① Light leaves morphisms, cellular structure

later

$\underline{x}$ expression $= s_1 s_2 \dots s_m$.

$\underline{e}$ subexpression ($e_i \in \{0,1\}$).

$$x_0 = id,\ x_1 = s_1^{e_1},\ x_2 = s_1^{e_1} s_2^{e_2},\ \dots,\ x_m = \underline{x}^{\underline{e}}.$$

Decorate sequences:

$x_{i-1} s_i > x_{i-1} \rightsquigarrow U e_i$

$x_{i-1} s_i < x_{i-1} \rightsquigarrow D e_i$.

Defect $= d(\underline{e}) =$ # U0's − # D0's.

$$\underline{H}_{\underline{w}} = \sum_{\underline{e} \text{ subexp}} v^{d(\underline{e})} H_{\underline{w}^{\underline{e}}}.$$

Today we will categorify this formula...

Reminders on generators and relations:

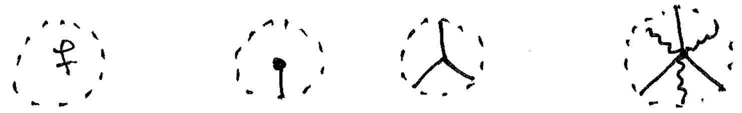
Fix a realisation $\mathfrak{h}, \{\alpha_s\}, \{\alpha_s^\vee\}$ of (W,S).
↑ set of "colours".

$\mathfrak{h}/\Bbbk$ only assumption:
$\alpha_s : \mathfrak{h}^* \to \Bbbk$
$\alpha_s^\vee : \mathfrak{h}^* \to \Bbbk$ are surjective for all $s \in S$.
Automatic if char $\Bbbk \neq 2$.

$\mathcal{D}$ monoidal category defined as follows:

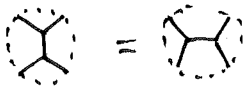
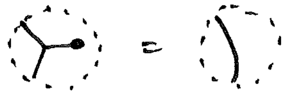
1) objects are isotopy classes of finitely many S-coloured points on $\mathbb{R}$.

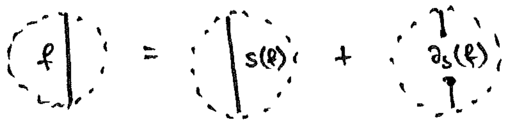
2) morphisms are $R = S(\mathfrak{h}^*)$ - linear combinations of isotopy classes of diagrams with generators:

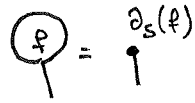


$f \in R$ "dot" "tri" "$2m_{st}$-valent vertex"

subject to the relations:

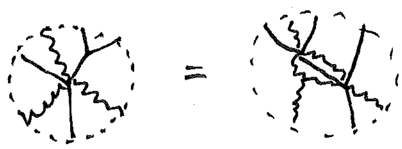
② Frob:  

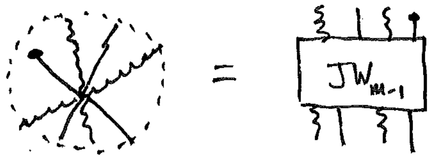
Poly:  $= 0$ 

$\Rightarrow$ 

2-colour:

(Assoc_2): $m=2$ 

$m=3$ 

(JW): 

3-colour: Zamolodchikov relations.

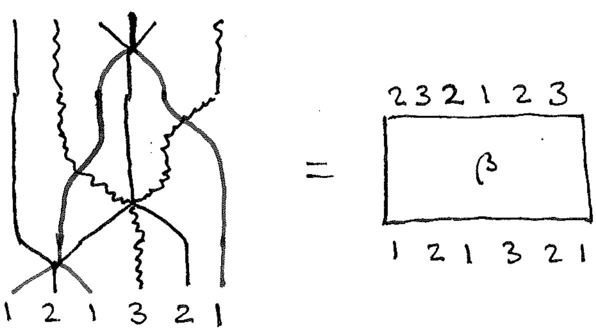
<u>Light leaves:</u>

~~Fix the following data:~~

Given two expressions $\underline{x}_1$ and $\underline{x}_2$ we write $\underline{x}_1 \xrightarrow{\beta} \underline{x}_2$ to mean that β a <u>braid move</u>, i.e. a sequence of braid relations transforming $\underline{x}_1$ to $\underline{x}_2$.

<u>Example:</u> $\underline{x}_1 = 121321 \to 212321 \to 213231 \to 231231 \to 231213 \to 232123 = \underline{x}_2$.

Given β as above we can also write it as a morphism in $\mathcal{D}$:



③

We now fix the following data arbitrarily:

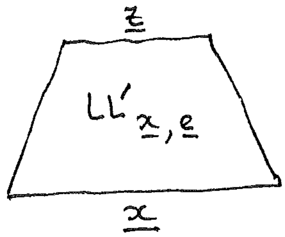
1) for all z a rex $\underline{z}_{fix}$ for z;

2) for any rex $\underline{z}$ for z a braid move $\underline{z} \xrightarrow{\beta_{fix}} \underline{z}_{fix}$.

3) for all pairs (z, s) with $s \in R(z) = \{s \in S \mid zs < z\}$ a rex $\underline{z}_{s\,\cancel{fix}}$ ending in s;

4) for all pairs $(\underline{z}, s)$ where $\underline{z}$ is a rex for z and $z \in R(z)$ a braid move $\underline{z} \xrightarrow{\beta_{\underline{z},s}} \underline{z}_s$.

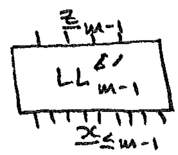
Now given any expression $\underline{x}$ and subexpression $\underline{e}$ we will construct a morphism

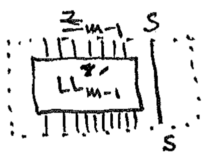


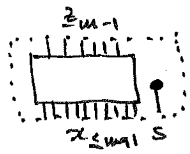
where $\underline{z}$ is a rex for $\underline{x}^{\underline{e}}$.

Construction is inductive. Suppose we have calculated $LL^{\mathcal{E}}_{m-1} := LL'_{\underline{x}_{\le m-1}, \underline{e}_{\le m-1}}$.

Assume $x_m = s$. There are four possibilities for e_m:



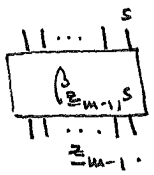
U1: $LL_m :=$ 

U0: $LL_m :=$ 

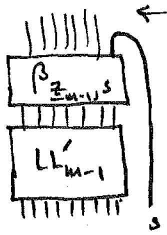
$\deg LL_m = \deg LL_{m-1}$

$\deg LL_m = \deg LL_{m-1} + 1$.

D: In this case $z_{m-1}s < z_{m-1} \Rightarrow \underline{z}_{m-1} \xrightarrow{\beta} (\underline{z}_{m-1})s$



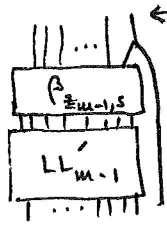
D1: $LL_m :=$



← rex for $z_{m-1}s$.

$\deg LL_m = \deg LL_{m-1}$

D0:

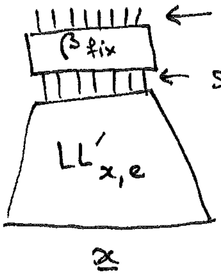


← $(\underline{z}_{m-1})s$

$\deg LL_m = \deg LL_{m-1} - 1.$

Finally, we set

$LL_{\underline{x},e} :=$

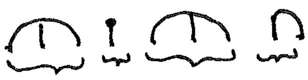


← ~~$(\underline{x}^{e})$~~ ← fix rex for $\underline{x}^{e}$.

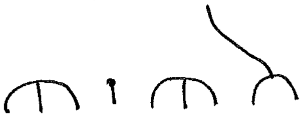
← some rex for $\underline{x}^{e}$.

Rmk: $\deg LL_{\underline{x},e} = d(e)$.

Example: 1) let $\underline{x} = ss\ldots s$ (m times). Show that all light leaves maps are of the form:

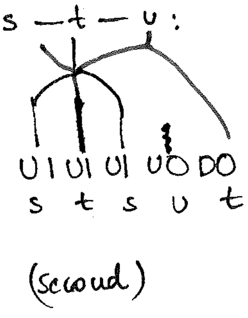


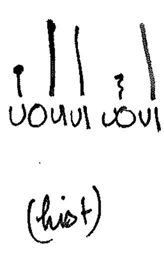
if $\underline{x}^{e} = id$.



if $\underline{x}^{e} = s$.

2) $s - t - u$: $\quad \underline{x} = stsut \quad$ subexpressions for $tst = sts$.





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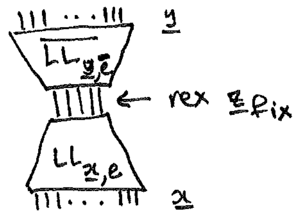
Double leaves: Given $\underline{x}, \underline{e}$ as above write

$\overline{LL_{\underline{x},\underline{e}}}$ for the vertical flip of $LL_{\underline{x},\underline{e}}$.

Write: $M(\underline{x}, \overset{z}{\cancel{\underline{e}}})$ for all subexpressions $\underline{e}$ of $\underline{x}$ with $\underline{x}^{\underline{e}} = z$.

Given $\underline{e} \in M(\underline{x}, z)$, $\underline{f} \in M(\underline{y}, z)$ we can form

$\mathbb{LL}_{\underline{e}, z, \underline{f}} :=$ 

$\in \mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$.

Double leaves thm: (Libedinsky for SBim, EW for $\mathcal{D}$).

$$\{ \mathbb{LL}_{\underline{e}, z, \underline{f}} \mid z \in W,\ \underline{e} \in M(\underline{x}, z),\ \underline{f} \in M(\underline{y}, z) \}$$

is ~~an~~ (left) R-basis for ~~SBim~~ $\mathrm{Hom}(\underline{x}, \underline{y})$.

Corollary: $\mathrm{Hom}(\underline{x}, \emptyset)$ is free with basis $\{ LL_{\underline{x},e} \mid e \in \underline{M}(\underline{x}, id) \}$.

Corollary: Since r maps a basis to a basis

$r : \mathcal{D} \xrightarrow{\sim} BSBim$

Hence $\mathcal{D}_{\oplus, Kar} \xrightarrow{\sim} SBim$.

Proof: ~~Soergel/Libedinsky shows that~~

From now on: $\boxed{\hat{\mathcal{D}} := \mathcal{D}_{\oplus, (m), Kar}.}$

Up until now we viewed $\mathcal{D}$ as a $\Bbbk$-linear cat enriched in graded R-bimodules.

From now on we view $\hat{\mathcal{D}}$ as a $\Bbbk$-linear cat with only degree zero morphisms.

Hence:

$$\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y}) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\hat{\mathcal{D}}}(\underline{x}, \underline{y}(m)).$$

Recall: $\mathcal{A}$ additive $\Bbbk$-linear. ~~[illegible]~~

$X \in \mathcal{A}$ is indec. if $X \cong Y \oplus Z \Rightarrow Y$ or $Z = 0$.

$\mathcal{A}$ is Krull-Remak-Schmidt if

1) any X is iso. to a fin. $\oplus$ of indecs.
2) $X \in \mathcal{A}$ is indec $\Leftrightarrow$ $\mathrm{End}(X)$ is local (i.e. $f \in \mathrm{End}(X)$ $\Downarrow$ f or $1-f$ is invertible).

Assume $\Bbbk$ is complete local Noetherian.

Thm (EW) 1) $\mathcal{D}$ is KRS.

2) for all $x \in W$ there exists a unique indecomposable $B_x \overset{\oplus}{\subset} \underline{x}$ which does not occur as a summand of $\underline{y}$ for any $y < x$, moreover B_x does not depend (up to isomorphism) on $\underline{x}$.

3) one has a bijection

$$\left\{ \begin{matrix} \text{indec. objects in} \\ \hat{\mathcal{D}} \end{matrix} \right\} \Big/ \begin{matrix} \text{shifts} \\ \text{and isomorphism} \end{matrix} \xleftarrow{\sim} W$$

$$B_x \longleftarrow\!\shortmid \; x$$

4) There exists a unique isomorphism of algebras

$$[\mathcal{D}] \cong \mathcal{H},$$

fixed by $[B_s] \mapsto \underline{H}_s$, with explicit inverse $\mathrm{ch} : [\mathcal{D}] \to \mathcal{H}$.

Rmk: The second half of the course will be devoted to proving

Thm (EW): if ~~char~~ $\Bbbk = \mathbb{R}$ and $\mathfrak{h}$ satisfies a positivity property, then

Soergel's conjecture: $$ch(B_x) = \underline{H}_x.$$

It is not difficult to see that $ch(B_x)$ only depends on the char of $\Bbbk$ if $\Bbbk$ is a field, or the char of res. field in general.

$$\rightsquigarrow \quad {}^p\underline{H}_x = ch(B_{x,\mathbb{F}_p}) \qquad \text{"p-canonical basis".}$$

Calculating ${}^p\underline{H}_x$ should be considered one of the big challenges for modern rep. theory. (IMHO).

Options for the rest of the course:

1) Hodge theory + positivity;
2) p-canonical bases, intersection forms.