RICCATI EQUATION AND VOLUME ESTIMATES

WERNER BALLMANN

Contents

1. Introduction

Gromov suggested the use of the Riccati equation in the discussion of Jacobi field and volume estimates under lower bounds on the curvature. Indeed, this approach lead him to an important improvement of the previously known estimates of the volume of geodesic balls, Lemma 5.3 in [3], now called the Bishop-Gromov inequality.

In these lecture notes, we will employ the Riccati equation to prove the Bishop-Gromov inequality and to refine the Heintze-Karcher inequalities for tubes about submanifolds in [4]. As an application and following [4], we obtain an improved version of Cheeger's injectivity radius estimate (Theorem 5.8 in [1]).

It is interesting to note that, in the derivation of our comparison results for Riccati equations, we return to estimating solutions of associated scalar Jacobi equations.

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2. Notions and notations

We let M be a Riemannian manifold of dimension m with Levi-Civita connection ∇, curvature tensor R, Ricci tensor $\mathrm{Ric} = \mathrm{Ric}_M$, and sectional curvature $K = K_M$.

For $m \geq 2$ and $\kappa \in \mathbb{R}$, the *model space* M^m_κ is the unique complete and simply connected Riemannian manifold of dimension m and constant sectional curvature κ. For $\kappa > 0$, we have $M^m_\kappa = S^m_\kappa$, the round sphere of radius $1/\sqrt{\kappa}$.

2.1. **Geodesics and Jacobi fields.** For a tangent vector v of M, we denote by γ_v the (maximal) geodesic in M with $\dot\gamma_v(0) = v$. For a geodesic γ in M, we write

$$R_\gamma X = R(X, \dot\gamma)\dot\gamma \tag{1}$$

and consider R_γ as a field of symmetric endomorphisms of the normal spaces of γ. A *Jacobi tensor field* along γ is a smooth field J of endomorphisms of the normal spaces of γ which solves the Jacobi equation

$$J'' + R_\gamma J = 0. \tag{2}$$

In terms of a parallel frame $(E_1, \dots, E_{m-1})$ of vector fields along γ and perpendicular to γ, this means that the $J_i = JE_i$ are Jacobi fields along γ and perpendicular to γ. If the E_i are orthonormal, J is represented by the field of matrices $\langle J_i, E_j\rangle$ with entries depending on the parameter of the geodesic γ.

2.2. **Submanifolds.** We let N be a submanifold of M of dimension n. Then we have an orthogonal decomposition

$$TM|_N = TN \oplus \nu N, \tag{3}$$

where $\nu N \subseteq TM$ is the normal bundle of N. For $p \in N$ and $v \in T_pM$, we write accordingly $v = v^\tau + v^\nu$. Sections of νN will be called normal fields. For smooth vector fields X and Y tangent to N, we have

$$\nabla_X Y = (\nabla_X Y)^\tau + (\nabla_X Y)^\nu = \nabla^N_X Y + II(X, Y), \tag{4}$$

where ∇^N and II denote the Levi-Civita connection and second fundamental form of N, respectively. For a smooth vector field X tangent to N and a normal field ξ, we have

$$\nabla_X \xi = (\nabla_X \xi)^\nu + (\nabla_X \xi)^\tau = \nabla^\nu_X \xi + S_\xi X, \tag{5}$$

where ∇^ν denotes the induced connection on νN and S_ξ is called the *Weingarten* or *shape operator* of N with respect to ξ. For smooth vector fields X and Y tangent to N and a smooth normal field ξ, the inner product $\langle \xi, Y\rangle$ vanishes and hence

$$\begin{aligned} II_\xi(X,Y) &= \langle \xi, II(X,Y)\rangle = \langle \xi, \nabla_X Y\rangle \\ &= -\langle \nabla_X \xi, Y\rangle = -\langle S_\xi X, Y\rangle. \end{aligned} \tag{6}$$

We conclude that $-S_\xi$ is the field of symmetric endomorphisms of TN corresponding to the second fundamental form II_ξ of N with respect to the normal field ξ. We have the following fundamental equations, named after Gauss, Codazzi, and Ricci, respectively:

$$\begin{aligned} \langle R(X,Y)U,V\rangle &= \langle R^N(X,Y)U,V\rangle \\ &\qquad + \langle II(X,U), II(Y,V)\rangle - \langle II(Y,U), II(X,V)\rangle, \quad (7) \\ \langle R(X,Y)U,\xi\rangle &= \langle (\nabla^\nu_X II)(Y,U) - (\nabla^\nu_Y II)(X,U), \xi\rangle, \quad (8) \\ \langle R(X,Y)\xi,\eta\rangle &= \langle R^\nu(X,Y)\xi,\eta\rangle - \langle [S_\xi, S_\eta]X, Y\rangle, \quad (9) \end{aligned}$$

for all vector fields X, Y, U, V tangent to N and normal fields ξ, η, where R^N and R^ν denote the curvature tensors of N and νN, respectively.

Given a (local) orthonormal frame $(E_1, \dots, E_n)$ of vector fields tangent to N, we obtain a normal field

$$\eta = \eta_N = \frac{1}{n}\big(II(E_1,E_1) + \cdots + II(E_n,E_n)\big), \tag{10}$$

called the *mean curvature field* of N. It does not depend on the choice of the frame $(E_1, \dots, E_n)$. The norm $h = |\eta|$ of η is called the *mean curvature* of N. For a unit vector $\xi \in \nu N$, $h(\xi) = \langle \xi, \eta\rangle$ is called the *mean curvature of* N *in the direction of* ξ. We have

$$h(\xi) = -\frac{1}{n}\operatorname{tr} S_\xi. \tag{11}$$

We say that $p \in N$ is an *umbilical point* or an *umbilic of* N if

$$II(v,w) = \langle v,w\rangle \eta(p) \tag{12}$$

for all $v, w \in T_pN$. Then $S_\xi = -\langle \eta, \xi\rangle \operatorname{id}$ for all $\xi \in \nu_pN$. We say that a subset P of N is *umbilical* if every point of P is an umbilic of N. Then

$$(\nabla^\nu_X II)(Y,Z) = \langle Y,Z\rangle \nabla^\nu_X \eta \tag{13}$$

in the interior of P.

If N is a curve, then any point of N is an umbilic of N. For all $1 < n < m$ and $h \ge 0$, a connected umbilical submanifold N of dimension n and mean curvature h in M^m_κ is an open subset of a standard subspace $M^n_{\kappa+h^2} \subseteq M^m_\kappa$. The crux of the proof is that the mean curvature field of such an N is parallel with respect to ∇^ν. This in turn follows from (13) since the left hand side of (8) vanishes for $M = M^m_\kappa$ and since the dimension of N is at least two.

2.3. **Cut locus.** In this subsection, we assume that M is complete and that N is proper, that is, the intersections of compact subsets of M with N are compact in N. Then N is a complete Riemannian manifold with respect to its first fundamental form. We have

$$\operatorname{rad} N = \sup\{d(q,N)\} \le \operatorname{diam} M \le \infty \tag{14}$$

for the *radius* $\operatorname{rad} N$ *of* N.

Since M is complete, the exponential map exp of M is defined on all of TM. The restriction of exp to νN is called the *normal exponential map of* N. Let SN be the S^{m-n-1}-bundle of unit vectors in νN. For $\xi \in SN$, set

$$t_c(\xi) = \sup\{t > 0 \mid d(\gamma_\xi(t), N) = t\} \in (0,\infty]. \tag{15}$$

As in the case where N is a point, we have that $t_c \colon SN \to (0,\infty]$ is a continuous function. If $t_c(\xi) < \infty$, then $\gamma_\xi(t_c(\xi)) = \exp(t_c(\xi)\xi)$ is called the *cut point of* N *along* γ_ξ. The closed subset

$$C_T(N) = \{t_c(\xi)\xi \mid \xi \in SN \text{ and } t_c(\xi) < \infty\} \tag{16}$$

of νN is called the *tangential cut locus of* N and its image $C(N)$ under the normal exponential map the *cut locus of* N. As in the case where N is a point, we have that the normal exponential map

$$\exp \colon \{t\xi \mid \xi \in SN \text{ and } 0 \le t < t_c(\xi)\} \to N \setminus C(N) \tag{17}$$

is a diffeomorphism. Recall that $t_c(\xi) \le t_b(\xi)$, where $t_b(\xi) \in (0,\infty]$ denotes the first positive time, when a focal point of N along γ_ξ occurs; see page 12 below for the definition of focal points.

3. Distance functions and Riccati equation

Let $f \colon W \to \mathbb{R}$ be a function, where W is an open subset of M. We say that f is a *distance function* if f is smooth with $|\operatorname{grad} f| \equiv 1$.

3.1. **Example.** Suppose that M is complete, and let $p \in M$. Then the distance $f = f(q) = d(p,q)$ from p is a distance function on the open subset $W = M \setminus (\{p\} \cup C_p)$ of M, where C_p denotes the cut locus of p in M.

More generally, if $N \subseteq M$ is a properly embedded submanifold, then the distance $f = f(q) = d(N,q)$ from N is a distance function on the open subset $W = M \setminus (N \cup C_N)$ of M, where C_N denotes the cut locus of N in M.

In what follows, let $f \colon W \to \mathbb{R}$ be a distance function and W_r be the level set of f of level r; that is, $W_r = \{p \in W \mid f(p) = r\}$. Since the gradient of f does not vanish on W, the level sets W_r are smooth hypersurfaces in W (if non-empty). Note that $\operatorname{grad} f$ is a unit normal field of the level sets W_r.

3.2. **Lemma.** *Let $c\colon [a,b] \to W$ be a piecewise smooth curve from $p \in W_r$ to $q \in W_s$, where $r \le s$. Then $L(c) \ge s - r$ with equality if and only if c solves $\dot c = \operatorname{grad} f \circ c$ up to monotonic reparametrization. In particular, the solution curves of the gradient field are unit speed geodesics which are minimal in W.*

Proof. Since $|\operatorname{grad} f| \equiv 1$, we have

$$\begin{aligned} L(c) = \int_a^b |\dot c| &\ge \int_a^b |\langle \operatorname{grad} f, \dot c\rangle| \\ &\ge \Big| \int_a^b \langle \operatorname{grad} f, \dot c\rangle \Big| \\ &= |f(q) - f(p)| = s - r. \end{aligned}$$

This shows the asserted inequality. Moreover, equality holds if and only if $\dot c(t)$ is a non-negative multiple of $\operatorname{grad} f|_{c(t)}$ for all $a \le t \le b$. $\square$

For $p \in W$, let γ_p be the solution curve of $\operatorname{grad} f$ with $\gamma_p(r) = p$, where $r = f(p)$. By Lemma 3.2, γ_p is a unit speed geodesic in W. The geodesics γ_p, for $p \in W$, will be called *f-geodesics*. Notice that $\gamma_p(t)$ depends smoothly on p and t.

Recall that the *Hessian* $\operatorname{Hess} f$ of f is the symmetric tensor field on W of type $(2,0)$ defined by

$$\operatorname{Hess} f(X,Y) = XYf - (\nabla_X Y)f = \langle \nabla_X \operatorname{grad} f, Y\rangle, \tag{18}$$

where X and Y are smooth vector fields on W. Since

$$2\langle \nabla_X \operatorname{grad} f, \operatorname{grad} f\rangle = X\langle \operatorname{grad} f, \operatorname{grad} f\rangle = 0, \tag{19}$$

we get that

$$\operatorname{Hess} f(X, \operatorname{grad} f) = 0 \tag{20}$$

for all vector fields X on W. In particular, the non-trivial information on $\operatorname{Hess} f$ comes from vector fields perpendicular to $\operatorname{grad} f$, that is, vector fields tangent to the level sets of f. Moreover,

$$UX = \nabla_X \operatorname{grad} f = S_{\operatorname{grad} f} X \tag{21}$$

is the Weingarten operator of the level sets W_r of f with respect to the unit normal field $\operatorname{grad} f$. For smooth vector fields X and Y tangent to the level sets of f, we obtain

$$\operatorname{Hess} f(X,Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle = \langle UX, Y\rangle. \tag{22}$$

Let $c = c(s)$ be a smooth curve in W_r through $c(0) = p$ with $\dot c(0) = v$. Then

$$J = \frac{\partial}{\partial s}\gamma_{c(s)}\big|_{s=0}$$

is a Jacobi field along $\gamma = \gamma_p$ with

$$J(0) = v \quad \text{and} \quad J'(0) = Uv, \tag{23}$$

where U denotes the Weingarten operator of W_r as in (21). The first equality of (23) is clear. As for the proof of the second, we have in fact that

$$\begin{aligned} J'(t) &= \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\gamma_{c(s)}(t)\big|_{s=0} \\ &= \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\gamma_{c(s)}(t)\big|_{s=0} = \nabla_{J(t)}\operatorname{grad} f = UJ(t). \end{aligned} \tag{24}$$

Jacobi fields J along any f-geodesic $\gamma = \gamma_p$ satisfying initial conditions as in (23) will be called *f-Jacobi fields*. By their definition, f-Jacobi fields along $\gamma = \gamma_p$ do not vanish anywhere along γ (as long as γ stays inside W). From (24), we also get that

$$U'J = (UJ)' - UJ' = J'' - U^2 J = -R_\gamma J - U^2 J, \tag{25}$$

where R_γ is defined as in (1). Since (25) holds for all f-Jacobi fields J, we conclude that the field of Weingarten operators U satisfies the *Riccati equation*

$$U' + U^2 + R_\gamma = 0 \tag{26}$$

along each f-geodesic γ. More generally, let γ be a geodesic through a point $p = c(0) \in M$ and U_0 be an endomorphism of the normal space of $\dot\gamma(0)$ in T_pM.

3.3. **Lemma.** *In the above situation, let J be the Jacobi tensor field along γ with initial condition*

$$J(0) = \mathrm{id} \quad \textit{and} \quad J'(0) = U_0.$$

Then the field $U = J'J^{-1}$ of endomorphisms of the normal spaces of γ is the solution of the Riccati equation with initial condition U_0. Moreover, U is a field of symmetric endomorphisms if and only if U_0 is symmetric.

4. Comparison theory for the Riccati equation

For a smooth function κ on some interval, let j solve the scalar Jacobi equation $j'' + \kappa j = 0$. Then $u = j'/j$ solves the scalar Riccati equation

$$u' + u^2 + \kappa = 0 \tag{27}$$

on its domain of definition, as we saw in greater generality in Lemma 3.3, and any solution of the scalar Riccati equation arises in this way.

If 0 belongs to the domain of definition of κ, then we denote by sn_κ and cs_κ the solutions of the scalar Jacobi equation with

$$\operatorname{sn}_\kappa(0) = 0,\ \operatorname{sn}'_\kappa(0) = 1 \quad \text{and} \quad \operatorname{cs}_\kappa(0) = 1,\ \operatorname{cs}'_\kappa(0) = 0, \tag{28}$$

respectively. Clearly, $u = \operatorname{sn}'_\kappa / \operatorname{sn}_\kappa$ is the unique solution of the Riccati equation which satisfies $\lim_{t\to 0} u(t) = \infty$.

If κ is constant, the case we need in our applications, then $\operatorname{sn}'_\kappa = \operatorname{cs}_\kappa$ and $\operatorname{cs}'_\kappa = -\kappa \operatorname{sn}_\kappa$, and we set

$$\operatorname{tn}_\kappa = \operatorname{sn}_\kappa / \operatorname{cs}_\kappa \quad \text{and} \quad \operatorname{ct}_\kappa = \operatorname{cs}_\kappa / \operatorname{sn}_\kappa . \tag{29}$$

Note that $u = \operatorname{ct}_\kappa$ is the unique solution of the scalar Riccati equation which satisfies $\lim_{t\to 0} u(t) = \infty$ and that tn_κ does not solve the scalar Riccati equation unless $\kappa = 1$.

4.1. **Lemma.** *Let $u, v\colon (a, b] \to \mathbb{R}$ be smooth with $u' + u^2 \le v' + v^2$ and assume that $u' + u^2$ and $v' + v^2$ extend smoothly to $[a, b]$. Then the limits $u(a) = \lim_{t\to a} u(t)$ and $v(a) = \lim_{t\to a} v(t)$ exist as extended real numbers in $(-\infty, \infty]$. If $u(a) \le v(a)$, then $u \le v$ on $(a, b]$ with equality $u = v$ if and only if $u(b) = v(b)$.*

Proof. The proof is motivated by the proof of the Sturm comparison theorem for solutions of the scalar Jacobi equation. Let $\kappa, \lambda\colon [a, b] \to \mathbb{R}$ be the smooth extensions of $-u' - u^2$ and $-v' - v^2$, respectively. Then $u = j'_\kappa/j_\kappa$, where j_κ solves the associated scalar Jacobi equation $j'' + \kappa j = 0$ with $j_\kappa(a) = 1$ and $j'_\kappa(a) = u(a)$ if $u(a) < \infty$ and $j_\kappa(a) = 0$ and $j'_\kappa(a) = 1$ if $u(a) = \infty$. Correspondingly, write $v = j'_\lambda/j_\lambda$. In both cases, $j_\kappa, j_\lambda > 0$ on $(a, b]$ since otherwise u and v would not be defined on $(a, b]$. We have

$$\begin{aligned} 0 &= \int_a^t \{j_\kappa(j''_\lambda + \lambda j_\lambda) - (j''_\kappa + \kappa j_\kappa) j_\lambda\} \\ &= \{j_\kappa j'_\lambda - j'_\kappa j_\lambda\}\big|_a^t + \int_a^t (\lambda - \kappa) j_\kappa j_\lambda. \end{aligned}$$

Therefore we obtain

$$j_\kappa(t) j'_\lambda(t) - j'_\kappa(t) j_\lambda(t) \ge j_\kappa(a) j'_\lambda(a) - j'_\kappa(a) j_\lambda(a) + \int_a^t (\kappa - \lambda) j_\kappa j_\lambda.$$

The first term on the right is nonnegative by the choice of j_κ and j_λ, the second term is nonnegative since $\kappa \ge \lambda$ and $j_\kappa j_\lambda > 0$ on $(a, b]$. Since $u = j'_\kappa/j_\kappa$ and $v = j'_\lambda/j_\lambda$, this implies the asserted inequality and criterion for the equality of u and v. ☐

There are two ways in which we arrive at scalar inequalities as in Lemma 4.1 and we discuss them in Examples 4.2 and 4.3 below. To that end, we let γ be a unit speed geodesic in M and U be a symmetric solution of the Riccati equation along γ (with domain of definition possibly smaller than the domain of definition of γ).

4.2. **Example.** Let E be a parallel field along γ with $|E| \equiv 1$ and perpendicular to γ. Then $u = \langle UE, E\rangle$ satisfies

$$\begin{aligned} u' &= \langle UE, E\rangle' = \langle U'E, E\rangle \\ &= -\langle U^2E, E\rangle - \langle R_\gamma E, E\rangle \\ &\le -\langle UE, E\rangle^2 - \langle R_\gamma E, E\rangle \\ &= -u^2 - K_M(\dot\gamma \wedge E), \end{aligned}$$

where we use the Schwarz inequality and that E has norm one and where $\dot\gamma \wedge E$ denotes the tangent plane in M spanned by $\dot\gamma$ and E. Hence if $\kappa = \kappa(t)$ is a smooth function such that the sectional curvature of tangent planes P of M containing $\dot\gamma(t)$ satisfies $K_M(P) \ge \kappa(t)$ for all t in the domain of U, then $u' + u^2 \le -\kappa$. Moreover, the equality $u' + u^2 = -\kappa$ holds if and only if $UE = uE$ and $\langle R_\gamma E, E\rangle = \kappa$, that is, if and only if $UE = uE$ and $R_\gamma E = \kappa E$. Lemma 4.1 now allows to compare u with smooth functions v solving $v' + v^2 = -\kappa$ with initial condition $u(a) \le v(a)$.

4.3. **Example.** Suppose that J is a Jacobi tensor field along γ with $\det J > 0$. Let $U = J'J^{-1}$ and set

$$u = \frac{1}{m-1}\frac{d}{dt}\ln\det J = \frac{1}{m-1}\operatorname{tr}(J'J^{-1}) = \frac{1}{m-1}\operatorname{tr} U.$$

Then we have

$$\begin{aligned} u' &= \frac{1}{m-1}(\operatorname{tr} U)' = \frac{1}{m-1}\operatorname{tr} U' \\ &= -\frac{1}{m-1}\operatorname{tr}(U^2) - \frac{1}{m-1}\operatorname{tr} R_\gamma \\ &\le -\frac{1}{(m-1)^2}(\operatorname{tr} U)^2 - \frac{1}{m-1}\operatorname{Ric}(\dot\gamma, \dot\gamma) \\ &= -u^2 - \frac{1}{m-1}\operatorname{Ric}(\dot\gamma, \dot\gamma), \end{aligned}$$

where we use the Schwarz inequality $(\operatorname{tr} U)^2 \le (m-1)\operatorname{tr}(U^2)$. Therefore, if $\kappa = \kappa(t)$ is a smooth function such that $\operatorname{Ric}(\dot\gamma, \dot\gamma) \ge (m-1)\kappa$ on the domain of definition of U, then $u'+u^2 \le -\kappa$. Moreover, the equality $u' + u^2 = -\kappa(t)$ holds if and only if $U = uI_\gamma$ and $\operatorname{Ric}(\dot\gamma, \dot\gamma) = (m-1)\kappa$, where I_γ denotes the field of identity endomorphisms along γ. Since $U'+U^2 = -R_\gamma$, this happens if and only if $U = uI_\gamma$ and $R_\gamma = \kappa I_\gamma$. As in the previous example, Lemma 4.1 now allows to compare u with smooth functions v solving $v' + v^2 = -\kappa$ with initial condition $u(a) \le v(a)$.

4.4. **Remark.** Section 1.6 in Karcher's survey article [5] contains a different approach to the comparison theory for solutions of the scalar Riccati equation. In [2], Eschenburg and Heintze develop an elegant comparison theory for solutions of the tensorial Riccati equation.

5. Bishop-Gromov inequalities

For simplicity, we assume throughout this section that M is complete. Let $p \in M$ and $\gamma\colon [0,b) \to M$ be a unit speed geodesic starting at p. Let J be the Jacobi tensor field along γ with initial condition $J(0) = 0$ and $J'(0) = \mathrm{id}$. Clearly, we have

$$\lim_{r\to 0} \frac{\det J(r)}{r^{m-1}} = 1. \tag{30}$$

Notice that $\det J(t) > 0$ for any $t > 0$ before the first conjugate point of p along γ. If $R_\gamma = \kappa I_\gamma$, where $\kappa \in \mathbb{R}$ and I_γ denotes the field of identity endomorphisms along γ as above, then $J = \mathrm{sn}_\kappa\, I_\gamma$ and $\det J = \mathrm{sn}_\kappa^{m-1}$.

The first version of the Bishop-Gromov inequality reads as follows.

5.1. **Theorem.** *Assume that* $\mathrm{Ric}(\dot\gamma,\dot\gamma) \ge (m-1)\kappa$ *for some* $\kappa \in \mathbb{R}$ *and that* $\det J > 0$ *on* $(0,b)$. *Then*

$$1 \ge \frac{\det J(r)}{\mathrm{sn}_\kappa^{m-1}(r)} \ge \frac{\det J(s)}{\mathrm{sn}_\kappa^{m-1}(s)}$$

for all $r < s$ *in* $(0,b)$. *The inequality on the left is strict unless* $R_\gamma = \kappa I_\gamma$ *on* $[0,r]$, *the inequality on the right is strict unless* $R_\gamma = \kappa I_\gamma$ *on* $[0,s]$.

Proof. This follows easily from Lemma 4.1, Example 4.3, and (30). □

5.2. **Corollary.** *In the situation of Theorem 5.1,* $b \le \pi/\sqrt{\kappa}$ *if* $\kappa > 0$.

Let S be the unit sphere in T_pM. For $v \in S$, denote by $t_b(v) \in (0,\infty]$ the first positive time, when a conjugate point of p along γ_v occurs, and by $t_c(v) \in (0,\infty]$ the cut point of p along γ_v. As usual, the value ∞ indicates that no conjugate or cut point occurs along $\gamma_v|_{(0,\infty)}$. By Theorem 5.1 and Jacobi's theorem, we have $t_b(v) \le \pi/\sqrt{\kappa}$ if $\kappa > 0$ and $t_c(v) \le t_b(v)$, respectively. Now for $0 \le t < t_b(v)$, the Jacobian of

$$(0,\infty) \times S \to M, \quad (t,v) \mapsto \exp(tv),$$

at (t,v) is given by $\det J_v(t)$, where J_v is the Jacobi tensor field along γ_v with initial conditions $J_v(0) = 0$ and $J_v'(0) = \mathrm{id}$. In particular, the volume $V_p(r)$ of the geodesic ball $B_p(r)$ of radius r about p in M is given by

$$V_p(r) = \int_S \int_0^{r \wedge t_c(v)} \det J_v(t)\, dt dv, \tag{31}$$

where we use the notation $a \wedge b = \min\{a, b\}$. We will compare $V_p(r)$ with the volume

$$V_\kappa(r) = \mathrm{vol}(S^{m-1}) \int_0^r \mathrm{sn}_\kappa^{m-1}(t)\, dt \tag{32}$$

of a geodesic ball $B_\kappa(r)$ of radius r in the model space M_κ^m. Here we only consider radii $r \leq \pi/\sqrt{\kappa}$ if $\kappa > 0$. By (30), we have

$$\lim_{r \to 0} \frac{V_p(r)}{V_\kappa(r)} = 1. \tag{33}$$

The global version of the Bishop-Gromov inequality reads as follows.

5.3. **Theorem.** *Assume that* $\mathrm{Ric}(\dot\gamma, \dot\gamma) \geq (m-1)\kappa$ *for all unit speed geodesics through* p *and some* $\kappa \in \mathbb{R}$. *Then*

$$1 \geq \frac{V_p(r)}{V_\kappa(r)} \geq \frac{V_p(s)}{V_\kappa(s)}$$

for all $0 < r < s \leq \max\{d(p,q)\}$ *(with* $s < \pi/\sqrt{\kappa}$ *if* $\kappa > 0$*). The inequality on the left is strict unless* $B_p(r)$ *is isometric to* $B_\kappa(r)$ *and the inequality on the right is strict unless* $B_p(s)$ *is isometric to* $B_\kappa(s)$.

Proof. By what we said above, we have

$$\frac{V_p(r)}{V_\kappa(r)} = \frac{1}{\mathrm{vol}(S^{m-1}) \int_0^r \mathrm{sn}_\kappa^{m-1}(t) dt} \int_S \int_0^r f(t,v)\, \mathrm{sn}_\kappa^{m-1}(t)\, dt dv, \tag{34}$$

where $f(t,v) = \det J_v(t) / \mathrm{sn}_\kappa^{m-1}(t)$ for $0 < t < t_c(v)$ and $f(t,v) = 0$ for $t \geq t_c(v)$. By (30), we have $\lim_{t \to 0} f(t,v) = 1$. By Theorem 5.1, f is monotonically decreasing, where $f(t,v) = 1$ can only hold if $t < t_c(v)$ and $R_{\gamma_v} = \kappa I_{\gamma_v}$ on $[0,t]$. Now the right hand side in (34) is the mean of f with respect to the volume form $\mathrm{sn}_\kappa^{m-1}\, dt dv$ on $(0,r] \times S$. $\square$

5.4. **Corollary.** *If* $\mathrm{Ric}_M \geq \kappa > 0$*, then*

$$\mathrm{diam}\, M \leq \mathrm{diam}\, S_\kappa^m = \pi/\sqrt{\kappa} \quad \textit{and} \quad \mathrm{vol}\, M \leq \mathrm{vol}\, S_\kappa^m,$$

and the inequalities are strict unless M *is isometric to* S_κ^m.

The assertion $\mathrm{diam}\, M \leq \pi/\sqrt{\kappa}$ is called the *theorem of Bonnet-Myers*, the characterization of S_κ^m by the equality $\mathrm{diam}\, M = \pi/\sqrt{\kappa}$ the *maximal diameter theorem of Cheng.*

Proof. Except for the maximal diameter theorem, the assertions are fairly straightforward consequences of Theorems 5.1 and 5.3. That the maximal diameter theorem follows from Theorem 5.3 was observed by Shiohama: Choose $p, q \in M$ of maximal distance $d(p,q) = \pi/\sqrt{\kappa}$. Then the geodesic balls $B_p(\pi/2\sqrt{\kappa})$ and $B_q(\pi/2\sqrt{\kappa})$ are disjoint and hence

$$\operatorname{vol} B_p(\pi/2\sqrt{\kappa}) + \operatorname{vol} B_q(\pi/2\sqrt{\kappa}) \leq \operatorname{vol} M. \tag{35}$$

Now Theorem 5.3 implies that

$$\frac{\operatorname{vol} B_p(\pi/2\sqrt{\kappa})}{\operatorname{vol} B_p(\pi/\sqrt{\kappa})}, \frac{\operatorname{vol} B_p(\pi/2\sqrt{\kappa})}{\operatorname{vol} B_p(\pi/\sqrt{\kappa})} \geq \frac{V_\kappa(\pi/2\sqrt{\kappa})}{V_\kappa(\pi/\sqrt{\kappa})} = \frac{1}{2}. \tag{36}$$

Since $M = B_p(\pi/\sqrt{\kappa}) = B_q(\pi/\sqrt{\kappa})$, by the theorem of Bonnet-Myers, we conclude that

$$\operatorname{vol} B_p(\pi/2\sqrt{\kappa}), \operatorname{vol} B_q(\pi/2\sqrt{\kappa}) \geq \operatorname{vol} M/2.$$

Hence we have equality in (35) and (36), hence $\operatorname{vol} M \geq \operatorname{vol} S^m_\kappa$ and therefore $M = S^m_\kappa$ by the equality case of the volume inequality. □

6. Heintze-Karcher inequalities

For simplicity, we assume throughout this section that M is complete and that N is a proper submanifold of M.

Let $\pi\colon \nu N \to N$ be the normal bundle of N. Associated to the induced connection ∇^ν on νN, there is a canonical decomposition of the tangent bundle of νN,

$$T\nu N = \mathcal{H} \oplus \mathcal{V}, \tag{37}$$

where the *vertical space* $\mathcal{V}_\xi = \ker d\pi_\xi$ at $\xi \in \nu_p N$ is canonically isomorphic to $\nu_p N$ and the *horizontal space* $\mathcal{H}_\xi$, defined to consist of all tangent vectors to parallel sections of νN through ξ, is canonically isomorphic to $T_p N$. If $\zeta = \zeta(s)$ is a smooth curve in νN through ξ with $\dot\zeta(0) = v$ and $\sigma = \pi \circ \zeta$, then

$$v^{\mathcal{H}} = \dot\sigma(0) = d\pi_\xi(v) \quad \text{and} \quad v^{\mathcal{V}} = \frac{\nabla^\nu}{\partial s}\zeta\big|_{s=0}. \tag{38}$$

The decomposition into vertical and horizontal subspaces induces a Riemannian metric on $T\nu N$ such that π is a Riemannian submersion. Note that this metric only depends on the Riemannian metric on N, the metric on the fibers of νN, and the decomposition.

6.1. **Lemma.** *For $v \in T_\xi \nu N$, we have*

$$d\exp_\xi(v) = J(1),$$

where J is the Jacobi field along γ_ξ with

$$J(0) = v^{\mathcal{H}} \quad \textit{and} \quad J'(0) = v^{\mathcal{V}} + S_\xi v^{\mathcal{H}}.$$

Proof. Let $\zeta = \zeta(s)$ be a smooth curve in νN through ξ with $\dot\zeta(0) = v$. Then

$$d\exp_\xi(v) = \frac{\partial}{\partial s}\exp(t\zeta(s))\big|_{t=1,s=0} = J(1),$$

where J is the Jacobi field along γ_ξ with

$$J(0) = \frac{\partial}{\partial s}\exp(t\zeta(s))\big|_{t=0,s=0} = \frac{\partial}{\partial s}\pi(\zeta(s))\big|_{s=0} = d\pi_\xi(v)$$

and

$$\begin{aligned} J'(0) &= \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\exp(t\zeta(s))\big|_{s=0,t=0} \\ &= \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\exp(t\zeta(s))\big|_{t=0,s=0} \\ &= \frac{\nabla}{\partial s}\zeta\big|_{s=0} = \frac{\nabla^\nu}{\partial s}\zeta\big|_{s=0} + S_\xi d\pi_\xi(v). \end{aligned}$$ □

For $\xi \in \nu_p N$, a Jacobi field J along γ_ξ will be called an N-Jacobi field if and only if it is associated to a geodesic variation (γ_s) of $\gamma_0 = \gamma_\xi$ such that $\sigma = \sigma(s) = \gamma_s(0)$ is a smooth curve in N with $\dot\gamma_s(0) \in \nu N$. By our discussion above, this holds precisely if

$$J(0) \in T_pN \quad \text{and} \quad J'(0)^{\mathcal{H}} = S_\xi J(0). \tag{39}$$

For $t > 0$, we say that $\gamma_\xi(t)$ is a *focal point of* N along γ_ξ if there is a non-trivial N-Jacobi field along γ_ξ that vanishes at t. If there is a $t > 0$ such that $\gamma_\xi(t)$ is a focal point of N along γ_ξ, then we let $t_b(\xi) > 0$ be the smallest such t and call $\gamma_\xi(t_b(\xi))$ the *first focal point of N along γ_ξ*. If there is no non-trivial N-Jacobi field that vanishes in positive time, then we set $t_b(\xi) = \infty$.

We say that a Jacobi field J along γ_ξ is a special N-Jacobi field if it is associated to a geodesic variation (γ_s) of $\gamma_0 = \gamma_\xi$ as above and if, in addition, $|\dot\gamma_s(0)|$ does not depend on s. This holds precisely if

$$J(0) \in T_pN \quad \text{and} \quad J'(0) = S_\xi J(0). \tag{40}$$

Special N-Jacobi fields will enter our discussion below. We will be concerned with volume and integration. To that end, we introduce polar coordinates; that is, we consider the diffeomorphism

$$F\colon (0,\infty) \times SN \to \nu N \setminus N, \quad F(t,\xi) = t\xi, \tag{41}$$

where we identify N with the zero section of νN. Since $\pi\colon SN \to N$ is a Riemannian submersion, we write $d\xi dp$ for the volume element of

SN, where dp denotes the volume element of N and $d\xi$ stands for the volume elements of the unit $(m-n-1)$-spheres S_pN.

Let now $\xi \in SN$ and $(e_1,\dots,e_n)$ be an orthonormal basis of T_pN consisting of eigenvectors of S_ξ with corresponding eigenvalues

$$\lambda_1 = \lambda_1(\xi) \le \cdots \le \lambda_n = \lambda_n(\xi). \tag{42}$$

Recall from (11) that the mean curvature $h(\xi)$ of N in the direction of ξ is given by $-nh(\xi) = \lambda_1 + \cdots + \lambda_n$. Extend $(e_1,\dots,e_n)$ to an orthonormal basis $(e_1,\dots,e_m)$ of T_pM with $e_m = \xi$. Let $(E_1,\dots,E_m)$ be the orthonormal frame along γ_ξ with $E_i(0) = e_i$ for all $1 \le i \le m$. Denote by $J_\xi = J_\xi(t)$ the Jacobi tensor field along γ_ξ which maps E_i to the Jacobi field J_i with $J_i(0) = e_i$ and $J_i'(0) = \lambda_i e_i$ for $1 \le i \le n$ and with $J_i(0) = 0$ and $J_i'(0) = e_i$ for $n < i < m$, respectively. For $1 \le i \le n$, J_i is the special N-Jacobi field with $J_i(0) = e_i$. We note that $\det J_\xi > 0$ up to $t_b(\xi)$.

6.2. **Lemma.** *The volume distortion of* $\exp \circ F$ *is given by*

$$(\exp \circ F)^*_{(t,\xi)} dV = \det J_\xi(t)\, dt d\xi dp$$

for all $0 < t < t_b(\xi)$, *where* dV *and* $d\xi dp$ *denote the volume forms of* M *and* SN, *respectively.*

Proof. By Lemma 6.1, the special N-Jacobi fields $J_1,\dots,J_n$ correspond to an orthonormal basis of $\mathcal{H}_\xi$. Furthermore, the Jacobi fields $J_{n+1},\dots,J_{m-1}$ together with the Jacobi field $J_m = J_m(t) = t\dot\gamma_\xi(t)$ correspond to an orthonormal basis of $\mathcal{V}_\xi$. Now exp is a radial isometry, that is, J_m represents a unit normal vector of SN at ξ and stays normal to $J_1,\dots,J_{m-1}$. Hence the volume distortion of $F \circ \exp$ is given by $\det J_\xi(t)$ □

6.3. **Lemma.** *As a field of matrices in terms of the* E_i *and up to* $O(t^2)$ *for* $t \to 0$, $J_\xi(t)$ *is the diagonal matrix with entries* $1 + \lambda_i t$ *for all* $1 \le i \le n$ *and* t *for all* $n+1 \le i \le m$. *In particular,*

$$\det J_\xi(t) = t^{m-n-1} + (\lambda_1 + \cdots + \lambda_n)t^{m-n} + O(t^{m-n+1}) \quad \textit{as } t \to 0.$$

We are after estimates of the volume distortion $\det J_\xi$ and of the volumina of tubes

$$U_N(r) = \{q \in M \mid d(q,N) < r\} \tag{43}$$

about N. Here the case of balls from Section 5 may be considered as the special case where $n = 0$, and we proceed in quite similar ways in the other cases. We want to compare $\det J_\xi$ with the corresponding quantity j_h in the case of an umbilical submanifold of dimension n in the model space M^m_κ, where $h = h(\xi)$. We have

$$j_h(t) = j_{m,n,\kappa,h}(t) = (\mathrm{cs}_\kappa(t) - h\,\mathrm{sn}_\kappa(t))^n\,\mathrm{sn}_\kappa^{m-n-1}(t). \tag{44}$$

The first positive zero $z(h)$ of j_h is determined by $\mathrm{ct}_\kappa(z(h)) = h$. Define

$$a_h(t) = a_{m,n,\kappa,h}(t) = \int_{S^{m-n-1}} \int_0^{t\wedge z(\langle\xi,\eta\rangle)} j(\langle\xi,\eta\rangle, t)\,dt d\xi, \tag{45}$$

where $\eta \in \mathbb{R}^{m-n}$ is a vector of length h. Clearly, a does not depend on the choice of η. In the case where N is an umbilical hypersurface in the model space M^m_κ with mean curvature h, $a_h(r)$ is the contribution of a fiber over N in a tube of radius r to the volume of the tube. We sometimes consider parameters as variables and write, e.g., $a(h,t)$ instead of $a_h(t)$ or $j_{m,n,\kappa}(h,t)$ instead of $j_{m,n,\kappa,h}(t)$.

6.4. **Lemma.** *The function a is monotonically increasing in h and t.*

Proof. The monotonicity in t is clear. As for the monotonicity in h, we follow the arguments in the proof of the corresponding Proposition 2.1.1 of [4]. We start by noting that $j_{\langle\xi,\eta\rangle}(t)$ is not monotonically increasing in $h = |\eta|$ for $\langle\xi,\eta\rangle \geq 0$. However, we may consider $\pm\xi$ simultaneously to conclude monotonicity of a in h.

Since the integrand of the inner integral in (45) vanishes at $z(\langle\xi,\eta\rangle)$, we only need to check the monotonicity of the intgrand in either case, $t\wedge z(\langle\xi,\eta\rangle) = t$ or $t\wedge z(\langle\xi,\eta\rangle) = z(\langle\xi,\eta\rangle)$. Assume now that $\langle\xi,\eta\rangle \geq 0$. Then we have $z(\langle\xi,\eta\rangle) \leq z(\langle-\xi,\eta\rangle)$ and

$$(x-hy)^n + (x+hy)^n = 2\sum_{2k\leq n} \binom{n}{2k} x^{n-2k} y^{2k} h^{2k},$$

where $x = \mathrm{cs}_\kappa(t)$, $y = \langle\xi,\eta\rangle\,\mathrm{sn}_\kappa(t)/|\eta|$, $h = |\eta|$, and $0 \leq t \leq z(\langle\xi,\eta\rangle)$. Now the right hand side in the above formula is monotonic in h. Furthermore, $j(\langle-\xi,\eta\rangle, t)$ is monotonic in h on $z(\langle\xi,\eta\rangle) \leq t \leq z(\langle-\xi,\eta\rangle)$. We conclude that $a = a(h,t)$ is monotonic in h. □

6.5. **Question.** Given $h < H$, is $a(h,t)/a(H,t)$ monotonic in t?

Up to $t_b(\xi)$, $J_\xi = J_\xi(t)$ is invertible and $U = J'_\xi J_\xi^{-1}$ solves the Riccati equation along γ_ξ. As in Example 4.3, we have the crucial relation

$$\frac{d}{dt}\ln\det J_\xi = \mathrm{tr}(J'_\xi J_\xi^{-1}) = \mathrm{tr}\,U. \tag{46}$$

To estimate $\mathrm{tr}\,U$, lower bounds for the Ricci curvature can be used efficiently in the cases $n = 0$, as we saw in Section 5, and $n = m-1$. In the general case, we will use lower bounds for the sectional curvature.

Using the results from Section 4, it would also be possible to discuss the more refined bounds on the curvature from Section 3.2 in [4]. However, for the sake of simplicity, we will only consider the traditional case of absolute lower bounds.

6.1. **The case of hypersurfaces.** In the next result, we refine the main inequality 3.2.1(d) from [4].

6.6. **Theorem.** *Assume that N is a hypersurface and that $\mathrm{Ric}_M(\dot\gamma_\xi) \geq (m-1)\kappa$ along γ_ξ, where $\xi \in SN$ and $\kappa \in \mathbb{R}$. Then we have*

$$1 \geq \frac{\det J_\xi(r)}{j(h(\xi), r)} \geq \frac{\det J_\xi(s)}{j(h(\xi), s)}$$

for all $0 < r < s < t_b(\xi)$, where j is as in (44). The inequality on the left is strict unless $R_{\gamma_\xi} = \kappa I_{\gamma_\xi}$ on $[0, r]$ and $S_\xi = -h(\xi)\,\mathrm{id}$, the inequality on the right is strict unless $R_{\gamma_\xi} = \kappa I_{\gamma_\xi}$ on $[0, s]$ and $S_\xi = -h(\xi)\,\mathrm{id}$.

Proof. With $\gamma = \gamma_\xi$, $J = J_\xi$, and $U = J_\xi' J_\xi^{-1}$, we choose u as in Example 4.3. Then u satisfies $u' + u^2 + \kappa \leq 0$ with initial condition

$$u(0) = \frac{1}{m-1} \operatorname{tr} U(0) = \frac{1}{m-1} \operatorname{tr} J_\xi'(0) = -h(\xi) = -h.$$

From Lemma 4.1 we get that $u \leq v$, where v solves $v' + v^2 + \kappa = 0$ with initial condition $v(0) = -h$. Then $v = j'/j$ with $j = \mathrm{cs}_\kappa - h\,\mathrm{sn}_\kappa$ and hence the asserted inequalities follow.

Equality can only happen if $(\ln\det J)'(t) = (m-1)(\ln j)'(t)$ for all $t \in (0, r)$ or $t \in (r, R)$, respectively. This implies $R_\gamma = \kappa I_\gamma$ on $[0, t]$ by the discussion in Example 4.3. □

6.2. **The general case.** In the following result, we refine the main inequality 3.2.1(c) from [4].

6.7. **Lemma.** *Assume that $K_M(\dot\gamma_\xi \wedge E) \geq \kappa$ for all parallel unit fields E along γ_ξ perpendicular to γ_ξ, where $\xi \in SN$ and $\kappa \in \mathbb{R}$. Then*

$$1 \geq \frac{\det J_\xi(r)}{j(r)} \geq \frac{\det J_\xi(s)}{j(s)}$$

for all $0 < r < s < t_b(\xi)$, where

$$j = (\mathrm{cs}_\kappa + \lambda_1(\xi)\,\mathrm{sn}_\kappa) \cdots (\mathrm{cs}_\kappa + \lambda_n(\xi)\,\mathrm{sn}_\kappa)\,\mathrm{sn}_\kappa^{m-n-1}.$$

The inequality on the left is strict unless $R_\gamma = \kappa I_\gamma$ on $[0, r]$, the inequality on the right is strict unless $R_\gamma = \kappa I_\gamma$ on $[0, s]$.

Proof. The trace of U is the sum of the terms $\langle UE_i, E_i\rangle$, where the E_i are chosen as above. Since $K_M(\dot\gamma_\xi \wedge E_i) \ge \kappa$, we have

$$\langle UE_i, E_i\rangle \le u_i = j_i'/j_i$$

by Example 4.2 and Lemma 4.1, where $j_i = \mathrm{cs}_\kappa + \lambda_i \,\mathrm{sn}_\kappa$ for $1 \le i \le n$ and $j_i = \mathrm{sn}_\kappa$ for $n < i < m$, respectively. Hence

$$\frac{d}{dt} \ln\det J \le \frac{j_1'}{j_1} + \cdots + \frac{j_{m-1}'}{j_{m-1}} = \frac{d}{dt}\ln(j_1 \cdots j_{m-1}) = \frac{d}{dt}\ln j.$$

Hence $\det J/j$ is monotonically decreasing. On the other hand, we also have $\lim_{r\to 0} \det J(r)/j(r) = 1$ by Lemma 6.3.

Equality in the asserted inequalities can only happen if $(\ln\det J)'(t) = j'(t)$ for all $t \in (0,r)$ or $t \in (r,R)$, respectively, and then $\langle UE_i, E_i\rangle = u_i$ for all $1 \le i \le m-1$. This implies $R_\gamma = \kappa I_\gamma$ on $[0,t]$ by the discussion in Example 4.2. □

Following the discussion in [4], we get rid of the explicit dependence on the eigenvalues $\lambda_i = \lambda_i(\xi)$ of S_ξ by weakening the inequality. Employing the inequality between geometric and arithmetic mean, we have

$$(\mathrm{cs}_\kappa + \lambda_1 \,\mathrm{sn}_\kappa)\dots(\mathrm{cs}_\kappa + \lambda_n \,\mathrm{sn}_\kappa) \le (\mathrm{cs}_\kappa - \langle \xi, \eta\rangle \,\mathrm{sn}_\kappa)^n \tag{47}$$

between $t = 0$ and the first positive zero of the left hand side, where η denotes the mean curvature vector field of N as in (10). We arrive at a refined version of Corollary 3.3.1 of [4].

6.8. **Theorem.** *Assume that $K_M(\dot\gamma_\xi \wedge E) \ge \kappa$ for all parallel unit fields E along γ_ξ perpendicular to γ_ξ, where $\xi \in SN$ and $\kappa \in \mathbb{R}$. Then*

$$1 \ge \frac{\det J_\xi(r)}{j(h(\xi), r)} \ge \frac{\det J_\xi(s)}{j(h(\xi), s)}$$

for all $0 < r < s < t_b(\xi)$, where $j = j_{m,n,\kappa}$ and $h(\xi) = \langle \xi, \eta\rangle$. The inequality on the left is strict unless $R_\gamma = \kappa I_\gamma$ on $[0,r]$ and $S_\xi = -h(\xi)\,\mathrm{id}$, the inequality on the right is strict unless $R_\gamma = \kappa I_\gamma$ on $[0,s]$ and $S_\xi = -h(\xi)\,\mathrm{id}$.

In particular, the first zero $z(\langle \xi, \eta\rangle)$ of $\mathrm{cs}_\kappa - \langle \xi, \eta\rangle \,\mathrm{sn}_\kappa$ is an upper bound for the first focal point $t_b(\xi)$ of N along γ_ξ.

Proof of Theorem 6.8. We show that the quotient

$$\frac{(\mathrm{cs}_\kappa + \lambda_1 \,\mathrm{sn}_\kappa)\dots(\mathrm{cs}_\kappa + \lambda_n \,\mathrm{sn}_\kappa)}{(\mathrm{cs}_\kappa - \langle \xi, \eta\rangle \,\mathrm{sn}_\kappa)^n} \tag{48}$$

is monotonically decreasing. Indeed, we have

$$\frac{d}{dt}\ln\left(\frac{(\mathrm{cs}_\kappa + \lambda_1 \,\mathrm{sn}_\kappa)\dots(\mathrm{cs}_\kappa + \lambda_n \,\mathrm{sn}_\kappa)}{(\mathrm{cs}_\kappa - \langle \xi, \eta\rangle \,\mathrm{sn}_\kappa)^n}\right) = u_1 + \cdots + u_n - nv,$$

where the u_i are as above and $v = j'/j$ with $j = \mathrm{cs}_\kappa - \langle \xi, \eta \rangle \mathrm{sn}_\kappa$. Now v solves $v' + v^2 + \kappa = 0$ and

$$u = \frac{u_1 + \cdots + u_n}{n}$$

satisfies $u' + u^2 + \kappa \leq 0$ with equality if all the u_i coincide. Furthermore, we have $u(0) = v(0)$. Hence $u \leq v$ by Lemma 4.1, and therefore the quotient in (48) is monotonically decreasing. □

6.3. **Main estimate for the volume of tubes.** For $r > 0$, the subset

$$U_N(r) = \{q \in M \mid d(q, N) < r\} \tag{49}$$

of M is called the *tube of radius* r *about* N. For an open subset P of N, let $U_P(r) = \{q \in U_N(r) \mid d(q, N) = d(q, P)\}$. The next result refines the global Heintze-Karcher inequality (Theorem 2.1 in [4]).

6.9. **Theorem.** *For a constant $\kappa \in \mathbb{R}$, assume that*
1) N is a hypersurface such that $\mathrm{Ric}_M(\dot\gamma_\xi) \geq (m-1)\kappa$ along γ_ξ for all unit normal vectors ξ of N or that
2) $K_M(\dot\gamma_\xi \wedge E) \geq \kappa$ for all unit normal vectors ξ of N and parallel unit vector fields E along γ_ξ perpendicular to γ_ξ.
Then we have

$$1 \geq \frac{\mathrm{vol}\, U_P(r)}{\int_P a(h(p), r)\, dp} \geq \frac{\mathrm{vol}\, U_P(s)}{\int_P a(h(p), s)\, dp}$$

for all $0 < r < s < \mathrm{rad}\, N$ and relatively compact open subsets P of N, where $a = a_{m,n,\kappa}$ and h denotes the mean curvature of N.

Proof. Consider the set

$$Z = \{(t, \xi) \in (0, \infty) \times SN \mid t < z(\xi)\},$$

endowed with the volume element $\omega = j_{h(\xi)}(t)\, dt d\xi dp$. Define a function $f \colon Z \to \mathbb{R}$ by

$$f(\xi, t) = \begin{cases} \det J_\xi(t)/j_h(t) & \text{for } 0 \leq t < t_c(\xi), \\ 0 & \text{for } t_c(\xi) \leq t < z(\xi). \end{cases}$$

Since the cut locus $C(N)$ has measure 0, we have

$$\frac{\mathrm{vol}\, U_P(r)}{\int_N a(h(p), r)\, dp} = \frac{1}{\int_N a(h(p), r)\, dp} \int_{SN|_P} \int_0^{r \wedge z(\xi)} f(\xi, t) j_{h(\xi)}(t)\, dt d\xi dp.$$

By the definition of a, the right hand side is the mean of f over the set

$$Z_r = \{(t, \xi) \in Z \mid t \leq r\}$$

with respect to the volume element ω. For each ξ, f is monotonically decreasing in t with $1 \geq f(\xi, t) > 0$ for $0 \leq t \leq t_c(\xi)$, by Theorem 6.6

and Theorem 6.8, respectively, and $f(\xi, t) = 0$ for $t \geq t_c(\xi)$. Since t_c is continuous and $\leq \operatorname{rad} N$, this proves the claimed inequalities. □

6.4. **Some consequences.** As a first consequence of Theorem 6.9, we obtain the global Heintze-Karcher inequality (Theorem 2.1 in [4]).

6.10. **Corollary.** *In the situation of Theorem 6.9, assume in addition that M and N are compact and that $h \leq \lambda$. Then we have*

$$\operatorname{vol} M \leq \int_N a_{m,n,\kappa}(h(p), \operatorname{diam} M) dp \leq a_{m,n,\kappa}(\lambda, \operatorname{rad} N) \operatorname{vol}(N).$$

Proof. The first inequality follows immediately from Theorem 6.9 since $U_N(\operatorname{diam} M) = M$, the second from the monotonicity of a in h. □

As a second consequence, we obtain Corollary 2.3.2 of [4], which improves Cheeger's injectivity radius estimate (Theorem 5.8 in [1]).

6.11. **Corollary.** *If M is compact with $K_M \geq \kappa$ and c is a simple closed geodesic in M, then*

$$\frac{L(c)}{\operatorname{vol} M} \geq \frac{2\pi}{\operatorname{vol} S^m} \operatorname{sn}_\kappa(\max\{d(q,c)\})^{1-m}.$$

Moreover, if $\kappa > 0$, then

$$\frac{L(c)}{\operatorname{vol} M} \geq \frac{2\pi/\sqrt{\kappa}}{\operatorname{vol} S^m_\kappa}.$$

Note that the second inequality is sharp in the case $M = M^m_\kappa$. For a compact hyperbolic surface S, we obtain $L(c) \geq |\chi(S)|/2\sinh(\operatorname{diam} S)$, where $\chi(S)$ denotes the Euler characteristic of S.

Proof of Corollary 6.11. As a submanifold, c is totally geodesic of dimension $n = 1$. Hence the comparison function j satisfies

$$j = j_{m,1,\kappa,0} = \operatorname{cs}_\kappa \operatorname{sn}_\kappa^{m-2} = \frac{1}{m-1}(\operatorname{sn}_\kappa^{m-1})'.$$

Therefore we have

$$a(h,r) = \frac{1}{m-1} \operatorname{vol}(S^{m-2}) \operatorname{sn}_\kappa^{m-1}(r) = \frac{1}{2\pi} \operatorname{vol}(S^m) \operatorname{sn}_\kappa^{m-1}(r),$$

and hence the first inequality follows from Corollary 6.13. As for the second, we note that $\operatorname{rad} c \leq \pi/2\sqrt{\kappa}$ if $\kappa > 0$. □

The next volume estimate is Theorem 2.2 of [4]. It generalizes the second estimate of $L(c)$ above.

6.12. **Corollary.** *In the situation of Theorem 6.9, assume in addition that M and N are compact and that $\kappa > 0$ and $h \le \lambda$. Then we have*

$$\frac{\operatorname{vol} N}{\operatorname{vol} M} \ge \frac{\operatorname{vol} S^n_{\kappa+\lambda^2}}{\operatorname{vol} S^m_\kappa} = \frac{\kappa^{m/2}}{(\kappa+\lambda^2)^{n/2}} \frac{\operatorname{vol} S^n}{\operatorname{vol} S^m}.$$

Heintze and Karcher show also that equality can only occur in the case where $N = S^n_{\kappa+\lambda^2} \subseteq S^m_\kappa = M$ (Theorem 4.6 in [4]).

Proof of Corollary 6.12. We recall that $a(h, r)$ is the contribution of a fibre to the volume of $U_P(r)$ for the standard $N = S^n_{\kappa+h^2}$ in $M = S^m_\kappa$. For $\kappa > 0$, we may choose $P = S^n_{\kappa+h^2}$ and get

$$a_{m,n,\kappa}(h, \pi/2\sqrt{\kappa}) = \frac{\operatorname{vol} S^m_\kappa}{\operatorname{vol} S^n_{\kappa+h^2}} = \frac{(\kappa+h^2)^{n/2}}{\kappa^{m/2}} \frac{\operatorname{vol} S^m}{\operatorname{vol} S^n} \tag{50}$$

since $\operatorname{rad} S^n_{\kappa+h^2} \le \pi/2\sqrt{\kappa}$. Now the monotonicity of a in h implies the asserted inequality. $\square$

The following estimate on the growth of the volume of tubes about submanifolds with constant mean curvature is a further immediate consequence of Theorem 6.9. It seems to be new.

6.13. **Corollary.** *In the situation of Theorem 6.9, if N has constant mean curvature $h = \lambda$, then*

$$\frac{\operatorname{vol} U_P(s)}{\operatorname{vol} U_P(r)} \le \frac{a(\lambda, s)}{a(\lambda, r)}$$

for all $0 < r < s \le \sup\{d(q, N)\}$ and relatively compact open subsets P of N, where $a = a_{m,n,\kappa}$.

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Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, and Hausdorff Center for Mathematics, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: ballmann@mpim-bonn.mpg.de