RICCATI EQUATION AND VOLUME ESTIMATES

WERNER BALLMANN

Contents
1. Introduction 1
2. Notions and notations 2
3. Distance functions and Riccati equation 4
4. Comparison theory for the Riccati equation 6
5. Bishop-Gromov inequalities 9
6. Heintze-Karcher inequalities 11
References 19

1. Introduction

Gromov suggested the use of the Riccati equation in the discussion of Jacobi field and volume estimates under lower bounds on the curvature. Indeed, this approach lead him to an important improvement of the previously known estimates of the volume of geodesic balls, Lemma 5.3 in [3], now called the Bishop-Gromov inequality.

In these lecture notes, we will employ the Riccati equation to prove the Bishop-Gromov inequality and to refine the Heintze-Karcher inequalities for tubes about submanifolds in [4]. As an application and following [4], we obtain an improved version of Cheeger’s injectivity radius estimate (Theorem 5.8 in [1]).

It is interesting to note that, in the derivation of our comparison results for Riccati equations, we return to estimating solutions of associated scalar Jacobi equations.

Date: March 9, 2016.

2010 Mathematics Subject Classification. 53C20.
Key words and phrases. Submanifold, Jacobi field, Riccati equation, volume.

I would like to thank Bogdan Georgiev and Anna Siffert for their careful reading of first versions of the manuscript and their helpful comments. I am grateful to the Max Planck Institute for Mathematics (MPIM) and Hausdorff Center for Mathematics (HCM) in Bonn for their support.
2. Notions and notations

We let $M$ be a Riemannian manifold of dimension $m$ with Levi-Civita connection $\nabla$, curvature tensor $R$, Ricci tensor $\text{Ric} = \text{Ric}_M$, and sectional curvature $K = K_M$.

For $m \geq 2$ and $\kappa \in \mathbb{R}$, the model space $M^m_\kappa$ is the unique complete and simply connected Riemannian manifold of dimension $m$ and constant sectional curvature $\kappa$. For $\kappa > 0$, we have $M^m_\kappa = S^m_\kappa$, the round sphere of radius $1/\sqrt{\kappa}$.

2.1. Geodesics and Jacobi fields. For a tangent vector $v$ of $M$, we denote by $\gamma_v$ the (maximal) geodesic in $M$ with $\dot{\gamma}_v(0) = v$. For a geodesic $\gamma$ in $M$, we write $R_\gamma X = R(X, \dot{\gamma})\dot{\gamma}$ and consider $R_\gamma$ as a field of symmetric endomorphisms of the normal spaces of $\gamma$. A Jacobi tensor field along $\gamma$ is a smooth field $J$ of endomorphisms of the normal spaces of $\gamma$ which solves the Jacobi equation

$$J'' + R_\gamma J = 0.$$  

(2)

In terms of a parallel frame $(E_1, \ldots, E_{m-1})$ of vector fields along $\gamma$ and perpendicular to $\gamma$, this means that the $J_i = JE_i$ are Jacobi fields along $\gamma$ and perpendicular to $\gamma$. If the $E_i$ are orthonormal, $J$ is represented by the field of matrices $\langle J_i, E_j \rangle$ with entries depending on the parameter of the geodesic $\gamma$.

2.2. Submanifolds. We let $N$ be a submanifold of $M$ of dimension $n$. Then we have an orthogonal decomposition

$$TM|_N = TN \oplus \nu N,$$  

(3)

where $\nu N \subseteq TM$ is the normal bundle of $N$. For $p \in N$ and $v \in T_pM$, we write accordingly $v = v^\tau + v^\nu$. Sections of $\nu N$ will be called normal fields. For smooth vector fields $X$ and $Y$ tangent to $N$, we have

$$\nabla_X Y = (\nabla_X Y)^\tau + (\nabla_X Y)^\nu = \nabla^N_X Y + II(X, Y),$$  

(4)

where $\nabla^N$ and $II$ denote the Levi-Civita connection and second fundamental form of $N$, respectively. For a smooth vector field $X$ tangent to $N$ and a normal field $\xi$, we have

$$\nabla_X \xi = (\nabla_X \xi)^\nu + (\nabla_X \xi)^\tau = \nabla^\nu_X \xi + S_\xi X,$$  

(5)

where $\nabla^\nu$ denotes the induced connection on $\nu N$ and $S_\xi$ is called the Weingarten or shape operator of $N$ with respect to $\xi$. For smooth
vector fields $X$ and $Y$ tangent to $N$ and a smooth normal field $\xi$, the inner product $\langle \xi, Y \rangle$ vanishes and hence
\[ II_\xi(X,Y) = \langle \xi, II(X,Y) \rangle = \langle \xi, \nabla_X Y \rangle = -\langle \nabla_X \xi, Y \rangle = -\langle S_\xi X, Y \rangle. \]
(6)

We conclude that $-S_\xi$ is the field of symmetric endomorphisms of $TN$ corresponding to the second fundamental form $II_\xi$ of $N$ with respect to the normal field $\xi$. We have the following fundamental equations, named after Gauss, Codazzi, and Ricci, respectively:
\[ \langle R(X,Y)U,V \rangle = \langle R^N(X,Y)U,V \rangle + \langle II(Y,U),II(X,V) \rangle - \langle II(Y,U),II(X,V) \rangle, \]
(7)
\[ \langle R(X,Y)U,\xi \rangle = \langle (\nabla^\nu X)II(Y,U) - (\nabla^\nu Y)II(X,U),\xi \rangle, \]
(8)
\[ \langle R(X,Y)\xi,\eta \rangle = \langle R^\nu(X,Y)\xi,\eta \rangle - \langle [S_\xi,S_\eta]X,Y \rangle, \]
(9)
for all vector fields $X,Y,U,V$ tangent to $N$ and normal fields $\xi,\eta$, where $R^N$ and $R^\nu$ denote the curvature tensors of $N$ and $\nu N$, respectively.

Given a (local) orthonormal frame $(E_1, \ldots, E_n)$ of vector fields tangent to $N$, we obtain a normal field
\[ \eta = \eta_N = \frac{1}{n} (II(E_1,E_1) + \cdots + II(E_n,E_n)), \]
(10)
called the mean curvature field of $N$. It does not depend on the choice of the frame $(E_1, \ldots, E_n)$. The norm $h = |\eta|$ of $\eta$ is called the mean curvature of $N$. For a unit vector $\xi \in \nu N$, $h(\xi) = \langle \xi, \eta \rangle$ is called the mean curvature of $N$ in the direction of $\xi$. We have
\[ h(\xi) = -\frac{1}{n} \text{tr} S_\xi. \]
(11)

We say that $p \in N$ is an umbilical point or an umbilic of $N$ if
\[ II(v,w) = \langle v,w \rangle \eta(p) \]
(12)
for all $v, w \in T_p N$. Then $S_\xi = -\langle \eta, \xi \rangle \text{id}$ for all $\xi \in \nu_p N$. We say that a subset $P$ of $N$ is umbilical if every point of $P$ is an umbilic of $N$. Then
\[ (\nabla^\nu_X II)(Y,Z) = \langle Y,Z \rangle \nabla^\nu_X \eta \]
(13)
in the interior of $P$.

If $N$ is a curve, then any point of $N$ is an umbilic of $N$. For all $1 < n < m$ and $h \geq 0$, a connected umbilical submanifold $N$ of dimension $n$ and mean curvature $h$ in $M^m$ is an open subset of a standard subspace $M_{n+h^2}^n \subseteq M^m$. The crux of the proof is that the mean curvature field of such an $N$ is parallel with respect to $\nabla^\nu$. This in turn follows from (13) since the left hand side of (8) vanishes for $M = M^m$ and since the dimension of $N$ is at least two.
2.3. Cut locus. In this subsection, we assume that $M$ is complete and that $N$ is proper, that is, the intersections of compact subsets of $M$ with $N$ are compact in $N$. Then $N$ is a complete Riemannian manifold with respect to its first fundamental form. We have
\[ \text{rad } N = \sup \{ d(q, N) \} \leq \text{diam } M \leq \infty \]  \hspace{1cm} (14)
for the radius $\text{rad } N$ of $N$.

Since $M$ is complete, the exponential map $\exp$ of $M$ is defined on all of $TM$. The restriction of $\exp$ to $\nu N$ is called the normal exponential map of $N$. Let $SN$ be the $S^{m-n-1}$-bundle of unit vectors in $\nu N$. For $\xi \in SN$, set
\[ t_c(\xi) = \sup \{ t > 0 \mid d(\gamma_\xi(t), N) = t \} \in (0, \infty]. \]  \hspace{1cm} (15)
As in the case where $N$ is a point, we have that $t_c: SN \to (0, \infty]$ is a continuous function. If $t_c(\xi) < \infty$, then $\gamma_\xi(t_c(\xi)) = \exp(t_c(\xi)\xi)$ is called the cut point of $N$ along $\gamma_\xi$. The closed subset
\[ C_T(N) = \{ t_c(\xi) \xi \mid \xi \in SN \text{ and } t_c(\xi) < \infty \} \]  \hspace{1cm} (16)
of $\nu N$ is called the tangential cut locus of $N$ and its image $C(N)$ under the normal exponential map the cut locus of $N$. As in the case where $N$ is a point, we have that the normal exponential map
\[ \exp: \{ t\xi \mid \xi \in SN \text{ and } 0 \leq t < t_c(\xi) \} \to N \setminus C(N) \]  \hspace{1cm} (17)
is a diffeomorphism. Recall that $t_c(\xi) \leq t_b(\xi)$, where $t_b(\xi) \in (0, \infty]$ denotes the first positive time, when a focal point of $N$ along $\gamma_\xi$ occurs; see page 12 below for the definition of focal points.

3. Distance functions and Riccati equation

Let $f: W \to \mathbb{R}$ be a function, where $W$ is an open subset of $M$. We say that $f$ is a distance function if $f$ is smooth with $|\text{grad } f| \equiv 1$.

3.1. Example. Suppose that $M$ is complete, and let $p \in M$. Then the distance $f = f(q) = d(p, q)$ from $p$ is a distance function on the open subset $W = M \setminus \{ \{p\} \cup C_p \}$ of $M$, where $C_p$ denotes the cut locus of $p$ in $M$.

More generally, if $N \subseteq M$ is a properly embedded submanifold, then the distance $f = f(q) = d(N, q)$ from $N$ is a distance function on the open subset $W = M \setminus (N \cup C_N)$ of $M$, where $C_N$ denotes the cut locus of $N$ in $M$.

In what follows, let $f: W \to \mathbb{R}$ be a distance function and $W_r$ be the level set of $f$ of level $r$; that is, $W_r = \{ p \in W \mid f(p) = r \}$. Since the gradient of $f$ does not vanish on $W$, the level sets $W_r$ are smooth
hypersurfaces in $W$ (if non-empty). Note that $\text{grad} \ f$ is a unit normal field of the level sets $W_r$.

3.2. Lemma. Let $c : [a, b] \to W$ be a piecewise smooth curve from $p \in W_r$ to $q \in W_s$, where $r \leq s$. Then $L(c) \geq s - r$ with equality if and only if $c$ solves $\dot{c} = \text{grad} \ f \circ c$ up to monotonic reparametrization. In particular, the solution curves of the gradient field are unit speed geodesics which are minimal in $W$.

Proof. Since $|\text{grad} \ f| \equiv 1$, we have

$$L(c) = \int_a^b |\dot{c}| \geq \int_a^b |\langle \text{grad} \ f, \dot{c} \rangle|$$

$$\geq |\int_a^b \langle \text{grad} \ f, \dot{c} \rangle|$$

$$= |f(q) - f(p)| = s - r.$$ 

This shows the asserted inequality. Moreover, equality holds if and only if $\dot{c}(t)$ is a non-negative multiple of $\text{grad} \ f|_{c(t)}$ for all $a \leq t \leq b$. \hfill \Box

For $p \in W$, let $\gamma_p$ be the solution curve of $\text{grad} \ f$ with $\gamma_p(r) = p$, where $r = f(p)$. By Lemma 3.2, $\gamma_p$ is a unit speed geodesic in $W$. The geodesics $\gamma_p$, for $p \in W$, will be called $f$-geodesics. Notice that $\gamma_p(t)$ depends smoothly on $p$ and $t$.

Recall that the Hessian $\text{Hess} \ f$ of $f$ is the symmetric tensor field on $W$ of type $(2, 0)$ defined by

$$\text{Hess} \ f(X, Y) = XYf - (\nabla X Y)f = \langle \nabla_X \text{grad} \ f, Y \rangle,$$ 

where $X$ and $Y$ are smooth vector fields on $W$. Since

$$2 \langle \nabla_X \text{grad} \ f, \text{grad} \ f \rangle = X \langle \text{grad} \ f, \text{grad} \ f \rangle = 0,$$ 

we get that

$$\text{Hess} \ f(X, \text{grad} \ f) = 0$$ 

for all vector fields $X$ on $W$. In particular, the non-trivial information on $\text{Hess} \ f$ comes from vector fields perpendicular to $\text{grad} \ f$, that is, vector fields tangent to the level sets of $f$. Moreover,

$$UX = \nabla_X \text{grad} \ f = S_{\text{grad} \ f}X$$ 

is the Weingarten operator of the level sets $W_r$ of $f$ with respect to the unit normal field $\text{grad} \ f$. For smooth vector fields $X$ and $Y$ tangent to the level sets of $f$, we obtain

$$\text{Hess} \ f(X, Y) = \langle \nabla_X \text{grad} \ f, Y \rangle = \langle UX, Y \rangle.$$
Let \( c = c(s) \) be a smooth curve in \( W_r \) through \( c(0) = p \) with \( \dot{c}(0) = v \). Then

\[
J = \frac{\partial}{\partial s} \gamma_{c(s)}\big|_{s=0}
\]

is a Jacobi field along \( \gamma = \gamma_p \) with

\[
J(0) = v \quad \text{and} \quad J'(0) = Uv,
\]

where \( U \) denotes the Weingarten operator of \( W_r \) as in (21). The first equality of (23) is clear. As for the proof of the second, we have in fact that

\[
J'(t) = \nabla_{\frac{\partial}{\partial t}} \gamma_{c(s)}(t)\big|_{s=0} = \nabla J(t) \text{grad } f = UJ(t).
\]

Jacobi fields \( J \) along any \( f \)-geodesic \( \gamma = \gamma_p \) satisfying initial conditions as in (23) will be called \( f \)-Jacobi fields. By their definition, \( f \)-Jacobi fields along \( \gamma = \gamma_p \) do not vanish anywhere along \( \gamma \) (as long as \( \gamma \) stays inside \( W \)). From (24), we also get that

\[
U'J = (UJ)' - UJ' = J'' - U^2J = -R_\gamma J - U^2J,
\]

where \( R_\gamma \) is defined as in (1). Since (25) holds for all \( f \)-Jacobi fields \( J \), we conclude that the field of Weingarten operators \( U \) satisfies the Riccati equation

\[
U' + U^2 + R_\gamma = 0
\]

along each \( f \)-geodesic \( \gamma \). More generally, let \( \gamma \) be a geodesic through a point \( p = c(0) \in M \) and \( U_0 \) be an endomorphism of the normal space of \( \dot{\gamma}(0) \) in \( T_pM \).

3.3. Lemma. In the above situation, let \( J \) be the Jacobi tensor field along \( \gamma \) with initial condition

\[
J(0) = \text{id} \quad \text{and} \quad J'(0) = U_0.
\]

Then the field \( U = J'J^{-1} \) of endomorphisms of the normal spaces of \( \gamma \) is the solution of the Riccati equation with initial condition \( U_0 \). Moreover, \( U \) is a field of symmetric endomorphisms if and only if \( U_0 \) is symmetric.

4. Comparison theory for the Riccati equation

For a smooth function \( \kappa \) on some interval, let \( j \) solve the scalar Jacobi equation \( j'' + \kappa j = 0 \). Then \( u = j'/j \) solves the scalar Riccati equation

\[
u' + u^2 + \kappa = 0
\]

on its domain of definition, as we saw in greater generality in Lemma 3.3, and any solution of the scalar Riccati equation arises in this way.
If 0 belongs to the domain of definition of $\kappa$, then we denote by $sn_\kappa$ and $cs_\kappa$ the solutions of the scalar Jacobi equation with
\[
    sn_\kappa(0) = 0, \quad sn'_\kappa(0) = 1 \quad \text{and} \quad cs_\kappa(0) = 1, \quad cs'_\kappa(0) = 0, \tag{28}
\]
respectively. Clearly, $u = sn'_\kappa / sn_\kappa$ is the unique solution of the Riccati equation which satisfies $\lim_{t \to 0} u(t) = \infty$.

If $\kappa$ is constant, the case we need in our applications, then $sn'_\kappa = cs_\kappa$ and $cs'_\kappa = -\kappa sn_\kappa$, and we set
\[
    tn_\kappa = sn_\kappa / cs_\kappa \quad \text{and} \quad ct_\kappa = cs_\kappa / sn_\kappa. \tag{29}
\]
Note that $u = ct_\kappa$ is the unique solution of the scalar Riccati equation which satisfies $\lim_{t \to 0} u(t) = \infty$ and that $tn_\kappa$ does not solve the scalar Riccati equation unless $\kappa = 1$.

**4.1. Lemma.** Let $u, v : (a, b] \to \mathbb{R}$ be smooth with $u' + u^2 \leq v' + v^2$ and assume that $u' + u^2$ and $v' + v^2$ extend smoothly to $[a, b]$. Then the limits $u(a) = \lim_{t \to a} u(t)$ and $v(a) = \lim_{t \to a} v(t)$ exist as extended real numbers in $( -\infty, \infty ]$. If $u(a) \leq v(a)$, then $u \leq v$ on $(a, b]$ with equality $u = v$ if and only if $u(b) = v(b)$.

**Proof.** The proof is motivated by the proof of the Sturm comparison theorem for solutions of the scalar Jacobi equation. Let $\kappa, \lambda : [a, b] \to \mathbb{R}$ be the smooth extensions of $-u' - u^2$ and $-v' - v^2$, respectively. Then $u = j'_\kappa / j_\kappa$, where $j_\kappa$ solves the associated scalar Jacobi equation $j'' + \kappa j = 0$ with $j_\kappa(a) = 1$ and $j'_\kappa(a) = u(a)$ if $u(a) < \infty$ and $j'_\kappa(a) = 0$ and $j''_\kappa(a) = 1$ if $u(a) = \infty$. Correspondingly, write $v = j'_\lambda / j_\lambda$. In both cases, $j_\kappa, j_\lambda > 0$ on $(a, b]$ since otherwise $u$ and $v$ would not be defined on $(a, b]$. We have
\[
    0 = \int_a^t \{ j_\kappa(j''_\lambda + \lambda j_\lambda) - (j''_\kappa + \kappa j_\kappa)j_\lambda \} \quad \text{and} \quad 0 = \left( j_\kappa j'_\lambda - j'_\kappa j_\lambda \right)|_a^t + \int_a^t (\lambda - \kappa)j_\kappa j_\lambda.
\]
Therefore we obtain
\[
    j_\kappa(t)j''_\lambda(t) - j'_\kappa(t)j_\lambda(t) \geq j_\kappa(a)j''_\lambda(a) - j'_\kappa(a)j_\lambda(a) + \int_a^t (\kappa - \lambda)j_\kappa j_\lambda.
\]
The first term on the right is nonnegative by the choice of $j_\kappa$ and $j_\lambda$, the second term is nonnegative since $\kappa \geq \lambda$ and $j_\kappa j_\lambda > 0$ on $(a, b]$. Since $u = j'_\kappa / j_\kappa$ and $v = j'_\lambda / j_\lambda$, this implies the asserted inequality and criterion for the equality of $u$ and $v$. \hfill \square

There are two ways in which we arrive at scalar inequalities as in Lemma 4.1 and we discuss them in Examples 4.2 and 4.3 below. To
that end, we let \( \gamma \) be a unit speed geodesic in \( M \) and \( U \) be a symmetric solution of the Riccati equation along \( \gamma \) (with domain of definition possibly smaller than the domain of definition of \( \gamma \)).

4.2. Example. Let \( E \) be a parallel field along \( \gamma \) with \(|E| \equiv 1\) and perpendicular to \( \gamma \). Then \( u = \langle UE, E \rangle \) satisfies
\[
\begin{align*}
    u' &= \langle UE', E \rangle = \langle U' E, E \rangle \\
    &= -\langle U^2 E, E \rangle - \langle R_\gamma E, E \rangle \\
    &\leq -\langle UE, E \rangle^2 - \langle R_\gamma E, E \rangle \\
    &= -u^2 - K_M(\dot{\gamma} \wedge E),
\end{align*}
\]
where we use the Schwarz inequality and that \( E \) has norm one and where \( \dot{\gamma} \wedge E \) denotes the tangent plane in \( M \) spanned by \( \dot{\gamma} \) and \( E \).

Hence if \( \kappa = \kappa(t) \) is a smooth function such that the sectional curvature of tangent planes \( P \) of \( M \) containing \( \dot{\gamma}(t) \) satisfies \( K_M(P) \geq \kappa(t) \) for all \( t \) in the domain of \( U \), then \( u' + u^2 \leq -\kappa \). Moreover, the equality \( u' + u^2 = -\kappa \) holds if and only if \( UE = uE \) and \( \langle R_\gamma E, E \rangle = \kappa \), that is, if and only if \( HE = uE \) and \( R_\gamma E = \kappa E \). Lemma 4.1 now allows to compare \( u \) with smooth functions \( v \) solving \( v' + v^2 = -\kappa \) with initial condition \( u(a) \leq v(a) \).

4.3. Example. Suppose that \( J \) is a Jacobi tensor field along \( \gamma \) with \( \det J > 0 \). Let \( U = J'J^{-1} \) and set
\[
    u = \frac{1}{m-1} \frac{d}{dt} \ln \det J = \frac{1}{m-1} \tr(J'J^{-1}) = \frac{1}{m-1} \tr U.
\]
Then we have
\[
\begin{align*}
    u' &= \frac{1}{m-1} (\tr U)' = \frac{1}{m-1} \tr U' \\
    &= -\frac{1}{m-1} \tr(U^2) - \frac{1}{m-1} \tr R_\gamma \\
    &\leq -\frac{1}{(m-1)^2} (\tr(U)^2 - \frac{1}{m-1} \Ric(\dot{\gamma}, \dot{\gamma})) \\
    &= -u^2 - \frac{1}{m-1} \Ric(\dot{\gamma}, \dot{\gamma}),
\end{align*}
\]
where we use the Schwarz inequality \( (\tr(U))^2 \leq (m-1) \tr(U^2) \). Therefore, if \( \kappa = \kappa(t) \) is a smooth function such that \( \Ric(\dot{\gamma}, \dot{\gamma}) \geq (m-1)\kappa \) on the domain of definition of \( U \), then \( u' + u^2 \leq -\kappa \). Moreover, the equality \( u' + u^2 = -\kappa(t) \) holds if and only if \( U = uI_\gamma \) and \( \Ric(\dot{\gamma}, \dot{\gamma}) = (m-1)\kappa \), where \( I_\gamma \) denotes the field of identity endomorphisms along \( \gamma \). Since \( U' + U^2 = -R_\gamma \), this happens if and only if \( U = uI_\gamma \) and \( R_\gamma = \kappa I_\gamma \). As in
the previous example, Lemma 4.1 now allows to compare \( u \) with smooth functions \( v \) solving \( v' + v^2 = -\kappa \) with initial condition \( u(a) \leq v(a) \).

4.4. **Remark.** Section 1.6 in Karcher’s survey article [5] contains a different approach to the comparison theory for solutions of the scalar Riccati equation. In [2], Eschenburg and Heintze develop an elegant comparison theory for solutions of the tensorial Riccati equation.

5. **Bishop-Gromov inequalities**

For simplicity, we assume throughout this section that \( M \) is complete. Let \( p \in M \) and \( \gamma : [0, b) \to M \) be a unit speed geodesic starting at \( p \). Let \( J \) be the Jacobi tensor field along \( \gamma \) with initial condition \( J(0) = 0 \) and \( J'(0) = \text{id} \). Clearly, we have

\[
\lim_{r \to 0} \frac{\det J(r)}{r^{m-1}} = 1. \tag{30}
\]

Notice that \( \det J(t) > 0 \) for any \( t > 0 \) before the first conjugate point of \( p \) along \( \gamma \). If \( R_\gamma = \kappa I_\gamma \), where \( \kappa \in \mathbb{R} \) and \( I_\gamma \) denotes the field of identity endomorphisms along \( \gamma \) as above, then \( J = \text{sn}_\kappa I_\gamma \) and \( \det J = \text{sn}_\kappa^{m-1} \).

The first version of the Bishop-Gromov inequality reads as follows.

5.1. **Theorem.** Assume that \( \text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq (m-1)\kappa \) for some \( \kappa \in \mathbb{R} \) and that \( \det J > 0 \) on \((0, b)\). Then

\[
1 \geq \frac{\det J(r)}{\text{sn}_\kappa^{m-1}(r)} \geq \frac{\det J(s)}{\text{sn}_\kappa^{m-1}(s)}
\]

for all \( r < s \) in \((0, b)\). The inequality on the left is strict unless \( R_\gamma = \kappa I_\gamma \) on \([0, r]\), the inequality on the right is strict unless \( R_\gamma = \kappa I_\gamma \) on \([0, s]\).

**Proof.** This follows easily from Lemma 4.1, Example 4.3, and (30). \( \square \)

5.2. **Corollary.** In the situation of Theorem 5.1, \( b \leq \pi/\sqrt{\kappa} \) if \( \kappa > 0 \).

Let \( S \) be the unit sphere in \( T_p M \). For \( v \in S \), denote by \( t_b(v) \in (0, \infty] \) the first positive time, when a conjugate point of \( p \) along \( \gamma_v \) occurs, and by \( t_c(v) \in (0, \infty] \) the cut point of \( p \) along \( \gamma_v \). As usual, the value \( \infty \) indicates that no conjugate or cut point occurs along \( \gamma_v \). By Theorem 5.1 and Jacobi’s theorem, we have \( t_b(v) \leq \pi/\sqrt{\kappa} \) if \( \kappa > 0 \) and \( t_c(v) \leq t_b(v) \), respectively. Now for \( 0 \leq t < t_b(v) \), the Jacobian of

\[
(0, \infty) \times S \to M, \quad (t, v) \mapsto \exp(tv),
\]

at \((t, v)\) is given by \( \det J_v(t) \), where \( J_v \) is the Jacobi tensor field along \( \gamma_v \) with initial conditions \( J_v(0) = 0 \) and \( J'_v(0) = \text{id} \). In particular, the
volume $V_p(r)$ of the geodesic ball $B_p(r)$ of radius $r$ about $p$ in $M$ is given by

$$V_p(r) = \int_S \int_0^{r \wedge t_c(v)} \det J_v(t) \, dt \, dv,$$

where we use the notation $a \wedge b = \min\{a, b\}$. We will compare $V_p(r)$ with the volume $V_\kappa(r) = \text{vol}(S^{m-1}) \int_0^r \text{sn}^{m-1}_\kappa(t) \, dt$ of a geodesic ball $B_\kappa(r)$ of radius $r$ in the model space $M^m_\kappa$. Here we only consider radii $r \leq \pi/\sqrt{\kappa}$ if $\kappa > 0$. By (30), we have

$$\lim_{r \to 0} \frac{V_p(r)}{V_\kappa(r)} = 1.$$  

The global version of the Bishop-Gromov inequality reads as follows.

5.3. **Theorem.** Assume that $\text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq (m - 1)\kappa$ for all unit speed geodesics through $p$ and some $\kappa \in \mathbb{R}$. Then

$$1 \geq \frac{V_p(r)}{V_\kappa(r)} \geq \frac{V_p(s)}{V_\kappa(s)}$$

for all $0 < r < s \leq \max\{d(p, q)\}$ (with $s < \pi/\sqrt{\kappa}$ if $\kappa > 0$). The inequality on the left is strict unless $B_p(r)$ is isometric to $B_\kappa(r)$ and the inequality on the right is strict unless $B_p(s)$ is isometric to $B_\kappa(s)$.

**Proof.** By what we said above, we have

$$\frac{V_p(r)}{V_\kappa(r)} = \frac{1}{\text{vol}(S^{m-1}) \int_0^r \text{sn}^{m-1}_\kappa(t) \, dt} \int_S \int_0^r f(t, v) \text{sn}^{m-1}_\kappa(t) \, dt \, dv,$$

where $f(t, v) = \det J_v(t)/\text{sn}^{m-1}_\kappa(t)$ for $0 < t < t_c(v)$ and $f(t, v) = 0$ for $t \geq t_c(v)$. By (30), we have $\lim_{t \to 0} f(t, v) = 1$. By Theorem 5.1, $f$ is monotonically decreasing, where $f(t, v) = 1$ can only hold if $t < t_c(v)$ and $R_{\gamma_v} = \kappa J_{\gamma_v}$ on $[0, t]$. Now the right hand side in (34) is the mean of $f$ with respect to the volume form $\text{sn}^{m-1}_\kappa \, dt \, dv$ on $(0, r] \times S$. □

5.4. **Corollary.** If $\text{Ric}_M \geq \kappa > 0$, then

$$\text{diam } M \leq \text{diam } S^m_\kappa = \pi/\sqrt{\kappa} \quad \text{and} \quad \text{vol } M \leq \text{vol } S^m_\kappa,$$

and the inequalities are strict unless $M$ is isometric to $S^m_\kappa$.

The assertion $\text{diam } M \leq \pi/\sqrt{\kappa}$ is called the *theorem of Bonnet-Myers*, the characterization of $S^m_\kappa$ by the equality $\text{diam } M = \pi/\sqrt{\kappa}$ the *maximal diameter theorem of Cheng*.
Proof. Except for the maximal diameter theorem, the assertions are fairly straightforward consequences of Theorems 5.1 and 5.3. That the maximal diameter theorem follows from Theorem 5.3 was observed by Shiohama: Choose \( p, q \in M \) of maximal distance \( d(p, q) = \pi/\sqrt{\kappa} \). Then the geodesic balls \( B_p(\pi/2\sqrt{\kappa}) \) and \( B_q(\pi/2\sqrt{\kappa}) \) are disjoint and hence

\[
\text{vol } B_p(\pi/2\sqrt{\kappa}) + \text{vol } B_q(\pi/2\sqrt{\kappa}) \leq \text{vol } M. \tag{35}
\]

Now Theorem 5.3 implies that

\[
\frac{\text{vol } B_p(\pi/2\sqrt{\kappa})}{\text{vol } B_p(\pi/\sqrt{\kappa})}, \quad \frac{\text{vol } B_q(\pi/2\sqrt{\kappa})}{\text{vol } B_q(\pi/\sqrt{\kappa})} \geq \frac{V_\kappa(\pi/2\sqrt{\kappa})}{V_\kappa(\pi/\sqrt{\kappa})} = \frac{1}{2}. \tag{36}
\]

Since \( M = B_p(\pi/\sqrt{\kappa}) = B_q(\pi/\sqrt{\kappa}) \), by the theorem of Bonnet-Myers, we conclude that

\[
\text{vol } B_p(\pi/2\sqrt{\kappa}), \text{vol } B_q(\pi/2\sqrt{\kappa}) \geq \text{vol } M/2.
\]

Hence we have equality in (35) and (36), hence \( \text{vol } M \geq \text{vol } S^m_\kappa \) and therefore \( M = S^m_\kappa \) by the equality case of the volume inequality. \( \square \)

6. Heintze-Karcher inequalities

For simplicity, we assume throughout this section that \( M \) is complete and that \( N \) is a proper submanifold of \( M \).

Let \( \pi: \nu N \rightarrow N \) be the normal bundle of \( N \). Associated to the induced connection \( \nabla^\nu \) on \( \nu N \), there is a canonical decomposition of the tangent bundle of \( \nu N \),

\[
T\nu N = \mathcal{H} \oplus \mathcal{V},
\]

where the vertical space \( \mathcal{V}_\xi = \ker d\pi_\xi \) at \( \xi \in \nu_p N \) is canonically isomorphic to \( \nu_p N \) and the horizontal space \( \mathcal{H}_\xi \), defined to consist of all tangent vectors to parallel sections of \( \nu N \) through \( \xi \), is canonically isomorphic to \( T_p N \). If \( \zeta = \zeta(s) \) is a smooth curve in \( \nu N \) through \( \xi \) with \( \dot{\zeta}(0) = v \) and \( \sigma = \pi \circ \zeta \), then

\[
v^\mathcal{H} = \dot{\sigma}(0) = d\pi_\xi(v) \quad \text{and} \quad v^\mathcal{V} = \frac{\nabla^\nu}{\partial s} \zeta|_{s=0}. \tag{38}
\]

The decomposition into vertical and horizontal subspaces induces a Riemannian metric on \( T\nu N \) such that \( \pi \) is a Riemannian submersion. Note that this metric only depends on the Riemannian metric on \( N \), the metric on the fibers of \( \nu N \), and the decomposition.

6.1. Lemma. For \( v \in T_\zeta \nu N \), we have

\[
d\exp_\zeta(v) = J(1),
\]
where $J$ is the Jacobi field along $\gamma_\xi$ with

$$J(0) = v^H \quad \text{and} \quad J'(0) = v^V + S_\xi v^H.$$  

**Proof.** Let $\zeta = \zeta(s)$ be a smooth curve in $\nu N$ through $\xi$ with $\dot{\zeta}(0) = v$.

Then

$$d\exp_\xi (v) = \frac{\partial}{\partial s} \exp(t\zeta(s)) \bigg|_{t=1,s=0} = J(1),$$

where $J$ is the Jacobi field along $\gamma_\xi$ with

$$J(0) = \frac{\partial}{\partial s} \exp(t\zeta(s)) \bigg|_{t=0,s=0} = \frac{\partial}{\partial s} \pi(\zeta(s)) \bigg|_{s=0} = d\pi(\xi)$$

and

$$J'(0) = \nabla_\xi \frac{\partial}{\partial t} \frac{\partial}{\partial s} \exp(t\zeta(s)) \bigg|_{s=0,t=0}$$

$$= \nabla_\xi \frac{\partial}{\partial s} \exp(t\zeta(s)) \bigg|_{t=0,s=0}$$

$$= \nabla_\xi \zeta \bigg|_{s=0} = \nabla_\xi \zeta \bigg|_{s=0} + S_\xi d\pi(\xi). \quad \Box$$

For $\xi \in \nu_p N$, a Jacobi field $J$ along $\gamma_\xi$ will be called an $N$-Jacobi field if and only if it is associated to a geodesic variation $(\gamma_s)$ of $\gamma_0 = \gamma_\xi$ such that $\sigma = \sigma(s) = \gamma_s(0)$ is a smooth curve in $N$ with $\dot{\gamma}_s(0) \in \nu N$.

By our discussion above, this holds precisely if

$$J(0) \in T_p N \quad \text{and} \quad J'(0)^H = S_\xi J(0). \quad (39)$$

For $t > 0$, we say that $\gamma_\xi(t)$ is a focal point of $N$ along $\gamma_\xi$ if there is a non-trivial $N$-Jacobi field along $\gamma_\xi$ that vanishes at $t$. If there is a $t > 0$ such that $\gamma_\xi(t)$ is a focal point of $N$ along $\gamma_\xi$, then we let $t_b(\xi) > 0$ be the smallest such $t$ and call $\gamma_\xi(t_b(\xi))$ the first focal point of $N$ along $\gamma_\xi$. If there is no non-trivial $N$-Jacobi field that vanishes in positive time, then we set $t_b(\xi) = \infty$.

We say that a Jacobi field $J$ along $\gamma_\xi$ is a special $N$-Jacobi field if it is associated to a geodesic variation $(\gamma_s)$ of $\gamma_0 = \gamma_\xi$ as above and if, in addition, $|\dot{\gamma}_s(0)|$ does not depend on $s$. This holds precisely if

$$J(0) \in T_p N \quad \text{and} \quad J'(0) = S_\xi J(0). \quad (40)$$

Special $N$-Jacobi fields will enter our discussion below. We will be concerned with volume and integration. To that end, we introduce polar coordinates; that is, we consider the diffeomorphism

$$F: (0, \infty) \times SN \to \nu N \setminus N, \quad F(t, \xi) = t\xi, \quad (41)$$

where we identify $N$ with the zero section of $\nu N$. Since $\pi: SN \to N$ is a Riemannian submersion, we write $d\xi dp$ for the volume element of
Let now \( \xi \in SN \) and \((e_1, \ldots, e_n)\) be an orthonormal basis of \( T_pN \) consisting of eigenvectors of \( S_\xi \) with corresponding eigenvalues

\[
\lambda_1 = \lambda_1(\xi) \leq \cdots \leq \lambda_n = \lambda_n(\xi).
\]

Recall from (11) that the mean curvature \( h(\xi) \) of \( N \) in the direction of \( \xi \) is given by \(-nh(\xi) = \lambda_1 + \cdots + \lambda_n\). Extend \((e_1, \ldots, e_n)\) to an orthonormal basis \((e_1, \ldots, e_m)\) of \( T_pM \) with \( e_m = \xi \). Let \((E_1, \ldots, E_m)\) be the orthonormal frame along \( \gamma_\xi \) with \( E_i(0) = e_i \) for all \( 1 \leq i \leq m \).

Denote by \( J_\xi = J_\xi(t) \) the Jacobi tensor field along \( \gamma_\xi \) which maps \( E_i \) to the Jacobi field \( J_i \) with \( J_i(0) = e_i \) and \( J_i'(0) = \lambda_i e_i \) for \( 1 \leq i \leq n \) and with \( J_i(0) = 0 \) and \( J_i'(0) = e_i \) for \( n < i < m \), respectively. For \( 1 \leq i \leq n \), \( J_i \) is the special \( N \)-Jacobi field with \( J_i(0) = e_i \). We note that \( \det J_\xi > 0 \) up to \( t_b(\xi) \).

6.2. Lemma. The volume distortion of \( \exp \circ F \) is given by

\[
(\exp \circ F)^*_{(t,\xi)}dV = \det J_\xi(t) dt d\xi dp
\]

for all \( 0 < t < t_b(\xi) \), where \( dV \) and \( d\xi dp \) denote the volume forms of \( M \) and \( SN \), respectively.

Proof. By Lemma 6.1, the special \( N \)-Jacobi fields \( J_1, \ldots, J_n \) correspond to an orthonormal basis of \( \mathcal{H}_\xi \). Furthermore, the Jacobi fields \( J_{n+1}, \ldots, J_{m-1} \) together with the Jacobi field \( J_m = J_m(t) = t\gamma_\xi(t) \) correspond to an orthonormal basis of \( \mathcal{V}_\xi \). Now \( \exp \) is a radial isometry, that is, \( J_m \) represents a unit normal vector of \( SN \) at \( \xi \) and stays normal to \( J_1, \ldots, J_{m-1} \). Hence the volume distortion of \( F \circ \exp \) is given by \( \det J_\xi(t) \) \( \square \)

6.3. Lemma. As a field of matrices in terms of the \( E_i \) and up to \( O(t^2) \) for \( t \to 0 \), \( J_\xi(t) \) is the diagonal matrix with entries \( 1 + \lambda_i t \) for all \( 1 \leq i \leq n \) and \( t \) for all \( n + 1 \leq i \leq m \). In particular,

\[
\det J_\xi(t) = t^{m-n} + (\lambda_1 + \cdots + \lambda_n) t^{m-n} + O(t^{m-n+1}) \quad \text{as } t \to 0.
\]

We are after estimates of the volume distortion \( \det J_\xi \) and of the volumina of tubes

\[
U_N(r) = \{ q \in M \mid d(q, N) < r \}
\]

about \( N \). Here the case of balls from Section 5 may be considered as the special case where \( n = 0 \), and we proceed in quite similar ways in the other cases. We want to compare \( \det J_\xi \) with the corresponding
quantity $j_h$ in the case of an umbilical submanifold of dimension $n$ in the model space $M^m_n$, where $h = h(\xi)$. We have

$$j_h(t) = j_{m,n,\kappa,k}(t) = (cs_n(t) - h \sn_n(t))^n \sn_m^{-n-1}(t).$$  \hfill (44)

The first positive zero $z(h)$ of $j_h$ is determined by $ct_n(z(h)) = h$. Define

$$a_h(t) = a_{m,n,\kappa,k}(t) = \int_{s^{m-n-1}} \int_0^{t \wedge z(\langle \xi, \eta \rangle)} j(\langle \xi, \eta \rangle, t) \, dt \, d\xi,$$  \hfill (45)

where $\eta \in \mathbb{R}^{m-n}$ is a vector of length $h$. Clearly, $a$ does not depend on the choice of $\eta$. In the case where $N$ is an umbilical hypersurface in the model space $M^m_n$ with mean curvature $h$, $a_h(r)$ is the contribution of a fiber over $N$ in a tube of radius $r$ to the volume of the tube. We sometimes consider parameters as variables and write, e.g., $a(h, t)$ instead of $a_h(t)$ or $j_{m,n,\kappa,k}(h, t)$ instead of $j_{m,n,\kappa,k}(t)$.

6.4. Lemma. The function $a$ is monotonically increasing in $h$ and $t$.

Proof. The monotonicity in $t$ is clear. As for the monotonicity in $h$, we follow the arguments in the proof of the corresponding Proposition 2.1.1 of [4]. We start by noting that $j_{\langle \xi, \eta \rangle}(t)$ is not monotonically increasing in $h = |\eta|$ for $\langle \xi, \eta \rangle \geq 0$. However, we may consider $\pm \xi$ simultaneously to conclude monotonicity of $a$ in $h$.

Since the integrand of the inner integral in (45) vanishes at $z(\langle \xi, \eta \rangle)$, we only need to check the monotonicity of the integrand in either case, $t \wedge z(\langle \xi, \eta \rangle) = t$ or $t \wedge z(\langle \xi, \eta \rangle) = z(\langle \xi, \eta \rangle)$. Assume now that $\langle \xi, \eta \rangle \geq 0$. Then we have $z(\langle \xi, \eta \rangle) \leq z(\langle -\xi, \eta \rangle)$ and

$$(x - hy)^n + (x + hy)^n = 2 \sum_{2k \leq n} \binom{n}{2k} x^{n-2k} y^{2k} h^{2k},$$

where $x = cs_n(t)$, $y = \langle \xi, \eta \rangle \sn_n(t)/|\eta|$, $h = |\eta|$, and $0 \leq t \leq z(\langle \xi, \eta \rangle)$. Now the right hand side in the above formula is monotonic in $h$. Furthermore, $j(\langle -\xi, \eta \rangle, t)$ is monotonic in $h$ on $z(\langle \xi, \eta \rangle) \leq t \leq z(\langle -\xi, \eta \rangle)$. We conclude that $a = a(h, t)$ is monotonic in $h$. \hfill \Box

6.5. Question. Given $h < H$, is $a(h, t)/a(H, t)$ monotonic in $t$?

Up to $t_0(\xi)$, $J_\xi = J_\xi(t)$ is invertible and $U = J_\xi J_\xi^{-1}$ solves the Riccati equation along $\gamma_\xi$. As in Example 4.3, we have the crucial relation

$$\frac{d}{dt} \ln \det J_\xi = \operatorname{tr}(J_\xi J_\xi^{-1}) = \operatorname{tr} U.$$  \hfill (46)

To estimate $\operatorname{tr} U$, lower bounds for the Ricci curvature can be used efficiently in the cases $n = 0$, as we saw in Section 5, and $n = m - 1$. In the general case, we will use lower bounds for the sectional curvature.
Using the results from Section 4, it would also be possible to discuss
the more refined bounds on the curvature from Section 3.2 in [4]. However,
for the sake of simplicity, we will only consider the traditional
case of absolute lower bounds.

6.1. The case of hypersurfaces. In the next result, we refine the
main inequality 3.2.1(d) from [4].

6.6. Theorem. Assume that \( N \) is a hypersurface and that \( \text{Ric}_M(\dot{\gamma}_\xi) \geq (m-1)\kappa \) along \( \gamma_\xi \), where \( \xi \in SN \) and \( \kappa \in \mathbb{R} \). Then we have

\[
1 \geq \frac{\det J_\xi(r)}{j(h(\xi), r)} \geq \frac{\det J_\xi(s)}{j(h(\xi), s)}
\]

for all \( 0 < r < s < t_b(\xi) \), where \( j \) is as in (44). The inequality on the
left is strict unless \( R_{\gamma_\xi} = \kappa I_{\gamma_\xi} \) on \([0, r]\) and \( S_\xi = -h(\xi) \text{id} \), the inequality
on the right is strict unless \( R_{\gamma_\xi} = \kappa I_{\gamma_\xi} \) on \([0, s]\) and \( S_\xi = -h(\xi) \text{id} \).

Proof. With \( \gamma = \gamma_\xi \), \( J = J_\xi \), and \( U = J_\xi J_\xi^{-1} \), we choose \( u \) as in
Example 4.3. Then \( u \) satisfies \( u' + u^2 + \kappa \leq 0 \) with initial condition

\[
u(0) = \frac{1}{m-1} \text{tr} U(0) = \frac{1}{m-1} \text{tr} J_\xi(0) = -h(\xi) = -h.
\]

From Lemma 4.1 we get that \( u \leq v \), where \( v \) solves \( v' + v^2 + \kappa = 0 \) with initial condition \( v(0) = -h \). Then \( v = j'/j \) with \( j = cs_n - h sn \kappa \)
and hence the asserted inequalities follow.

Equality can only happen if \((\ln \det J)'(t) = (m-1)(\ln j)'(t)\) for all \( t \in (0, r) \) or \( t \in (r, R) \), respectively. This implies \( R_\gamma = \kappa I_\gamma \) on \([0, t]\) by
the discussion in Example 4.3. \( \square \)

6.2. The general case. In the following result, we refine the main
inequality 3.2.1(c) from [4].

6.7. Lemma. Assume that \( K_M(\dot{\gamma}_\xi \wedge E) \geq \kappa \) for all parallel unit fields
\( E \) along \( \gamma_\xi \) perpendicular to \( \gamma_\xi \), where \( \xi \in SN \) and \( \kappa \in \mathbb{R} \). Then

\[
1 \geq \frac{\det J_\xi(r)}{j(r)} \geq \frac{\det J_\xi(s)}{j(s)}
\]

for all \( 0 < r < s < t_b(\xi) \), where

\[
j = (cs_n + \lambda_1(\xi) sn_\kappa) \cdots (cs_n + \lambda_n(\xi) sn_\kappa) sn_m^{m-n-1}.
\]

The inequality on the left is strict unless \( R_\gamma = \kappa I_\gamma \) on \([0, r]\), the in-
equality on the right is strict unless \( R_\gamma = \kappa I_\gamma \) on \([0, s]\).
Proof. The trace of $U$ is the sum of the terms $\langle UE_i, E_i \rangle$, where the $E_i$ are chosen as above. Since $K_M(\gamma \xi \wedge E_i) \geq \kappa$, we have

$$\langle UE_i, E_i \rangle \leq u_i = j'_i / j_i$$

by Example 4.2 and Lemma 4.1, where $j_i = cs_\kappa + \lambda_i sn_\kappa$ for $1 \leq i \leq n$ and $j_i = sn_\kappa$ for $n < i < m$, respectively. Hence

$$\frac{d}{dt} \ln \det J \leq \frac{j'_1}{j_1} + \cdots + \frac{j'_{m-1}}{j_{m-1}} = \frac{d}{dt} \ln(j_1 \cdots j_{m-1}) = \frac{d}{dt} \ln j.$$

Hence $\det J / j$ is monotonically decreasing. On the other hand, we also have $\lim_{r \to 0} \det J(r) / j(r) = 1$ by Lemma 6.3.

Equality in the asserted inequalities can only happen if $(\ln \det J)'(t) = j'(t)$ for all $t \in (0, r)$ or $t \in (r, R)$, respectively, and then $\langle UE_i, E_i \rangle = u_i$ for all $1 \leq i \leq m - 1$. This implies $R_\gamma = \kappa I_\gamma$ on $[0, t]$ by the discussion in Example 4.2.

Following the discussion in [4], we get rid of the explicit dependence on the eigenvalues $\lambda_i = \lambda_i(\xi)$ of $S_\xi$ by weakening the inequality. Employing the inequality between geometric and arithmetic mean, we have

$$\left( cs_\kappa + \lambda_1 sn_\kappa \right) \cdots \left( cs_\kappa + \lambda_n sn_\kappa \right) \leq \left( cs_\kappa - \langle \xi, \eta \rangle sn_\kappa \right)^n \quad (47)$$

between $t = 0$ and the first positive zero of the left hand side, where $\eta$ denotes the mean curvature vector field of $N$ as in (10). We arrive at a refined version of Corollary 3.3.1 of [4].

6.8. Theorem. Assume that $K_M(\gamma \xi \wedge E) \geq \kappa$ for all parallel unit fields $E$ along $\gamma \xi$ perpendicular to $\gamma \xi$, where $\xi \in SN$ and $\kappa \in \mathbb{R}$. Then

$$1 \geq \frac{\det J_\xi(r)}{j(h(\xi), r)} \geq \frac{\det J_\xi(s)}{j(h(\xi), s)}$$

for all $0 < r < s < t_b(\xi)$, where $j = jm_{n,n}$ and $h(\xi) = \langle \xi, \eta \rangle$. The inequality on the left is strict unless $R_\gamma = \kappa I_\gamma$ on $[0, r]$ and $S_\xi = -h(\xi)$ id, the inequality on the right is strict unless $R_\gamma = \kappa I_\gamma$ on $[0, s]$ and $S_\xi = -h(\xi)$ id.

In particular, the first zero $z(\langle \xi, \eta \rangle)$ of $cs_\kappa - \langle \xi, \eta \rangle sn_\kappa$ is an upper bound for the first focal point $t_b(\xi)$ of $N$ along $\gamma \xi$.

Proof of Theorem 6.8. We show that the quotient

$$\frac{\left( cs_\kappa + \lambda_1 sn_\kappa \right) \cdots \left( cs_\kappa + \lambda_n sn_\kappa \right)}{\left( cs_\kappa - \langle \xi, \eta \rangle sn_\kappa \right)^n}$$

(48)

is monotonically decreasing. Indeed, we have

$$\frac{d}{dt} \ln \left( \frac{\left( cs_\kappa + \lambda_1 sn_\kappa \right) \cdots \left( cs_\kappa + \lambda_n sn_\kappa \right)}{\left( cs_\kappa - \langle \xi, \eta \rangle sn_\kappa \right)^n} \right) = u_1 + \cdots + u_n - nv,$$
where the $u_i$ are as above and $v = j'/j$ with $j = cs_\kappa - \langle \xi, \eta \rangle s_\kappa$. Now $v$ solves $v' + v^2 + \kappa = 0$ and
\[ u = \frac{u_1 + \cdots + u_n}{n} \]
satisfies $u' + u^2 + \kappa \leq 0$ with equality if all the $u_i$ coincide. Furthermore, we have $u(0) = v(0)$. Hence $u \leq v$ by Lemma 4.1, and therefore the quotient in (48) is monotonically decreasing. \hfill \Box

6.3. Main estimate for the volume of tubes. For $r > 0$, the subset
\[ U_N(r) = \{ q \in M \mid d(q, N) < r \} \]
(49)
of $M$ is called the tube of radius $r$ about $N$. For an open subset $P$ of $N$, let $U_P(r) = \{ q \in U_N(r) \mid d(q, N) = d(q, P) \}$. The next result refines the global Heintze-Karcher inequality (Theorem 2.1 in [4]).

6.9. Theorem. For a constant $\kappa \in \mathbb{R}$, assume that
1) $N$ is a hypersurface such that $\text{Ric}_M(\dot{\gamma}_\xi) \geq (m-1)\kappa$ along $\gamma_\xi$ for all unit normal vectors $\xi$ of $N$ or that
2) $K_M(\dot{\gamma}_\xi \wedge E) \geq \kappa$ for all unit normal vectors $\xi$ of $N$ and parallel unit vector fields $E$ along $\gamma_\xi$ perpendicular to $\gamma_\xi$.
Then we have
\[ 1 \geq \frac{\text{vol} U_P(r)}{\int_P a(h(p), r) \, dp} \geq \frac{\text{vol} U_P(s)}{\int_P a(h(p), s) \, dp} \]
for all $0 < r < s < \text{rad} N$ and relatively compact open subsets $P$ of $N$, where $a = a_{m,n,\kappa}$ and $h$ denotes the mean curvature of $N$.

Proof. Consider the set
\[ Z = \{ (t, \xi) \in (0, \infty) \times SN \mid t < z(\xi) \}, \]
endowed with the volume element $\omega = j_{h(\xi)}(t) \, dtd\xi dp$. Define a function $f: Z \rightarrow \mathbb{R}$ by
\[ f(\xi, t) = \begin{cases} 
\det J_\xi(t)/j_h(t) & \text{for } 0 \leq t < t_c(\xi), \\
0 & \text{for } t_c(\xi) \leq t < z(\xi).
\end{cases} \]
Since the cut locus $C(N)$ has measure 0, we have
\[ \frac{\text{vol} U_P(r)}{\int_N a(h(p), r) \, dp} = \frac{1}{\int_N a(h(p), r) \, dp} \int_{SN \setminus \{0\}} \int_0^{r \wedge z(\xi)} f(\xi, t) j_{h(\xi)}(t) \, dtd\xi dp. \]
By the definition of $a$, the right hand side is the mean of $f$ over the set
\[ Z_r = \{ (t, \xi) \in Z \mid t \leq r \} \]
with respect to the volume element $\omega$. For each $\xi$, $f$ is monotonically decreasing in $t$ with $1 \geq f(\xi, t) > 0$ for $0 \leq t \leq t_c(\xi)$, by Theorem 6.6.
and Theorem 6.8, respectively, and \( f(\xi, t) = 0 \) for \( t \geq t_c(\xi) \). Since \( t_c \) is continuous and \( \leq \text{rad} \ N \), this proves the claimed inequalities.

### 6.4. Some consequences.

As a first consequence of Theorem 6.9, we obtain the global Heintze-Karcher inequality (Theorem 2.1 in [4]).

### 6.10. Corollary.

In the situation of Theorem 6.9, assume in addition that \( M \) and \( N \) are compact and that \( h \leq \lambda \). Then we have

\[
\text{vol } M \leq \int_N a_{m,n,\kappa}(h(p), \text{diam } M) dp \leq a_{m,n,\kappa}(\lambda, \text{rad } N) \text{ vol}(N).
\]

**Proof.** The first inequality follows immediately from Theorem 6.9 since \( U_N(\text{diam } M) = M \), the second from the monotonicity of \( a \) in \( h \). \( \square \)

As a second consequence, we obtain Corollary 2.3.2 of [4], which improves Cheeger’s injectivity radius estimate (Theorem 5.8 in [1]).

### 6.11. Corollary.

If \( M \) is compact with \( K_M \geq \kappa \) and \( c \) is a simple closed geodesic in \( M \), then

\[
\frac{L(c)}{\text{vol } M} \geq \frac{2\pi}{\text{vol } S^m} \text{ sn}_\kappa(\max\{d(q, c)\})^{1-m}.
\]

Moreover, if \( \kappa > 0 \), then

\[
\frac{L(c)}{\text{vol } M} \geq \frac{2\pi}{\sqrt{\kappa}} \frac{1}{\text{vol } S^m}.
\]

Note that the second inequality is sharp in the case \( M = M^m_\kappa \). For a compact hyperbolic surface \( S \), we obtain \( L(c) \geq |\chi(S)|/2 \sinh(\text{diam } S) \), where \( \chi(S) \) denotes the Euler characteristic of \( S \).

**Proof of Corollary 6.11.** As a submanifold, \( c \) is totally geodesic of dimension \( n = 1 \). Hence the comparison function \( j \) satisfies

\[
j = j_{m,1,\kappa,0} = cs_\kappa \text{ sn}_\kappa^{m-2} = \frac{1}{m-1} (\text{ sn}_\kappa^{m-1})'.
\]

Therefore we have

\[
a(h, r) = \frac{1}{m-1} \text{ vol}(S^{m-2}) \text{ sn}_\kappa^{m-1}(r) = \frac{1}{2\pi} \text{ vol}(S^m) \text{ sn}_\kappa^{m-1}(r),
\]

and hence the first inequality follows from Corollary 6.13. As for the second, we note that \( \text{rad } c \leq \pi/2\sqrt{\kappa} \) if \( \kappa > 0 \). \( \square \)

The next volume estimate is Theorem 2.2 of [4]. It generalizes the second estimate of \( L(c) \) above.
6.12. **Corollary.** In the situation of Theorem 6.9, assume in addition that $M$ and $N$ are compact and that $\kappa > 0$ and $h \leq \lambda$. Then we have
\[
\frac{\text{vol } N}{\text{vol } M} \geq \frac{\text{vol } S^n_{\kappa + \lambda^2}}{\text{vol } S^n_{\kappa}} = \frac{\kappa^{m/2}}{(\kappa + \lambda^2)^{n/2}} \frac{\text{vol } S^n}{\text{vol } S^m}.
\]
Heintze and Karcher show also that equality can only occur in the case where $N = S^n_{\kappa + \lambda^2} \subseteq S^m_{\kappa} \subseteq M$ (Theorem 4.6 in [4]).

**Proof of Corollary 6.12.** We recall that $a(h, r)$ is the contribution of a fibre to the volume of $U_P(r)$ for the standard $N = S^n_{\kappa + h^2}$ in $M = S^m_{\kappa}$. For $\kappa > 0$, we may choose $P = S^n_{\kappa + h^2}$ and get
\[
a_{m, n, \kappa}(h, \pi/2\sqrt{\kappa}) = \frac{\text{vol } S^m_{\kappa}}{\text{vol } S^n_{\kappa + h^2}} = \frac{(\kappa + h^2)^{n/2}}{\kappa^{m/2}} \frac{\text{vol } S^n}{\text{vol } S^m}
\]
(50)
since $S^n_{\kappa + h^2} \leq \pi/2\sqrt{\kappa}$. Now the monotonicity of $a$ in $h$ implies the asserted inequality. \( \square \)

The following estimate on the growth of the volume of tubes about submanifolds with constant mean curvature is a further immediate consequence of Theorem 6.9. It seems to be new.

6.13. **Corollary.** In the situation of Theorem 6.9, if $N$ has constant mean curvature $h = \lambda$, then
\[
\frac{\text{vol } U_P(s)}{\text{vol } U_P(r)} \leq \frac{a(\lambda, s)}{a(\lambda, r)}
\]
for all $0 < r < s \leq \sup\{d(q, N)\}$ and relatively compact open subsets $P$ of $N$, where $a = a_{m, n, \kappa}$.

**References**


Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, and Hausdorff Center for Mathematics, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: ballmann@mpim-bonn.mpg.de