The Hodge theory of the Hecke category

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Abstract. Ideas from Hodge theory have found important applications in representation theory. We give a survey of joint work with Ben Elias which uncovers Hodge theoretic structure in the Hecke category ("Soergel bimodules"). We also outline similarities and differences to other combinatorial Hodge theories.

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1. Introduction

In representation theory the setting is often highly structured. Thus it is surprising that several central theorems and conjectures may be interpreted as stating that a given situation behaves as if it were generic.[1] (An example: a general bilinear form on a vector space is non-degenerate, however establishing that a specific form is non-degenerate might be very difficult.)

The notion of a Hodge structure first arose in complex algebraic geometry. The existence of Hodge structures is the extra data that distinguishes the cohomology groups of algebraic varieties (or Kähler manifolds) from those of general spaces. Over the last twenty years it has been discovered that some kind of combinatorial Hodge structure exists in at least four other situations (polytopes, Coxeter groups, matroids, tropical geometry). The existence of these structures has lead to the solutions of several conjectures which have elementary formulations, but no elementary proof. (An example of such an elementary question: how many faces of each dimension can a polytope have?) It is an interesting question as to whether there is a unifying framework for these various "Hodge theories".

In this paper we survey these Hodge theories and then concentrate on Soergel bimodules, which gives rise to Hodge structures in three distinct ways (global, relative and local). Hodge structures are powerful enough to deduce the genericity statements alluded to above, and thus to prove several positivity conjectures about the Kazhdan-Lusztig basis. They also provide new algebraic proofs of difficult theorems in representation theory (Kazhdan-Lusztig conjecture, Jantzen conjecture).

The structure of his survey is as follows. In §2 we define what we mean by a combinatorial Hodge theory and in §3 we give some examples. In §4 we recall the Hecke algebra, its Kazhdan-Lusztig basis and its categorification by Soergel bimodules. Finally, in §5 we survey the Hodge structures arising from Soergel bimodules.

[1] I learnt this point of view from W. Soergel and P. Fiebig.

2. Combinatorial Hodge theory

Here we outline what we mean by combinatorial Hodge theory. So far most of the structures that show up outside complex algebraic geometry are of a very simple form (they are of "Hodge-Tate type"). This means that much of the linear algebra simplifies greatly.

2.1. Lefschetz data. Let $\Lambda \subset \mathbb{R}$ denote a subfield. Examples to keep in mind are $\Lambda = \mathbb{Q}, \mathbb{R}$ or some finite extension of $\mathbb{Q}$. By *Lefschetz data* we mean the following:

(1) A finite dimensional graded Λ-vector space $H = \bigoplus_{i \in \mathbb{Z}} H^i$ which vanishes in either even or odd degree.

(2) A non-degenerate graded symmetric bilinear form

$$\langle -, - \rangle : H \times H \to \Lambda$$

("graded" means that $\langle H^i, H^j \rangle = 0$ if $i \neq -j$).

(3) A graded vector space V concentrated in degree 2 together with a map

$$a : S^\bullet(V) \to \mathrm{End}(H)$$

where $S^\bullet(V)$ denotes the graded symmetric algebra on V (i.e. we are given commuting endomorphisms $a(\gamma) : H^\bullet \to H^{\bullet+2}$ depending linearly on $\gamma \in V$). Via a we may view H as a graded $S^\bullet(V)$-module. From now on we forget a and, for any $p \in S^\bullet(V)$, simply write

$$p \cdot h := a(p)(h).$$

We require that the action if compatible with $\langle -, - \rangle$ in the sense that

$$\langle p \cdot h, h' \rangle = \langle h, p \cdot h' \rangle \quad \text{for all } p \in S^\bullet(V) \text{ and } h, h' \in H.$$

(4) An open convex cone $V_{\mathrm{ample}} \subset V$ ("cone" means that V_{ample} is closed under multiplication by $\Lambda_{>0} := \mathbb{R}_{>0} \cap \Lambda$.)

We say that $\gamma \in V$ *satisfies hard Lefschetz* if for all $i \geq 0$ action by γ^i yields an isomorphism

$$\gamma^i : H^{-i} \xrightarrow{\sim} H^i.$$

We say that Lefschetz data as above *satisfies hard Lefschetz* if all $\gamma \in V_{\mathrm{ample}}$ satisfy hard Lefschetz.

Let us fix $\gamma \in V$ which satisfies hard Lefschetz. We define the γ-*primitive subspace* (or simply *primitive subspace*) to be

$$P_\gamma^{-i} := \ker(\gamma^{i+1} : H^{-i} \to H^{i+2}) \subset H^{-i}.$$

For any such γ the map induced by the inclusions $P_\gamma^{-i} \hookrightarrow H^{-i}$ gives an isomorphism of $\Lambda[\gamma]$-modules (the *primitive decomposition*):

$$\bigoplus_{i \geq 0} \Lambda[\gamma]/(\gamma^{i+1}) \otimes_\Lambda P_\gamma^{-i} \xrightarrow{\sim} H.$$

Remark 2.1. Consider the Lie algebra $\mathfrak{sl}_2 := \Lambda f \oplus \Lambda h \oplus \Lambda e$ over Λ with

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\gamma \in V$ satisfies hard Lefschetz if and only if there is a (necessarily unique) $\mathfrak{sl}_2$-action on H with $e(x) = \gamma \cdot x$ and $h(x) = jx$ for all $x \in H^j$. With respect to this action, the primitive decomposition is the isotypic decomposition and the primitive subspaces are the lowest weight spaces (i.e. $P_\gamma^{-i} = \ker(f : H^{-i} \to H^{-i-2})$).

For any $\gamma \in V$ and $i \geq 0$ we can consider the symmetric form

$$\begin{aligned} (-,-)_\gamma^{-i} : H^{-i} \times H^{-i} &\to \Lambda \\ (h, h') &\mapsto \langle h, \gamma^i \cdot h' \rangle \end{aligned}$$

It is non-degenerate if and only if γ satisfies hard Lefschetz on H. Let m denote the minimal non-zero degree in H. We say that $\gamma \in V$ *satisfies the Hodge-Riemann bilinear relations* if for all $i \geq 0$ the restriction of $(-,-)_\gamma^i$ to $P_\gamma^{-i} \subset H^{-i}$ is $(-1)^{(-i-m)/2}$-definite.[2] In other words, we require that

- $(-,-)_\gamma^m$ is positive definite on $P_\gamma^m = H^m$,
- $(-,-)_\gamma^{m+2}$ is negative definite on $P_\gamma^{m+2} \subset H^{m+2}$,
- $(-,-)_\gamma^{m+4}$ is positive definite on $P_\gamma^{m+4} \subset H^{m+4}$, etc.

If γ satisfies the Hodge-Riemann bilinear relations then it satisfies hard Lefschetz. We say that Lefschetz data as above *satisfies the Hodge-Riemann bilinear relations* if all $\gamma \in V_{\text{ample}}$ do.

Remark 2.2. The set of $\gamma \in V$ which satisfy hard Lefschetz is Zariski open and stable under multiplication by $\Lambda^\times$. The set of $\gamma \in V$ which satisfy the Hodge-Riemann bilinear relations is an open semi-algebraic set stable under multiplication by $\Lambda_{>0}$.

2.2. The case of a graded algebra. Many examples of Lefschetz data arise from Frobenius algebras. Consider

(1) A positively graded commutative Λ-algebra $A = \bigoplus_{i \geq 0} A^i$ which vanishes in odd degree.

(2) A linear functional ("Frobenius form")

$$\text{tr} : A \to \Lambda(-2m)$$

which does not vanish on any non-zero ideal of A. (Here $\Lambda(-2m)$ denotes the one-dimensional graded vector space concentrated in degree $2m$, for some $m \geq 0$.)

[2] The exponent $(-i-m)/2$ is always an integer if $H^{-i} \neq 0$ by our assumption that H vanishes in even or odd degrees.

(3) An open convex cone $A^2_{\mathrm{ample}} \subset A^2$.

To this we may associate a Lefschetz datum as follows:

(1) We set $H := A(m)$, i.e. $H = \bigoplus H^i$ where $H^i := A^{i+m}$.

(2) We define $\langle a, a' \rangle := \mathrm{tr}(aa')$ (non-degenerate by assumption (2)).

(3) $V_{\mathrm{ample}} := A^2_{\mathrm{ample}} \subset V := A^2$ with action induced by the multiplication in A.

3. Some examples

In this section we give some examples of Lefschetz data.

3.1. Classical Hodge theory. Let X be a connected smooth projective variety (or more generally a compact Kähler manifold) of complex dimension m. Let $A := H^*(X, \mathbb{Q})$ denote its (singular) cohomology ring. It is equipped with the trace $\mathrm{tr} : A^{2m} \to \mathbb{Q}$ given by pairing with the fundamental class of X and the corresponding Poincaré form $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha \cup \beta)$. Let $A_{\mathrm{ample}} \subset A^2 = H^2(X, \mathbb{Q})$ denote the ample cone (the convex hull of all $\mathbb{Q}_{>0}$-multiples of Chern classes of ample line bundles).

Assume that the Hodge decomposition of $H^*(X, \mathbb{C})$ involves only type (p, p).[3] By classical Hodge theory (see e.g. [47]) the corresponding Lefschetz datum (see §2.2) satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.

If X is still projective but no longer smooth we can instead consider its intersection cohomology $H := IH^*(X, \mathbb{Q})$. This is a graded $H^*(X)$-module concentrated in degrees between $-m$ and m, and is equipped with a non-degenerate intersection form $\langle -, - \rangle$. Suppose as before that the Hodge decomposition of $H \otimes_{\mathbb{Q}} \mathbb{C}$ only involves type (p, p). Then the Lefschetz datum $(H, \langle -, - \rangle, A_{\mathrm{ample}})$ satisfies hard Lefschetz and the Hodge-Riemann bilinear relations [4, 12].

In examples coming from algebraic geometry, all Lefschetz data are naturally defined over $\mathbb{Q}$.

3.2. Polytopes. Recall that a polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$. A polytope is simplicial if all of its maximal dimensional faces are simplices. Any n-dimensional polytope determines a collection Δ of polyhedral cones (a "fan") by moving P so that its interior contains the origin and considering the cones spanned by all faces $F \neq P$ of P. To P we may associate $A(P)$, the algebra of functions on $\mathbb{R}^n$ which are piecewise polynomial on all cones in Δ, and its quotient $\Pi(P)$ by the action of polynomials of positive degree on $\mathbb{R}^n$. The algebras $A(P)$ and $\Pi(P)$ are graded by degree. Inside $\Pi(P)^1$ we can consider the convex cone of $\Pi(P)^1_{\mathrm{ample}}$ given by (the image of) strictly convex piecewise linear functions on Δ. If P is simplicial then there exists a trace map $\mathrm{tr} : \Pi(P)^n \to \mathbb{R}$

[3] We ignore here the more general form of the Hodge-Riemann bilinear relations (see e.g. [47]) when Hodge types other than (p, p) are present.

[10]. (In this case $\Pi(P)$ is isomorphic to the *polytope algebra* of McMullen [36], and under this isomorphism tr is given by volume.)

If P is simplicial then (after doubling degrees and applying the recipe of §2.2) the associated Lefschetz data satisfies hard Lefschetz and the Hodge-Riemann bilinear relations. The hard Lefschetz theorem can be used to deduce the necessity of McMullen's conditions on the number of faces of P of each dimension ("face numbers") [35]. The Hodge-Riemann relations imply generalisations of the Aleksandrov-Fenchel inequalities in convex geometry. The hard Lefschetz theorem and Hodge-Riemann bilinear relations were proved by Stanley by "wiggling the vertices" of P so that they become rational numbers, and then identifying $\Pi(P)$ with the cohomology ring of a rationally smooth projective toric variety [45]. A direct proof (entirely within the world of convexity) was given by McMullen [37].

If P is no longer simplicial then Bressler and Lunts [9] described a combinatorial recipe to produce a graded $\Pi(P)$-module $H(P)$, which agrees with the intersection cohomology of a projective (but no longer necessarily rationally smooth) toric variety if the vertices of P are rational numbers (similar results were obtained by Barthel, Brasselet, Fieseler and Kaup [2]). The graded module $H(P)$ is equipped with a symmetric non-degenerate form $\langle -, - \rangle : H(P) \times H(P) \to \mathbb{R}$. It is a theorem of Karu [26] (see also [7]) that the Lefschetz data $(H(P), \langle -, - \rangle, \Pi(P)^1_{\text{ample}})$ satisfies hard Lefschetz and the Hodge-Riemann bilinear relations. The hard Lefschetz theorem is used to deduce the unimodality of Stanley's generalised h-vector, which gives necessary conditions on the face numbers of arbitrary polytopes.

The (initially counterintuitive) fact that there exist polytopes whose combinatorial type is not that of any polytope with rational vertices (see e.g. [50]) means that the trick of "wiggling vertices" cannot be used to deduce Karu's theorem from the hard Lefschetz theorem and Hodge-Riemann relations for algebraic varieties.

Note that, if P has vertices defined over some subfield $\Lambda \subset \mathbb{R}$ then $\Pi(P)$ and $H(P)$ are defined over Λ. Hence non-rational polytopes give many examples of Lefschetz data with no natural $\mathbb{Q}$-structure.

3.3. Coxeter groups. This is the subject of this survey. The theory of Soergel bimodules yields Lefschetz data satisfying hard Lefschetz and the Hodge-Riemann bilinear relations in three distinct ways (which we refer to as *global*, *relative* and *local*). The simplest interesting example of a vector space underlying Lefschetz data that arises from this theory is the *coinvariant algebra* (see Remark 5.7)

$$C := R/\langle R^W_+ \rangle.$$

Here $W \subset GL(V)$ denotes a finite reflection group; R denotes the polynomial functions on V, graded so that $\deg V^* = 2$; R^W_+ denotes the W-invariant polynomial functions of positive degree.

Examples arising from Soergel bimodules are defined over finite extensions of $\mathbb{Q}$ obtained by adjoining algebraic integers of the form $2\cos(\pi/m_{st})$ where $m_{st} \in \mathbb{Z}_{\geq 2}$. Thus there are many examples with no natural $\mathbb{Q}$-structure (and probably no $\mathbb{Q}$-structure at all). Thus it seems unlikely that they arise from complex algebraic varieties in any straightforward way.

3.4. Matroids. Let M be a matroid of rank $r+1$ on a set of cardinality $n+1$. (For example, if $\{v_0, \dots, v_n\} \subset V$ is a spanning set of vectors in an $r+1$-dimensional space then we obtain a matroid which tells us which subsets of $\{v_0, \dots, v_n\}$ are linearly independent, of maximal size and fixed rank, etc. Matroids arising from such arrangements are said to be *realisable* over k.) To M one may associate its *Chow ring* $A^*(M)$, a graded commutative $\mathbb{R}$-algebra. Under mild assumptions ("loopless") on M, the Chow ring is equipped with an isomorphism $\deg : A^r(M) \xrightarrow{\sim} \mathbb{Q}$ and a cone $A^1(M)_{\text{conv}} \subset A^1(M)$ of "strictly convex functions" (see [1] and the references therein).

In [1], it is proved (after doubling degrees and applying the recipe of §2.2) that the associated Lefschetz datum satisfies hard Lefschetz and the Hodge-Riemann bilinear relation. From the Hodge-Riemann relations the authors deduce the log concavity of the absolute value of the coefficients of the characteristic polynomial of M, an old conjecture in matroid theory. This generalises a similar log concavity property for the chromatic polynomial of a graph [23].

The Chow Ring $A^*(M)$ is defined over $\mathbb{Q}$. In [1, §5.4] it is proved that $A^r(M)$ is naturally isomorphic to the Chow ring of a rationally smooth projective variety over k if and only if the matroid is realisable over k. If $k = \mathbb{C}$ then the hard Lefschetz and Hodge-Riemann relations for $A^*(M)$ may be deduced from classical Hodge theory; for general k they are related to Grothendieck's standard conjectures.

Remark 3.1. The reader is referred to [24] for another remarkable example (of quite a different flavour to those discussed above) of combinatorial Hodge theory, this time arising from tropical geometry.

4. The Hecke category

4.1. Coxeter systems. Let (W, S) denote a Coxeter system. That is, W is a group together with a distinguished finite generating set $S \subset W$ of *simple reflections* such that W admits a *Coxeter presentation*

$$W = \langle s \in S \mid (st)^{m_{st}} = \text{id for all } s, t \in S \rangle$$

for certain $m_{st} \in \mathbb{Z}_{\geq 0}$ such that $m_{ss} = 1$ for all $s \in S$ and $m_{st} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for $s \neq t$ ($m_{st} = \infty$ means that we do not impose any relation on st). By definition the *reflections* is the subset $T \subset W$ consisting of the conjugates of S:

$$T := \bigcup_{w \in W} wSw^{-1}.$$

Remark 4.1. Coxeter groups are so called because of Coxeter's theorem that any finite reflection group (i.e. finite subgroup of linear transformations of a real vector space generated by reflections) may be given a Coxeter presentation.

For any $x \in W$ a reduced expression is an expression $\underline{x} = (s_1, \dots, s_m)$ for x in the generating set S (i.e. $x = s_1 \dots s_m$ with all $s_i \in S$) which is of minimal length.

We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function on W (i.e. $\ell(w)$ is the length of a reduced expression for x). Let $\leq$ denote the Bruhat order on W.

Central to the theory of Soergel bimodules is a certain reflection[4] representation of W. To this end consider a finite dimensional real vector space $\mathfrak{h}$ together with subsets $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ of *coroots* and *roots* satisfying the following two conditions:

(1) the subsets $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ are linearly independent;

(2) under the natural pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ we have
$$\langle \alpha_s, \alpha_t^\vee \rangle = -2\cos(\pi/m_{st}). \tag{1}$$

Then the assignment $s \mapsto \phi_s^\vee \in GL(\mathfrak{h})$ (resp. $s \mapsto \phi_s \in GL(\mathfrak{h}^*)$) where
$$\phi_s^\vee(v) := v - \langle \alpha_s, v\rangle \alpha_s^\vee \qquad (\text{resp.} \quad \phi_s(\lambda) := \lambda - \langle \lambda, \alpha_s^\vee\rangle \alpha_s)$$
defines a representation of W on $\mathfrak{h}$ (resp. $\mathfrak{h}^*$).

Remark 4.2. Some remarks are in order:

(1) The above definition mimics that action of the Weyl group of a complex semi-simple Lie algebra. The only difference to the Weyl group case is that we have scaled all roots so as to have the same length.

(2) Central to the study of general Coxeter sytems is a "geometric" representation first defined by Tits. One sets $\mathfrak{h} := \bigoplus_{s\in S} \mathbb{R}\alpha_s^\vee$ and defines $\alpha_s \in \mathfrak{h}^*$ by (1). With this definition $\{\alpha_s\}_s \in \mathfrak{h}^*$ will be linearly independent if and only if W is finite. Thus the representation considered above usually does not agree with Tits' representation. The need to "enlarge" so that both roots and coroots are linearly independent is important in the theory of Kac-Moody Lie algebras [25]. It is also aesthetically pleasing to have $\mathfrak{h}$ and $\mathfrak{h}^*$ play symmetric roles.

By our assumptions above the intersections of half-spaces
$$\mathfrak{h}^+_{\mathrm{reg}} := \bigcap_{s\in S} \{v \in \mathfrak{h} \mid \langle \alpha_s, v\rangle > 0\} \quad \text{and} \quad \mathfrak{h}^{*+}_{\mathrm{reg}} := \bigcap_{s\in S} \{\lambda \in \mathfrak{h} \mid \langle \lambda, \alpha_s^\vee\rangle > 0\}$$
are non-empty. Borrowing terminology from Lie theory we refer to elements in either set as *dominant regular*.

4.2. The Hecke algebra and Kazhdan-Lusztig basis. The Hecke algebra is a deformation of the group algebra of a Coxeter group which plays a fundamental role in Lie theory. It is a free $\mathbb{Z}[v^{\pm 1}]$-algebra H with basis $\{h_x\}_{x\in W}$ and multiplication determined by the rules (for $s \in S$ and $x \in W$)
$$h_s h_x = \begin{cases} h_{sx} & \text{if } sx > x, \\ (v^{-1} - v)h_x + h_{sx} & \text{if } sx < x. \end{cases}$$

[4] i.e. reflections act as reflections

The basis $\{h_x \mid x \in W\}$ is the *standard basis* of H. The algebra H posesses an involution $h \mapsto \overline{h}$ determined by $v \mapsto v^{-1}$ and $h_x \mapsto h_{x^{-1}}^{-1}$.

The algebra H possesses a remarkable basis, discovered by Kazhdan and Lusztig [27]. The Kazhdan-Lusztig basis is the unique basis $\{b_x\}$ for H such that:

(1) $\overline{b_x} = b_x$ ("self-duality");

(2) $b_x \in h_x + \sum_{y<x} v\mathbb{Z}[v]h_y$.

For example $b_s = h_s + vh_{\mathrm{id}}$. If we write

$$b_x = \sum_{y \in W} p_{y,x} h_y$$

the polynomials $p_{y,x}$ are *Kazhdan-Lusztig polynomials.*

The Kazhdan-Lusztig basis appears to satisfy numerous (a priori very mysterious) positivity properties. Here is an incomplete list:

(1) *Positivity of Kazhdan-Lusztig polynomials:*

$$p_{y,x} \in \mathbb{Z}_{\geq 0}[v]. \tag{2}$$

(2) *Positivity of inverse Kazhdan-Lusztig polynomials:* If we write

$$h_x = \sum (-1)^{\ell(x)-\ell(y)} g_{y,x} b_y$$

then

$$g_{y,x} \in \mathbb{Z}_{\geq 0}[v]. \tag{3}$$

(3) *Positivity of structure constants:* If we write

$$b_x b_y = \sum \mu_{x,y}^z b_z$$

then

$$\mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]. \tag{4}$$

(4) *Unimodality of structure constants:* If we set

$$[m] := \frac{v^m - v^{-m}}{v - v^{-1}} = v^{-m+1} + v^{-m+3} + \cdots + v^{m-3} + v^{m-1} \tag{5}$$

and, for all $x, y, z \in W$, write

$$\mu_{x,y}^z = \sum_{m \geq 1} a_{x,y}^{z,m}[m]$$

(this is possible since $\overline{\mu_{x,y}^z} = \mu_{x,y}^z$), then

$$a_{x,y}^{z,m} \in \mathbb{Z}_{\geq 0}. \tag{6}$$

(In other words, $\mu_{x,y}^z$ is the character of a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module.)

Remark 4.3. As the reader has probably already noticed, (6) is strictly stronger than (4). We separate them here because (as we will see) the explanation for (6) lies deeper than that of (4).

One of the main purposes of this note is to state theorems of a Hodge theoretic nature about Soergel bimodules which imply:

Theorem 4.4 ([16] and [19]). *Positivity properties* (2), (3), (4) *and* (6) *hold for any Coxeter system.*

Remark 4.5. That (2) holds for Weyl and affine Weyl groups is due to Kazhdan-Lusztig [28]. That (3) is true for affine and Weyl groups appears to be due to Springer, MacPherson and Brylinsky [46]. Since the construction of Kac-Moody flag varieties for any generalised Cartan matrix it was understood that the arguments of [28, 46] can be used to show that (2) and (3) hold for any "crystallographic" Coxeter group (i.e. $m_{st} \in \{2, 3, 4, 6, \infty\}$ for all $s \neq t$).

Remark 4.6. There are almost no cases so far where there exists combinatorial proofs of the above positivity properties. Notable exceptions include the the case of "Grassmannian" permutations in the symmetric group [31], and the case of "universal" Coxeter systems [14]. Let us repeat Bernstein's opinion on the subject of combinatorial formulas [5]: "In some cases one can get explicit formulas for [Kazhdan-Lusztig polynomials]. For instance, one can calculate intersection cohomology for Schubert varieties on usual Grassmannians (see Lascoux and Schützenberger). But Zelevinsky showed that in this case it is possible to construct small resolutions of singularities. I would say that if you can compute a polynomial P for intersection cohomologies in some case without a computer, then probably there is a small resolution which gives it."

Remark 4.7. The reader is referred to [15] for a list of further positivity properties and to [21] for some recent results.

4.3. Bimodules. Let R denote the regular functions on $\mathfrak{h}$. (After choosing a basis $x_1, \dots, x_n$ for $\mathfrak{h}^*$, R is simply the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$.) Let R-bim denote the category of graded R-bimodules, with morphisms those R-bimodule maps which preserve the grading (i.e. are of "degree zero"). Given bimodules $M, N \in R$-bim we denote their tensor product by juxtaposition:

$$MN := M \otimes_R N \in R\text{-bim}.$$

This gives R-bim the structure of monoidal category with unit R. For $M = \bigoplus_{i \in \mathbb{Z}} M^i \in R$-bim and $m \in \mathbb{Z}$ we denote the shifted bimodule by $M(m)$

$$M(m)^i := M^{m+i}.$$

(So $M(m)$ is the same bimodule with grading shifted "down" by m.) Given bimodules $M, N \in R$-bim we denote by $\mathrm{Hom}^\bullet(M, N)$ the graded R-bimodule consisting of morphisms of all degrees:

$$\mathrm{Hom}^\bullet(M, N) := \bigoplus_{m \in \mathbb{Z}} (M, N(m)).$$

(We emphasise that this is simply notation: it does not refer to the homomorphisms in any category we consider in this paper.)

Let $R\text{-bim}^{fg}$ denote the full subcategory of R-bim consisting of bimodules which are finitely generated both as left and right R-modules. Then $R\text{-bim}^{fg}$ is a monoidal subcategory of R-bim. If $M, N \in R\text{-bim}^{fg}$ then $\mathrm{Hom}(M, N)$ is finite dimensional. It follows that the additive category $R\text{-bim}^{fg}$ is Krull-Schmidt: any object admits a decomposition into indecomposable objects and an object is indecomposable if and only if its endomorphism ring is local; it follows that the Krull-Schmidt theorem holds in $R\text{-bim}^{fg}$.

4.4. Soergel bimodules. Recall that R is the polynomial ring of regular functions on $\mathfrak{h}$. Thus our Coxeter group W acts on R (by functoriality). Given $x \in W$ we denote by R^x the subring of invariants under x. For $s \in S$ we consider the bimodule

$$B_s := R \otimes_{R^s} R(1).$$

It is easy to see that B_s belongs to $R\text{-bim}^{fg}$. We denote by $\mathcal{H}$ the smallest strict, graded, additive, monoidal and Karoubian[5] subcategory of $R\text{-bim}^{fg}$ containing B_s for all $s \in S$. In formulas:

$$\mathcal{H} := \langle B_s \mid s \in S \rangle_{\cong,(\pm 1),\oplus,\otimes,\mathrm{Kar}} \subset R\text{-bim}^{fg}.$$

By the Krull-Schmidt theorem, the indecomposable objects of $\mathcal{H}$ are the indecomposable direct summands of the *Bott-Samelson bimodules*

$$BS(\underline{x}) = B_s B_t \dots B_u(m) \in \mathcal{H}$$

for all expressions $\underline{x} = (s, t, \dots, u)$ in the simple reflections and all $m \in \mathbb{Z}$. Bimodules belonging to $\mathcal{H}$ are called *Soergel bimodules*. We call the category $\mathcal{H}$ the category of Soergel bimodules or the *Hecke category* (see Remark 4.15).

Remark 4.8. The reader is warned that $\mathcal{H}$ is additive but never abelian.

It is a remarkable theorem of Soergel that $\mathcal{H}$ categorifies the Hecke algebra. Let $[\mathcal{H}]$ denote the split Grothendieck group of $\mathcal{H}$: it is the free abelian group generated by symbols $[M]$ for $M \in \mathcal{H}$ modulo the relation $[M] = [M'] + [M'']$ if $M \cong M' \oplus M''$. We view $[\mathcal{H}]$ as an $\mathbb{Z}[v^{\pm 1}]$-algebra via:

$$[M][N] := [M \otimes N'],$$
$$v[M] := [M(1)].$$

Theorem 4.9 ("Soergel's categorification theorem", [43])**.** *There exists an (obviously unique) isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras:*

$$\phi : H \xrightarrow{\sim} [\mathcal{H}] : b_s \mapsto [B_s]. \tag{7}$$

[5] i.e. $\mathcal{H}$ is closed under taking direct summands: if $M \oplus M' = N \in \mathcal{H}$ then $M, M' \in \mathcal{H}$

To establish the existence of ϕ it is enough to verify certain isomorphisms among tensor products of B_s and B_t for pairs $s, t \in S$ (see the the examples below). In the words of Soergel [42]: "This is a bit tricky, but not deep". To show that ϕ is an isomorphism we must control the size of $[\mathcal{H}]$, which comes as a corollary of the following theorem:

Theorem 4.10 ([43])**.** *For each $x \in W$ there exists a unique indecomposable bimodule B_x (well-defined up to isomorphism) which occurs as a summand of the Bott-Samelson bimodule $BS(\underline{x})$ for any reduced expression $\underline{x}$ of x, and does not occur as a summand of $BS(\underline{y})$ for any shorter expression $\underline{y}$. The set*

$$\{B_x(m) \mid x \in W, m \in \mathbb{Z}\}$$

gives a set of representatives for the indecomposable Soergel bimodules up to isomorphism.

It follows that the classes of the bimodules B_x give a basis for $[\mathcal{H}]$:

$$[\mathcal{H}] = \bigoplus_{x \in W} \mathbb{Z}[v^{\pm 1}][B_x]. \tag{8}$$

Once one has established the existence of ϕ, Soergel's categorification theorem is easily deduced from (8).

Example 4.11. (1) Suppose that $S = \{s\}$ so that $W = \mathbb{Z}/2\mathbb{Z}$ acting on $R = \mathbb{R}[\alpha]$ via $s(\alpha) = -\alpha$, and $R^s = \mathbb{R}[\alpha^2]$. In this case one may easily calculate that $b_s = h_s + v$ satisfies

$$b_s b_s = (v + v^{-1}) b_s.$$

In this case Soergel's categorification theorem follows easily from the isomorphism

$$\begin{aligned} B_s B_s &= R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &= B_s(1) \oplus B_s(-1). \end{aligned} \tag{9}$$

where for the middle step we use that $R = R^s \oplus \alpha R^s$ as an R^s-bimodule: "any polynomial can be written as the sum of an even and an odd polynomial". Thus the indecomposable Soergel bimodules are (up to shift) given by

$$\{R := B_{\mathrm{id}}, B_s\}.$$

(2) Suppose that $W = \langle s, t \rangle$ with $m_{st} = 3$ so that W is isomorphic to the symmetric group on three letters. In this case one may check easily that

$$b_s b_t = b_{st}, \quad b_s b_t b_s = b_{sts} + b_s \quad \text{and} \quad b_{sts} b_s = (v + v^{-1}) b_{sts}.$$

(and similarly with s and t interchanged). In this case Soergel's categorification amounts to the isomorphism (9) for B_s and B_t as well as the statements (which can be checked by hand):

$$B_{st} := B_s B_t \quad \text{is indecomposable as an } R\text{-bimodule},$$
$$B_s B_t B_s \cong B_{sts} \oplus B_s \quad \text{where } B_{sts} := R \otimes_{R^{s,t}} R(3),$$
$$B_{sts} B_s \cong B_{sts}(1) \oplus B_{sts}(-1).$$

(and similarly with s and t interchanged). Indeed once one knows (9) and the above two statements then it is easy to show by induction that any indecomposable summand of any tensor product of B_s and B_t is isomorphic to a shift of one of the indecomposable bimodules

$$\{B_{\mathrm{id}} := R, B_s, B_t, B_{st}, B_{ts}, B_{sts}\}.$$

Remark 4.12. The above examples are deceptive: in general there is no explicit description of the indecomposable Soergel bimodules (just as there is no explicit description of the Kazhdan-Lusztig polynomials). Soergel's proof [43] is highly non-constructive. It formally ressembles the proof giving uniqueness of tilting objects in highest weight categories [39, 13].

In [43], Soergel also constructs an explicit inverse to ϕ ("the character of a Soergel bimodule"):

$$\mathrm{ch} : [\mathcal{H}] \xrightarrow{\sim} H.$$

Its definition (which we do not give here) involves taking the graded rank of the subquotients of certain filtrations. It is manifestly positive on bimodules, i.e.

$$\mathrm{ch}(M) := \mathrm{ch}([M]) \in \bigoplus_{x \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] H_x \quad \text{for } M \in \mathcal{H}.$$

Thus $\{\mathrm{ch}(B_x) \mid x \in W\}$ gives a basis for H which is positive (i.e. belongs to $\bigoplus \mathbb{Z}_{\geq 0}[v^{\pm 1}] H_x$) and has structure constants in $\mathbb{Z}_{\geq 0}[v^{\pm 1}]$ (the structure constants give the graded multiplicity of B_z as a summand of $B_x B_y$).

Conjecture 4.13 (Soergel [43])**.** *For all $x \in W$, $\mathrm{ch}(B_x) = b_x$.*

This conjecture is proved in [16] using Hodge theoretic ideas, as we will discuss below. It immediately implies the positivity properties (2) and (4). The method of proof also establishes (3) (see [16, Remark 6.10]). A variant of these methods establishes (6).

Remark 4.14. This theorem was proved for Weyl groups and dihedral groups by Soergel in [41]. It is proved for the Weyl group of any Kac-Moody Lie algebra by Härterich in [22]. Outside dihedral groups, these proofs rely on the decomposition theorem: the bimodule B_x is realised as the equivariant intersection cohomology of a Schubert variety. Soergel's original motivation for studying these bimodules came from attempts to better understand the Kazhdan-Lusztig conjecture [27]; indeed his conjecture implies the Kazhdan-Lusztig conjecture [40].

Remark 4.15. In geometric settings the category $\mathcal{H}$ has other incarnations (as semi-complexes on the flag variety, or as a variant of Harish-Chandra bimodules, ...). In [18], a monoidal category is defined by explicit diagrammatic generators and relations and it is proved that this category is equivalent to $\mathcal{H}$. The upshot is that the category of Soergel bimodules is one incarnation of a more fundamental object, often referred to as the Hecke category. This point of view is particularly useful when one wishes to study variants of the theory over fields of positive characteristic, where the theory of Soergel bimodules becomes unwieldy. The reader is referred to the introduction to [18] for more on this point of view. The Braden-MacPherson and Fiebig theory of sheaves on moment graphs can be seen as another incarnation of the Hecke category [8, 20].

5. Hodge theory of Soergel bimodules

5.1. Global theory. Recall the indecomposable Soergel bimodules B_x introduced in the previous section. In the global theory a central role is played by the corresponding *Soergel modules*, which are obtained by quotienting by the action of positive degree polynomials on the right:

$$\overline{B}_x := B_x \otimes_R \mathbb{R}.$$

Each $\overline{B}_x$ is a graded R-module, which is finite dimensional over $\mathbb{R}$ and vanishes in degree of parity different to that of $\ell(x)$.

Remark 5.1. As already remarked above, in geometric settings B_x may be obtained as the equivariant intersection cohomology of a Schubert variety. In such cases $\overline{B}_x$ is isomorphic to the ordinary (i.e. non-equivariant) intersection cohomology. See [17] for a detailed discussion of Soergel modules and their connection to intersection cohomology.

Any indecomposable Soergel bimodule carries an *intersection form*. This is a graded symmetric bilinear form

$$\langle -, - \rangle : B_x \times B_x \to R$$

which is characterised (up to multiplication by $\mathbb{R}_{>0}$) by the following properties (see [16, Lemmas 3.7 and 3.10]):

(1) For all $b, b' \in B_x$ and $r \in R$ we have:

$$\langle rb, b' \rangle = \langle b, rb' \rangle, \tag{10}$$

$$\langle br, b' \rangle = \langle b, b' \rangle r = \langle b, b'r \rangle. \tag{11}$$

(2) By (11), $\langle -, - \rangle$ descends to a form on the corresonding Soergel module

$$\langle -, - \rangle_{\overline{B}_x} : \overline{B}_x \times \overline{B}_x \to R/R^{>0} = \mathbb{R}$$

such that $\langle rm, m' \rangle = \langle m, rm' \rangle$ for all $r \in R$ and $m, m' \in \overline{B}_x$. This form is non-degenerate.

(3) If $b_{\min} \in B_x$ denotes a non-zero element of degree $-\ell(x)$ (the minimal non-zero degree) then

$$\langle \lambda^{\ell(x)} b_{\min}, b_{\min} \rangle_{B_x} > 0 \tag{12}$$

for any (equivalently every) dominant regular $\lambda \in \mathfrak{h}^*$.

Remark 5.2. The reader may well wonder where the intersection form on B_x comes from. Recall that B_x occurs as a direct summand of $BS(\underline{x})$ for any reduced expression $\underline{x}$. It is not difficult to equip $BS(\underline{x})$ with a symmetric non-degenerate R-valued form (see [16, §3.4]). Somewhat miraculously, this form restricts to yield the desired form on B_x (for any choice of embedding).

Remark 5.3. In geometric settings the intersection form may be identified with the (topological) intersection form on equivariant intersection cohomology.

Thus $(\overline{B}_x, \langle -, - \rangle_{\overline{B}_x}, \mathfrak{h}^{*+}_{\mathrm{reg}} \subset \mathfrak{h}^*)$ gives Lefschetz data (see §2.1). The main theorem of [16] is the following:

Theorem 5.4. *For any $x \in W$ and $\lambda \in \mathfrak{h}^{*+}_{reg}$, the action of λ on $\overline{B}_x$ satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.*

Remark 5.5. This theorem is not new in geometric settings, where it may be deduced from the hard Lefschetz theorem and Hodge-Riemann bilinear relations for intersection cohomology.

Remark 5.6. We will not go into how Theorem 5.4 is related to Soergel's conjecture (Conjecture 4.13). In rough outline the strategy of proof in [16] is to establish Soergel's conjecture and Theorem 5.4 "at the same time" by an induction over the Bruhat order. Let us simply mention that ideas from de Cataldo and Migliorini's proof of the decomposition theorem play a key role [11, 12, 49]. In particular, the definiteness provided by the Hodge-Riemann relations is central to the proof.

Remark 5.7. If W is finite with longest element w_0 then $B_{w_0} = R \otimes_{R^W} R(\ell(w_0))$ and hence

$$\overline{B}_{w_0} = R/\langle R^W_+ \rangle(\ell(w_0)).$$

If W is a finite Weyl group then it is well-known that this algebra is (up to shift) isomorphic to the cohomology ring of the flag variety and Theorem 5.4 follow from classical Hodge theory. However if W is not a Weyl group then there is no known geometric proof of Theorem 5.4 for $x = w_0$. (This case seems like it should be much simpler than the general case.) The surprisingly interesting example of dihedral groups is discussed in detail in [17, §6].

Remark 5.8. For a smooth projective variety X Looijenga-Lunts [32] considered the Lie subalgebra of endomorphisms of $H^*(X, \mathbb{R})$ generated all copies of $\mathfrak{sl}_2(\mathbb{R})_\lambda$ associated to all ample classes $\lambda \in H^2(X, \mathbb{R})$ (see Remark 2.1). They show (using the Hodge-Riemann bilinear relations) that one always obtains a reductive Lie algebra in this way, and compute several examples. By Theorem 5.4 the definition of this Lie algebra also makes sense for any Soergel module. Recently, Patimo [38] has shown that for many $x \in W$ this Lie algebra is the full Lie algebra of symmetries of an orthogonal or symplectic form built from $\langle -, - \rangle_{\overline{B}_x}$.

5.2. Relative theory. We now turn to the relative Hodge theory of Soergel bimodules. In part this theory is motivated by the unimodality property (6) of the structure constants $\mu_{x,y}^z$ of multiplication in the Kazhdan-Lusztig basis. For all $x, y \in W$ we can find an isomorphism

$$B_x B_y \cong \bigoplus_{z \in W} V_{x,y}^z \otimes_{\mathbb{R}} B_z \tag{13}$$

for some graded vector space $V_{x,y}^z$. The structure constant $\mu_{x,y}^z$ is equal to the graded dimension of the vector spaces $V_{x,y}^z$. Recall that the unimodality property (6) is equivalent to the fact that $\mu_{x,y}^z$ is the character of a finite dimensional $\mathfrak{sl}_2$-module. Of course this would be the case if we can establish that $V_{x,y}^z$ is actually an $\mathfrak{sl}_2$-module. This is equivalent to the existence of an operator $L : V_{x,y}^z \to V_{x,y}^z$ of degree 2 which satisfies hard Lefschetz (see Remark 2.1).

The problem is that the decomposition (13) is not canonical, and hence it is difficult to produce endomorphisms of the multiplicity spaces $V_{x,y}^z$. This problem is overcome as follows: any Soergel bimodule carries a canonical increasing *perverse filtration* (see [16, §6.2])

$$\dots \hookrightarrow \tau_{\le i}(B) \hookrightarrow \tau_{i+1}(B) \hookrightarrow \dots$$

such that all maps are split inclusions of Soergel bimodules (i.e. the filtration is non-canonically split). The associated graded (non-canonically isomorphic to B) admits a canonical isotypic decomposition

$$\mathrm{gr}(B) := \bigoplus \tau_{\le i}(B)/\tau_{\le i-1}(B) = \bigoplus_{z \in W} H_z(B) \otimes_{\mathbb{R}} B_z$$

for certain graded vectors spaces $H_z(B)$. Moreover any degree d endomorphism of B induces a degree d endomorphism of each $H_z(B)$.

Applying this construction to the tensor product $B_x B_y$, we may redefine the vector spaces $V_{x,y}^z$ above as follows:

$$V_{x,y}^z := H_z(B_x B_y).$$

These graded vector spaces vanish in degrees of parity different from $\ell(x) + \ell(y) + \ell(z)$ and are equipped with the following structure:

(1) A graded, symmetric, non-degenerate form induced by an "intersection form" on $B_x B_y$ (see [19, §2.2]).

(2) The structure of a graded R-module defined as follows: any $r \in R$ of degree d gives a morphism

$$B_x B_y \to B_x B_y(d) : b \otimes b' \mapsto br \otimes b' = b \otimes rb' \tag{14}$$

and hence induces a degree d endomorphism of $V_{x,y}^z$.

The main theorem of [19] is the following:

Theorem 5.9. *For all $x, y, z \in W$ and $\lambda \in \mathfrak{h}^{*+}_{reg}$ the action of λ on $V^z_{x,y}$ satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.*

By the discussion above, this theorem immediately implies the unimodality property (6) of the structure constants of the Kazhdan-Lusztig basis.

Remark 5.10. In geometric settings Theorem 5.9 can be deduced from the relative hard Lefschetz theorem [4] and the relative Hodge-Riemann bilinear relations [12].

Remark 5.11. Suppose that W is finite with longest element w_0. Then one has a canonical isomorphism

$$\overline{B}_x \cong V^{w_0}_{x,w_0}$$

compatible with forms and the R-module structure. (This is a categorification of the fact that if χ denotes the "trivial" character of H, i.e. $H_s \mapsto v^{-1}$ for all $s \in S$, then we have $b_x b_{w_0} = \chi(b_x) b_{w_0}$ for all $x \in W$.) In this case Theorem 5.4 is a special case of Theorem 5.9.

Remark 5.12. Theorem 5.9 can be used to prove that certain tensor categories associated by Lusztig (see [33, 34]) to any two-sided cell in W are rigid [19].

5.3. Local theory. We finish our discussion with a brief overview of the local theory, contained in [48]. Here the motivation cannot be given strictly in terms of positivity properties in the Hecke algebra (although a better understanding of the local route might yield an alternative proof of Soergel's conjecture, see [48, §7.5]).

Remark 5.13. In geometric settings, the global and relative theory discussed above have "easy" translations into statements about intersection cohomology (see Remarks 4.15 and 5.1). In the local setting the translation (via the "fundamental example" of Bernstein-Lunts [6]) is available, but is more complicated. We will not comment on this further below and instead refer the reader to the introduction of [48], where the connection is discusssed in detail.

For a Soergel bimodule B and $y \in W$, let $\Gamma_y(B)$ (resp. $\Gamma^y(B)$) denote the largest submodule (resp. quotient) on which one has the relation $m \cdot r = y(r) \cdot m$ for all $r \in R$. (Thus, for example, $\Gamma_{\mathrm{id}}(B)$ and $\Gamma^{\mathrm{id}}(B)$ are the Hochschild cohomology and homology of the bimodule B.)

Both $\Gamma_y(B)$ and $\Gamma^y(B)$ are free graded R-modules. The evident morphism

$$i_y : \Gamma_y(B) \to \Gamma^y(B)$$

is injective and becomes an isomorphism if we tensor with $R[1/\alpha_s]_{s \in S}$.

Any $\mu^\vee \in \mathfrak{h}$ determines a specialisation $R \to \mathbb{R}[z]$ ("restriction to the line $\mathbb{R}\mu^\vee \subset \mathfrak{h}$") and if $\langle \mu^\vee, \alpha_s \rangle \neq 0$ for all $s \in S$ then $\mathbb{R}[z] \otimes_R i_y$ is an inclusion of graded $\mathbb{R}[z]$-modules. For any $x \in W$ we define an $\mathbb{R}[z]$-module $H^\mu_{y,x}$ via the short exact sequence

$$0 \to \mathbb{R}[z] \otimes_R \Gamma_y(B_x) \to \mathbb{R}[z] \otimes_R \Gamma^y(B_x) \to H^\mu_{y,x}(1) \to 0.$$

If we write $v^{\ell(x)-\ell(y)}p_{y,x} = \sum_{i\geq 1} a^i_{y,x}v^i$ then the graded rank of $H^\mu_{y,x}$ is

$$\sum_{i\geq 1} a^i_{y,x}[i]$$

(see (5) for the definition of $[i]$). In particular it vanishes in degrees of the same parity as $\ell(x)$. The intersection form B_x induces a symmetric non-degenerate form on $H^\mu_{y,x}$. The main theorem of [48] is the following:

Theorem 5.14. *For any $x < y \in W$ and any $\mu^\vee \in \mathfrak{h}^+_{reg}$, multiplication by z on $H^z_{y,x}$ satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.*

Remark 5.15. The main motivation for establishing Theorem 5.14 is that (by work of Soergel [44] and Kübel [29, 30]) it gives an algebraic proof of the Jantzen conjectures on the Jantzen filtration on Verma modules for complex semi-simple Lie algebras. The first proof of the Jantzen conjectures was given by Beilinson and Bernstein [3].

Remark 5.16. The statements and proofs in the local case are less intuitive than in the previous two settings (global and relative). This might be because we do not yet have a good framework for discussing the hard Lefschetz theorem and Hodge-Riemann bilinear relations in equivariant cohomology.

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