is simply connected and complete, L satisfies the assumption of Theorem 3.2. This leads to the theory of (affinely) symmetric spaces.

3) Let M be a simply connected complete affine manifold with $DT = 0$ and $R = 0$. Since M is simply connected with $R = 0$, parallel translation is independent of the path and defines a global trivialization of TM by parallel vector fields. Hence by Theorem 3.2, for any two points $p, q \in M$, there is a natural affine transformation $f_{q,p}$ of M mapping p to q, namely the one whose differential leaves the parallelization of M invariant. It is clear that $f_{q,p}$ depends smoothly on p and q and that

$$f_{p,p} = \mathrm{id}\,, \quad f_{p,q} = f_{q,p}^{-1} \quad \text{and} \quad f_{r,q} \circ f_{q,p} = f_{r,p}\,.$$

It follows easily that M is a Lie group, where the parallelization corresponds to the Lie algebra of left invariant vector fields and the maps $f_{q,p}$ correspond to left translations. Vice versa, if M is a Lie group, then requiring left invariant vector fields to be parallel defines a connection on M with $DT = 0$ and $R = 0$.

3.4. Remark. That M has parallel torsion and curvature can be restated in a different way. For this we assume that a connection D on M is given. We ask when the connection $\bar{D} = D - S$ has parallel torsion $\bar{T}$ and curvature $\bar{R}$, where S is a given or looked for $(2,1)$–tensor. We note first that torsion and curvature tensors of D and $\bar{D}$ are related by

$$\begin{aligned} T(X,Y) &= \bar{T}(X,Y) + S_X Y - S_Y X\,; \\ R(X,Y)Z &= \bar{R}(X,Y)Z + [S_X, S_Y]\cdot Z + S(\bar{T}(X,Y),Z) \\ &\qquad + (\bar{D}_X S)(Y,Z) - (\bar{D}_Y S)(X,Z)\,. \end{aligned}$$

Hence if S is $\bar{D}$–parallel, then $\bar{T}$ and $\bar{R}$ are $\bar{D}$–parallel if and only if T and R are. Now S, T and R are parallel with respect to $\bar{D}$ if and only if the following three equations are satisfied,

$$\begin{aligned} (D_X S)_Y &= [S_X, S_Y] - S_{S_X Y}\,; \\ (D_X T)_Y &= [S_X, T_Y] - T_{S_X Y}\,; \\ (D_X R)_{YZ} &= [S_X, R_{YZ}] - R_{S_X Y Z} - R_{Y S_X Z}\,. \end{aligned}$$

The condition that S be parallel with respect to $\bar{D}$ ensures that $\bar{D}$–affine transformations of M are also D–affine. Thus M is affinely homogeneous with respect to the given connection D if it is affinely homogeneous with respect to the connection $\bar{D}$, compare with Corollary 5.13 and Remark 6.9.

4. Homogeneous Structures: Local Data

Let M be a affine manifold with parallel torsion and curvature.

Although corresponding local formulas can be derived for a general affine manifold M with parallel torsion and curvature, the exposition is smoother in the case when M is simply connected and complete. This we assume in the rest of this section. We emphasize, however, that the computations are local and that we will need the formulas in other cases further on (without repeating the computations).

We fix an origin $o \in M$.

Since torsion and curvature of M are parallel, the first and second Bianchi identity (1.1),(1.2) become

$$\mathcal{S}\{R(X,Y)Z + T(X,T(Y,Z))\} = 0 \tag{4.1}$$

$$\mathcal{S}\{R(X,T(Y,Z))\} = 0\,. \tag{4.2}$$

Let A be an endomorphism of T_oM. Then the 1–parameter subgroup $\exp(tA)$, $t \in \mathbb{R}$, of automorphisms of T_oM generated by A preserves torsion and curvature if and only if

$$\begin{aligned} AT(u,v) &= T(Au,v) + T(u,Av) \\ AR(u,v)w &= R(Au,v)w + R(u,Av)w + R(u,v)Aw \end{aligned} \tag{4.3}$$

for all $u,v,w \in T_oM$. The last equation can also be viewed as an equation on the commutator $[A, R(u,v)]$.

We say that an endomorphism A of T_oM *preserves torsion and curvature as a derivation* if A satisfies (4.3). A similar definition applies to fields of endomorphisms.

4.4. Lemma. *For all vector fields X, Y on M, the field $R(X,Y)$ of endomorphisms preserves T and R as a derivation.*

Proof. This is immediate from the definition of R and the hypothesis that T and R are parallel. □

We denote by $\mathfrak{g}^*$ the Lie algebra of affine vector fields on M. The reason for the star in the notation is explained by (5.2).

We first consider affine vector fields whose flow preserves o, that is, which vanish at o,

$$\mathfrak{k}^* = \{X^* \in \mathfrak{g}^* \mid X^*(o) = 0\}\,. \tag{4.5}$$

Clearly $\mathfrak{k}^*$ is a Lie subalgebra of $\mathfrak{g}^*$. Let $X^* \in \mathfrak{k}^*$ and denote by f_t the flow of X^*. The isotropy representation at o sends f_t to the differential $df_t(o)$. To compute the differential of the isotropy representation, we choose a vector $v \in T_oM$ and represent v by a smooth curve $\sigma = \sigma(s)$ with $\sigma(0) = o$ and $\dot\sigma(0) = v$. Then

$$\begin{aligned}\partial_t\big(df_t|_o(v)\big)|_{t=0} &= \partial_t\partial_s\big(f_t(\sigma(s))\big)|_{s=t=0}\\ &= D_t\partial_s\big(f_t(\sigma(s))\big)|_{s=t=0}\\ &= D_s\partial_t\big(f_t(\sigma(s))\big)|_{t=s=0} + T(X^*(o), v)\\ &= D_s\partial_t\big(f_t(\sigma(s))\big)|_{t=s=0} = D_vX^* .\end{aligned} \tag{4.6}$$

Hence the linearized isotropy representation sends X^* to the covariant derivative $DX^*(o)$. Since X^* is affine, $DX^*(o)$ preserves torsion and curvature as a derivation.

4.7. THEOREM. *Let $\mathfrak{k}$ be the Lie algebra of endomorphisms of T_oM which preserve torsion and curvature as derivations. Then the linearized isotropy representation*

$$\mathfrak{k}^* \to \mathfrak{k}\,, \quad X^* \mapsto DX^*(o)\,,$$

is an anti–isomorphism of Lie algebras.

Proof. Injectivity is clear since an affine vector field on a connected affine manifold is determined by its value and covariant derivative at a point.

In the proof of surjectivity, we use that M is simply connected and complete. Let $A \in \mathfrak{k}$ and let

$$L_t = \exp(tA) = e^{tA} = 1 + A + \frac{1}{2}A^2 + \cdots\,,\ t \in \mathbb{R}\,,$$

be the 1–parameter subgroup of automorphisms of T_oM associated to A. Then $L_t^*T = T$ and $L_t^*R = R$ since $A \in \mathfrak{k}$. Hence by Theorem 3.2 there exist affine transformations f_t of M with $f_t(o) = o$ and $df_t(o) = L_t$. It follows that f_t, $t \in \mathbb{R}$, is a smooth 1–parameter group of affine transformations of M. It is immediate from the construction that the covariant derivative at o of the affine vector field X^* of M associated to (f_t) is equal to A. This shows that the linearized isotropy representation is surjective.

Let $X^*, Y^* \in \mathfrak{k}^*$ and set $A = DX^*(o)$, $B = DY^*(o)$. For any $u \in T_oM$ we have

$$\begin{aligned}&D_u[X^*, Y^*](o) = D_u\{D_{X^*}Y^* - D_{Y^*}X^*\}\\ &= D^2Y^*(u, X^*(o)) + D_{D_uX^*}Y^* - D^2X^*(u, Y^*(o)) - D_{D_uY^*}X^* .\end{aligned}$$

Now $X^*(o) = Y^*(o) = 0$, hence the right hand side is equal to

$$D_{D_u X^*} Y^* - D_{D_u Y^*} X^* = BAu - ABu = -[A,B]\,u\,.$$

We see that the linearized isotropy representation reverses the sign of Lie brackets and hence it is an anti–homomorphism of Lie algebras. □

We now discuss the second important class of infinitesimal affine transformations associated to the choice of origin $o \in M$. Let $u \in T_o M$ and c be the geodesic through o with $\dot c(0) = u$. Let $P_t : T_o M \to T_{c(t)} M$ be parallel translation along c from $o = c(0)$ to $c(t)$. Then P_t is a linear isomorphism, and since T and R are parallel we have $P_t^* T = T$ and $P_t^* R = R$. Hence by Theorem 3.2, there are affine transformations f_t of M with $f_t(o) = c(t)$ and $df_t(o) = P_t$, so called *transvections* along the geodesic c.

Now c is a geodesic and hence $P_t v = \dot c(t)$. Hence the geodesic c is invariant under f_t, that is, $f_t(c(s)) = c(s+t)$ for all $s \in \mathbb{R}$. Since f_t is affine, it maps parallel vector fields along c to parallel vector fields along c. By our choice of $df_t(o)$, f_t simply reparameterizes these parallel vector fields. Hence $df_t(c(s))$ is parallel translation along c from $c(s)$ to $c(s+t)$. It follows that $f_{s+t} = f_s \circ f_t$ for all $s, t \in \mathbb{R}$. Hence the family f_t, $t \in \mathbb{R}$, of transvections along c constitutes a smooth 1–parameter group. The corresponding affine vector field X^* is called the *infinitesimal transvection* along c.

4.8. Theorem. *Let $u \in T_o M$ and X^* be the infinitesimal transvection along the geodesic c through o with $\dot c(0) = u$. Then $X^*(o) = u$ and $D_v X^* = T(v,u)$ for all $v \in T_o M$. More generally, $X^*(c(t)) = \dot c(t)$ and $D_v X^* = T(v, \dot c(t))$ for all $t \in \mathbb{R}$ and $v \in T_{c(t)} M$.*

Proof. Let $v \in T_o M$ and $\sigma = \sigma(s)$ be a curve through o with $\dot\sigma(0) = v$. Let (f_t) be the flow of X^*. Then

$$\begin{aligned} D_v X^* &= D_s \partial_t \big(f_t(\sigma(s))\big)|_{t=s=0} \\ &= T(v,u) + D_t \partial_s \big(f_t(\sigma(s))\big)|_{s=t=0} \\ &= T(v,u) + D_t (df_t \cdot v)|_{t=0} \\ &= T(v,u) + D_t (P_t \cdot v) = T(v,u)\,. \end{aligned}$$

As for the more general assertion, note that X^* is independent of the choice of origin, at least as long as it lies on c. □

4.9. Corollary. *The space $\mathfrak{p}^*$ of infinitesimal transvections along geodesics through o is a linear subspace of $\mathfrak{g}^*$,*

$$\mathfrak{p}^* = \{X^* \in \mathfrak{g}^* \mid D_v X^* = T(v, X^*(o)) \textit{ for all } v \in T_o M\}\,.$$

For $X^* \in \mathfrak{k}^*$ *and* $Y^* \in \mathfrak{p}^*$ *with* $Y^*(o) = v$ *we have*

$$[X^*, Y^*](o) = -D_v X^* \,.$$

Proof. Recall that an affine vector field is determined by its value and covariant derivative at a point. Now the equation for $DX^*(o)$ in Theorem 4.8 is linear in $X^*(o) = u$, and therefore the set of infinitesimal transvections X^* along geodesics through o is a linear subspace of $\mathfrak{g}^*$. The asserted formula for the Lie bracket is clear since $X^*(o) = 0$. □

Since any vector $u \in T_oM$ determines a geodesic through o and thus a 1–parameter group of transvections, we obtain a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^* \,. \tag{4.10}$$

Accordingly, any $X^* \in \mathfrak{g}^*$ has components $X^*_{\mathfrak{k}}$ in $\mathfrak{k}^*$ and $X^*_{\mathfrak{p}}$ in $\mathfrak{p}^*$. Since affine vector fields from $\mathfrak{k}^*$ vanish at o, the $\mathfrak{p}^*$–component $X^*_{\mathfrak{p}}$ of X^* is the unique transvection in $\mathfrak{p}^*$ equal to $X^*(o)$ at o.

If f is an affine transformation of M and f_t, $t \in \mathbb{R}$, is the 1–parameter group of transvections along a geodesic c through o, then ff_tf^{-1}, $t \in \mathbb{R}$, is the 1–parameter group of transvections along the geodesic $f \circ c$ through $f(o)$. In particular, for any affine transformation f of M fixing o, ff_tf^{-1}, $t \in \mathbb{R}$, is the 1–parameter group of transvections along the geodesic $f \circ c$ through $f(o) = o$. Hence for any such f,

$$f_*(\mathfrak{p}^*) \subset \mathfrak{p}^* \,, \tag{4.11}$$

where for any vector field X^* of M, f_*X^* denotes the push forward of X^* along f as in (1.5).

4.12. Theorem. *Let* $u, v, w \in T_oM$ *and* $X^*, Y^*, Z^* \in \mathfrak{p}^*$ *be the infinitesimal transvections with* $X^*(o) = u$, $Y^*(o) = v$ *and* $Z^*(o) = w$. *Then*

$$\begin{aligned}
[X^*, Y^*](o) &= T(u, v) \,, \\
D_w[X^*, Y^*]_{\mathfrak{p}} &= T(w, T(u, v)) \,, \\
D_w[X^*, Y^*]_{\mathfrak{k}} &= -[[X^*, Y^*]_{\mathfrak{k}}, Z^*](o) = R(u, v)w \,.
\end{aligned}$$

Proof. By Corollary 4.9 above we have

$$\begin{aligned}
[X^*, Y^*](o) &= D_{X^*}Y^*(o) - D_{Y^*}X^*(o) - T(u, v) \\
&= T(u, v) - T(v, u) - T(u, v) = T(u, v) \,.
\end{aligned}$$

Since $[X^*, Y^*]_{\mathfrak{k}}(o) = 0$ we also have

$$D_w[X^*, Y^*]_{\mathfrak{k}} = -[[X^*, Y^*]_{\mathfrak{k}}, Z^*](o) \,.$$

Now the torsion tensor T is parallel, therefore an affine vector field X^* satisfies the differential equation

$$D^2X^*(Y,Z) + R(X^*,Y)Z = T(Z, D_Y X^*)\,,$$

see (1.18). Hence

$$\begin{aligned} D_w[X^*,Y^*] &= D_w\{D_{X^*}Y^* - D_{Y^*}X^* - T(X^*,Y^*)\} \\ &= D^2Y^*(w,u) - D^2X^*(w,v) \\ &= -R(v,w)u + T(u,T(w,v)) \\ &\qquad + R(u,w)v - T(v,T(w,u)) \\ &= R(u,v)w + T(w,T(u,v))\,. \end{aligned}$$

Now since $[X^*,Y^*]_{\mathfrak{p}}(o) = [X^*,Y^*](o)$, the formulas for the covariant derivatives $D_w[X^*,Y^*]_{\mathfrak{p}}$ and $D_w[X^*,Y^*]_{\mathfrak{k}}$ follow immediately from Corollary 4.9 and the formula for $[X^*,Y^*](o)$. □

The following theorem is an immediate consequence of (4.1) and (4.2). Note that as input we really only need a vector space $\mathfrak{p}$, here $\mathfrak{p} = T_oM$, together with a torsion and curvature tensor on $\mathfrak{p}$ satisfying the Bianchi identities as in (4.1) and (4.2).

4.13. THEOREM. *Let $\mathfrak{k}$ be the space of all endomorphisms of T_oM preserving torsion and curvature as a derivation. Let $\mathfrak{p} = T_oM$ and set $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then*

$$[(A,u),(B,v)] = ([A,B] - R(u,v), Av - Bu - T(u,v))\,,$$

where $[A,B]$ denotes the usual commutator of endomorphisms, turns $\mathfrak{g}$ into a Lie algebra.

Theorem 4.7, Corollary 4.9 and Theorem 4.12 show that the evaluation map

$$\mathfrak{g}^* \to \mathfrak{g}\,, \quad X^* \mapsto (DX^*(o), X^*(o))$$

is an anti–isomorphism of Lie algebras. It follows that the simply connected Lie group G with Lie algebra $\mathfrak{g}$ acts on M such that for any $X \in \mathfrak{g}$, the associated vector field X^* from (5.1) corresponds to X via the above evaluation map. In particular, the stabilizer K of o in G has Lie algebra $\mathfrak{k}$.

Recall that the action of a Lie group G on a homogeneous space G/K is called *reductive* if there is an Ad_K–invariant splitting $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K. Note that the action of the above Lie group G on $M = G/K$ is reductive in this sense.

In Theorem 5.12 we will see that the splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ determines a G–invariant connection on G/K with parallel torsion and curvature.

With respect to this connection, G/K is locally affinely equivalent to the manifold M we started with.

A final remark is in order. Let $\mathfrak{p}$ be a vector space together with a torsion tensor T and a curvature tensor R satisfying the Bianchi identities (4.1) and (4.2). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Lie algebra as in Theorem 4.13. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the connected Lie subgroup K of G with Lie algebra $\mathfrak{k}$ need not be closed, see [Kw90]. I am grateful to Claudia Meusers for pointing this out to me.

5. Homogeneous Structures: Existence

We now consider the homogeneous case more closely. We start with a few general remarks on smooth actions of Lie groups. Let G be a Lie group and $\lambda : G \times M \to M$ be a smooth (left) action of G on a smooth manifold M. Depending on readability, we use the notation $\lambda(g,p) = \lambda_g(p) = g(p) = gp$. To distinguish between the diffeomorphism λ_g of M and left translation of g on G, we denote the latter by Λ_g.

The Lie algebra of left invariant vector fields of G is denoted $\mathfrak{g}$. As usual, we identify $\mathfrak{g} = T_eG$, where e is the neutral element of G. For any $X \in \mathfrak{g}$, the 1–parameter group $g_t = \exp(tX)$, $t \in \mathbb{R}$, of G generates a vector field X^* on M,

$$X^*(p) = \partial_t(g_t(p))|_{t=0} = \partial_t(e^{tX}p)|_{t=0}\,. \tag{5.1}$$

By definition, $\exp(tX)$ is the flow of X^*. We have

$$[X,Y]^* = -[X^*,Y^*]\,. \tag{5.2}$$

Thus

$$\mathfrak{g}^* = \{X^* \mid X \in \mathfrak{g}\} \tag{5.3}$$

is a Lie subalgebra of the Lie algebra of all vector fields on M, and the map $\mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto X^*$, is an anti–homomorphism.

From now on we assume that the action of G on M is transitive. We fix a point $o \in M$ and let

$$K = \{g \in G \mid g(o) = o\} \tag{5.4}$$

be the stabilizer of o. The Lie algebra of K is

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X^*(o) = 0\}\,. \tag{5.5}$$

The smooth map $\pi : G \to M$, $\pi(g) = go$, is a submersion. Note that

$$\lambda_g \circ \pi = \pi \circ \Lambda_g \tag{5.6}$$

for all $g \in G$. The fiber of π over $p = go \in M$ is gK and with respect to the right action of K on G, π is a principal bundle over M with structure group K. For any $g \in G$, the kernel of the differential of π at g,

$$\mathcal{V}(g) = \ker d\pi(g) = \Lambda_{g*}(\mathfrak{k}) , \tag{5.7}$$

is called the *vertical subspace* of G at g. We have $\mathcal{V}(e) = \mathfrak{k}$. The distribution $\mathcal{V}$ of G is called the *vertical distribution*. It is left invariant, in particular smooth.

A distribution $\mathcal{H}$ of G is called *horizontal* (with respect to π) if $\mathcal{H}(g)$ is a linear complement of $\mathcal{V}(g)$ for all $g \in G$. For a horizontal distribution $\mathcal{H}$ and $g \in G$, $\mathcal{H}(g)$ is called the *horizontal subspace* of G at g. Below we will be concerned with left invariant horizontal distributions of G. Namely, we assume from now on that the action of G on M is reductive and fix an Ad_K–invariant complement $\mathfrak{p}$ of $\mathfrak{k} = \mathcal{V}(e)$ in $\mathfrak{g} = T_eG$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} , \quad \mathrm{Ad}_K(\mathfrak{p}) \subset \mathfrak{p} . \tag{5.8}$$

The corresponding left invariant horizontal distribution

$$\mathcal{H}(g) = \Lambda_{g*}(\mathfrak{p}) \tag{5.9}$$

on G is also right invariant under Ad_K. Hence it is a connection on the principal bundle $\pi : G \to M$.

For any $k \in K$ we have $k(o) = o$ by the definition of K. By a straightforward computation we get

$$d\lambda_k(o) \circ d\pi(e) = d\pi(e) \circ \mathrm{Ad}_k . \tag{5.10}$$

This equation implies that the tangent bundle TM of M is isomorphic to the bundle E associated to the K–principal bundle $\pi : G \to M$ via the adjoint representation of K on $\mathfrak{p}$: the map

$$\Phi : E \to TM , \quad \Phi([g, X)]) = \lambda_{g*}\pi_*(X) \tag{5.11}$$

establishes a canonical isomorphism. Thus the above connection $\mathcal{H}$ on the principal bundle $\pi : G \to M$ defines a connection D on M, that is, on the tangent bundle of M. This is the connection we are after.

We use the isomorphism $\pi_* : \mathfrak{p} \to T_oM$ to identify $\mathfrak{p} \cong T_oM$. This simplifies some of the statements and formulas below. For example, via this identification restriction to $\mathfrak{p}$ gives a 1–1 correspondence between G–invariant tensor fields on M and Ad_K–invariant tensors on $\mathfrak{p}$.

5.12. THEOREM. *The connection D above is the unique G–invariant connection on M such that for any $X \in \mathfrak{p}$, $c(t) = \exp(tX)o$, $t \in \mathbb{R}$, is the geodesic through o with initial velocity $\pi_*(X) = X^*(o)$ and such that* $\exp(tX)$, *$t \in \mathbb{R}$, is the 1–parameter group of transvections along c. Moreover,*

(1) *G–invariant tensor fields on M are parallel with respect to D. In particular, D has parallel torsion T and curvature R;*
(2) *for $X, Y, Z \in \mathfrak{p} \cong T_pM$, we have*

$$T(X,Y) = -[X,Y]_{\mathfrak{p}}, \quad R(X,Y)Z = -[[X,Y]_{\mathfrak{k}}, Z],$$

where $[X,Y]_{\mathfrak{k}}$ and $[X,Y]_{\mathfrak{p}}$ denote the components of $[X,Y]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.
(3) *any automorphism φ of G leaving K and $\mathfrak{p}$ invariant descends to an affine transformation f of M, and f leaves those G–invariant tensor fields on M invariant whose restriction to $T_oM \cong \mathfrak{p}$ is invariant under φ.*

Proof. Connection D is G–invariant since the connection $\mathcal{H}$ of the principal bundle $\pi : G \to M$ is left invariant. For any $X \in \mathfrak{p}$, the 1–parameter subgroup $g_t = \exp(tX)$ is horizontal, that is, $\dot{g}_t \in \mathcal{H}_{g_t}$ for all $t \in \mathbb{R}$. By definition of D, this means that g_{t*} is parallel translation along the curve $c(t) = g_t(o)$. Since g_t is a 1–parameter group, $g_{t*}\dot{c}(0) = \dot{c}(t)$ for all $t \in \mathbb{R}$. Hence $\dot{c}$ is parallel along c, that is, c is a geodesic. Therefore D has the first properties asserted in the theorem. These properties determine D at o, hence the asserted uniqueness follows from G–invariance.

Now let A be a G–invariant tensor field on M. Let $X \in \mathfrak{p}$ and set $g_t = \exp(tX)$, $c(t) = g_t(o)$. Then

$$A(c(t)) = g_{-t}^* A(o).$$

Since g_{t*} is parallel translation along the geodesic c, we conclude that A is parallel along c. Since this holds for any X in $\mathfrak{p}$ we get that $DA(o) = 0$. Now A and D are G–invariant, therefore A is parallel.

Since D is G–invariant, T and R are G–invariant. Hence T and R are parallel. The formulas for T and R are immediate consequences of (5.2) and the formulas in Theorem 4.12.

If φ is an automorphism of G leaving K and $\mathfrak{p}$ invariant, then φ descends to a diffeomorphism f of M. Since φ leaves the data which determine D invariant, f is an affine transformation of M. Clearly, tensor fields on M which are invariant under G and f correspond to tensors on $T_oM \cong \mathfrak{p}$ which are invariant under Ad_K and φ. □

5.13. Corollary. *For G, M and D as in Theorem 5.12, a connection $\tilde{D} = D + S$ on M is G–invariant if and only if the restriction of S to $T_oM \cong \mathfrak{p}$ is Ad_K–invariant.*

5.14. Remark. For G and M as in Theorem 5.12, not all G–invariant connections on M with parallel torsion and curvature are given by a splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Note that the $\mathfrak{p}$ there corresponds to the space of infinitesimal transvections along geodesics through the chosen origin o. Now $\mathfrak{g}$ might be too small to contain this particular space $\mathfrak{p}$.

A typical example occurs in the case of the upper half plane H with the hyperbolic metric. Then the group

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \,\Big|\, \alpha > 0 \right\} \subset Sl(2, \mathbb{R})$$

operates simply transitively and isometrically on H (by linear fractional transformations). In particular, G leaves the Levi-Civita connection D of H invariant. Now D is torsionfree and of constant Gauß curvature (and in particular has parallel torsion and curvature). However, through any point $o \in H$, there is only one transvection in G, namely the one along the vertical line through o.

5.15. Remark. Let M be a manifold with a transitive action by a Lie group G. Then G–invariant connections need not be complete. Easy examples are the non–zero real numbers $\mathbb{R}^*$ respectively non–zero complex numbers $\mathbb{C}^*$, both of them considered as Lie groups with respect to multiplication. The standard Euclidean connection on $\mathbb{R}^*$ and $\mathbb{C}^*$ is invariant under left and right multiplication by non–zero real respectively complex numbers, but is not complete.

However, the connection D in Theorem 5.12 is complete: the 1–parameter groups of G are defined on $\mathbb{R}$, hence all maximal geodesics through o are defined on all of $\mathbb{R}$. Now D is G–invariant, hence all maximal geodesics are defined on all of $\mathbb{R}$.

6. Riemannian Homogeneous Spaces

Let M be a Riemannian manifold with metric g, Levi-Civita connection D and curvature tensor R. We say that a map $f : M \to M$ is an *isometry* if $f^*g = g$. We say that a vector field X on M is a *Killing field* if the (local) flow $f = f_t$ of X consists of (local) isometries of M. To characterize Killing fields by a differential equation we define the Lie derivative of the metric g with respect a to vector field X by

$$\begin{aligned} \mathcal{L}_X(g) = &= \lim_{t \to 0} \frac{1}{t}((f_t^* g)(Y, Z) - g(Y, Z)) \\ &= \langle D_Y X, Z\rangle + \langle Y, D_Z X\rangle\,. \end{aligned} \tag{6.1}$$

In contrast to affine fields, Killing fields are characterized by a differential equation of first order.

6.2. Proposition. *For a vector field X on M the following properties are equivalent:*

(1) *X is a Killing field;*

(2) *$\mathcal{L}_X g = 0$;*
(3) *for all $p \in M$, $DX(p)$ is a skew symmetric endomorphism of T_pM.*

The space of Killing fields is a subalgebra of the Lie algebra of all vector fields.

The proof is similar to the one of Proposition 1.17. Moreover, the global version of the Jacobi equation (1.18) simplifies since D is torsion free.

6.3. PROPOSITION. *If X is a Killing field, then for all vector fields Y and Z*

$$D^2X(Y,Z) + R(X,Y)Z = 0\,.$$

Obviously Killing fields are complete if M is complete. The space of complete Killing fields is a subalgebra of the Lie algebra of all Killing fields and is anti–isomorphic to the Lie algebra $\mathfrak{i}(M)$ of $\mathfrak{I}(M)$.

We are interested in the question whether M is homogeneous. Note that unlike in the affine case, Riemannian homogeneous spaces are complete.

6.4. THEOREM. *If M is simply connected and complete, then if M has a metric connection $\bar{D}$ with $\bar{D}$–parallel torsion and curvature, then M is homogeneous.*

Proof. Suppose that M has a metric connection $\bar{D}$ with $\bar{D}$–parallel torsion and curvature. Let $p, q \in M$ and c be a piecewise smooth curve from p to q. Then parallel translation $\bar{P}_c : T_pM \to T_qM$ with respect to $\bar{D}$ is an isomorphism preserving the metric. By Theorem 3.2, there is a $\bar{D}$–affine transformation $f : M \to M$ with $f(p) = q$ and $df(p) = P_c$. Now $f_p^* g_q = P_c^* g_q = g_p$, and therefore $f^*g = g$ by Corollary 1.14. Hence f is an isometry. □

The other direction in Theorem 6.4 also holds, actually a somewhat stronger statement. Recall that the group $\mathfrak{I}(M)$ of isometries of a Riemannian manifold M is a Lie group when endowed with the compact–open topology. In the case where $\mathfrak{I}(M)$ is transitive on M — the case we are interested in — this is a rather immediate consequence of the fact that a closed subgroup of a Lie group is a Lie subgroup.

We consider now the following situation: G is a Lie group and $\lambda : G \times M \to M$ is an isometric, locally effective and transitive action on our Riemannian manifold M. By definition of locally effective, the closed normal subgroup

$$H = \{g \in G \mid \lambda_g = \mathrm{id}\} \tag{6.5}$$

is a discrete subgroup of G. Again we denote by K the stabilizer of the chosen origin $o \in M$.

6.6. Lemma. *The Killing form $B = B_{\mathfrak{g}}$ of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$.*

Proof. The isotropy representation $K \to O(T_oM)$ has kernel H, hence we may view K/H as a subgroup of the compact group $O(n)$, $n = \dim M$. It follows that the Killing form of K/H is negative semidefinite. Since H is discrete, $\mathfrak{k}$ is isomorphic to the Lie algebra of K/H. Hence the Killing form $B_{\mathfrak{k}}$ of $\mathfrak{k}$ is negative semidefinite.

Now let $X \in \mathfrak{k}$. Then $\mathrm{ad}_X(\mathfrak{k}) \subset \mathfrak{k}$ and $\mathrm{ad}_X(\mathfrak{p}) \subset \mathfrak{p}$. Hence

$$B(X, X) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X |\mathfrak{k}) + \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X |\mathfrak{p})\,.$$

The first term on the right hand side is the Killing form $B_{\mathfrak{k}}$ of $\mathfrak{k}$, hence is ≤ 0. On the other hand, via the isomorphism $\pi_* : \mathfrak{p} \to T_oM$, the second term on the right hand side is equal to

$$\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X |\mathfrak{p}) = \mathrm{tr}(DX^*(o) \circ DX^*(o))\,,$$

see (4.6), (5.10). By Proposition 6.2, $DX^*(o)$ is a skew symmetric endomorphism of T_oM. Hence $\mathrm{tr}(DX^*(o) \circ DX^*(o)) \leq 0$ with equality if and only if $DX^*(o) = 0$. Now X^* is a Killing field and $X^*(o) = 0$ since $X \in \mathfrak{k}$. Hence $DX^*(o) = 0$ implies $X^* = 0$. Since the action of G is locally effective, this implies $X = 0$. □

6.7. Remark. The proof shows that the bilinear form $\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y |\mathfrak{p})$ is also negative definite on $\mathfrak{k}$. For our purposes below, it would be as well to use this form. However, it seems more natural to use the Killing form.

6.8. Theorem. *Let G be a Lie group acting isometrically, locally effectively and transitively on M. Choose an origin $o \in M$ and let K be the stabilizer of o in G. Then the Lie algebra $\mathfrak{k}$ of K has an Ad_K–invariant complement $\mathfrak{p}$ in the Lie algebra $\mathfrak{g}$ of G. In particular, M has a metric connection $\bar{D}$ with parallel torsion and curvature such that G contains the $\bar{D}$–transvections.*

Proof. The first assertion is immediate from Lemma 6.6, choose $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. As for the second assertion, we let $\bar{D}$ be the connection determined by $\mathfrak{p}$, see Theorem 5.12. Now the pull back of g_o to $\mathfrak{p}$ via the identification $\mathfrak{p} \cong T_oM$ is Ad_K–invariant since K acts isometrically on T_oM. Hence $\bar{D}$ is metric. □

6.9. Remark. That M has a metric connection with parallel torsion and curvature can be restated in a different way. To that end we set

$\bar{D} = D - S$ (where D is the Levi–Civita connection of M) and ask for the conditions on S which are equivalent to $\bar{D}$ being metric and having parallel torsion $\bar{T}$ and curvature $\bar{R}$. Now

$$\begin{aligned} \bar{T}(X,Y) &= S_Y X - S_X Y \,; \\ \bar{R}(X,Y)Z &= R(X,Y)Z - [S_X, S_Y]\cdot Z - S(\bar{T}(X,Y),Z) \\ &\qquad + (\bar{D}_Y S)(X,Z) - (\bar{D}_X S)(Y,Z)\,, \end{aligned}$$

where $S_X Y := S(X,Y)$. We see that $\bar{D}\bar{T} = 0$ and $\bar{D}\bar{R} = 0$ if $\bar{D}S = 0$ and $\bar{D}R = 0$. Now $\bar{D}$ is metric with $\bar{D}S = 0$ and $\bar{D}R = 0$ if

$$\begin{aligned} \langle S_X Y, Z\rangle &= -\langle Y, S_X Z\rangle \,; \\ (D_X S)(Y,Z) &= [S_X, S_Y]\cdot Z - S(S_X Y, Z)\,; \\ (D_X R)(Y,Z) &= [S_X, R(Y,Z)] - R(S_X Y, Z) - R(Y, S_X Z)\,. \end{aligned} \tag{6.10}$$

Hence we conclude that $\bar{D}$ is metric and has parallel torsion and curvature if Equations 6.10 hold.

Vice versa, suppose that $\bar{D}$ is metric with parallel torsion and curvature. Then after what we just said, we may ask whether the difference tensor S is parallel with respect to $\bar{D}$. Now by the first of the above equations, the torsion tensor $\bar{T}$ determines the skew symmetric part of S. On the other hand, since $\bar{D}$ is metric, the symmetric part of S is also determined by $\bar{T}$,

$$\langle S(X,Y) + S(Y,X), Z\rangle = \langle X, \bar{T}(Y,Z)\rangle + \langle Y, \bar{T}(X,Z)\rangle\,,$$

and hence S is parallel with respect to $\bar{D}$. In fact,

$$2\langle S(X,Y), Z\rangle = \langle X, \bar{T}(Y,Z)\rangle + \langle Y, \bar{T}(X,Z)\rangle - \langle Z, \bar{T}(X,Y)\rangle\,.$$

Using this explicit formula for S and the formula for $\bar{T}$ from Theorem 5.12, there is an explicit formula for the sectional curvature of M, very similar to the one for left–invariant metrics on Lie groups.

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