Tuesday 15/10

Example of a diagrammatic presentation

$\mathrm{Rep}_f\, \mathfrak{sl}_2(\mathbb{C})$ = $\otimes$-cat of f.d. $\mathfrak{sl}_2$-reps. V natural representation.

$\cup$

$\langle V \rangle_\otimes$ = full subcategory with objects $V^{\otimes m}$, $m \geq 0$

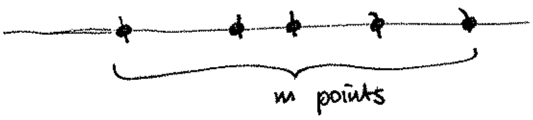
Then $\mathrm{Rep}_f\, \mathfrak{sl}_2(\mathbb{C})$ is the additive, ~~Karoubian~~ Karoubian envelope of $\langle V \rangle_{\otimes}$.

Note: In $\langle V \rangle_\otimes$ objects admit an easy combinatorial description $\leftrightarrow \mathbb{Z}_{\geq 0}$.

$\boxed{\text{This is not the case for } \mathrm{Rep}_f\, \mathfrak{sl}_2(\mathbb{C})}$

Consider the following diagrammatic category: TL

Objects: isotopy classes of finitely many ~~dots~~ points on a line



shorthand: m.

Morphisms: $\mathbb{C}$-linear combinations of isotopy classes of non-intersecting
$a \to b$ 1-manifolds in $\mathbb{R} \times [0,1]$ with bottom (resp. top) boundary a (resp. b).

Eg: 

Relations: $\bigcirc = -2$.

Monoidal category with tensor product given by horizontal concatenation.

~~Claim~~ Proposition: $TL \cong \langle V \rangle_\otimes$. Hence $(TL)_{\oplus,\text{Karoubian}} \cong \mathrm{Rep}_f\, \mathfrak{sl}_2(\mathbb{C})$.

Proof: 1) There exists a functor $r : TL \to \langle V \rangle_\otimes$.

Set $r(\underbrace{\text{-}\circ\circ\circ\text{-}}_{m \text{ dots}}) = V^{\otimes m}$. Morphisms in TL are generated by

 and . Send  $\mapsto \varepsilon : \begin{array}{c}\mathbb{C}\\ \uparrow\\ V\otimes V\end{array} = \begin{array}{c}\mathbb{C}\\ \uparrow \mathrm{Tr}\\ \mathrm{End}(V)\end{array}$ $\quad \Delta :$  $\begin{array}{c}V\otimes V = \mathrm{End}(V)\\ \uparrow \text{identity}\\ \mathbb{C}\end{array}$ $e_1 \otimes e_2 + e_2 \otimes e_1$.

(Some funny business with signs here I don't completely understand!)

Claim: ε, Δ are ~~units~~ counits and units in a ~~do~~ biadjunction (V,V).

Check zig-zag relations by hand.

$\Rightarrow$ isotopy classes give well-defined morphisms.

$\mathrm{Tr}(\mathrm{id}) = -\dim V = -2 \quad \Rightarrow \quad \bigcirc = -2$ holds. 1) $\square$

Step 2: r is an equivalence.

$(\bullet,\bullet)$ biadjoint, (V,V) biadjoint: r ~~maps~~ matches units and counits

$\Rightarrow$ enough to check that $r : \mathrm{Hom}_{TL}(\emptyset, \underset{m \text{ dots}}{\cdots}) \xrightarrow{\sim} \mathrm{Hom}_{Rep}(\mathbb{C}, V^{\otimes m})$.

Exercise: 1) $\dim \mathrm{Hom}_{TL}(-, \underset{2m \text{ dots}}{\cdots}) = \#$ crossingless matchings of $2m$ dots $= \frac{1}{m+1}\binom{2m}{m}$

$\quad \| $

mult. of $\mathbb{C}$ inside $V^{\otimes m}$. $\square$

2) r is injective.

Rmks: a) establishing relations was (relatively) straightforward, whereas getting that we had an equivalence was much harder (think about presenting groups).

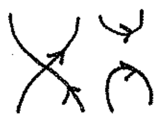
b) semi-simplicity of category $\Rightarrow$ non-degeneracy of the form: quite non-trivial. $\boxed{\text{NOW: JW proj.}}$

Generalizations: 1) $U_q(\mathfrak{sl}_2)$, $\bigcirc = \pm[2]_q$. Also works at a root of unity.

2) $GL_n(\overset{V}{\mathbb{C}}) = \langle V, V^* \rangle_{\otimes, \oplus, \text{Karoubian}}$.

$\boxed{\text{Discuss the fact that TL and } GL_\delta \text{ are almost ~~free~~ "free constructions".}}$

$GL_\delta :=$ objects $\uparrow \downarrow$

morphisms:  (first fundamental thm).

rels: 

Then: if $\delta \notin \mathbb{Z}_{\geq 0}$, GL_δ is semi-simple.

Partition algebra $\rightsquigarrow$ Rep S_δ for $\delta \in \mathbb{C}$.

$\mathrm{Rep}\, GL(V) \cong (GL_{\dim V}) / \mathrm{rad}$.

... back to $\mathfrak{sl}_2(\mathbb{C})$.

<u>Jones-Wenzl projectors:</u> For $m \geq 0$ let V_m denote the simple h.w. module for $\mathfrak{sl}_2(\mathbb{C})$ with h.w. m.

Then $V_m \subset V^{\otimes m}$ with multiplicity 1.

The Jones-Wenzl projector is the idempotent $JW_m \in \mathrm{End}_{TL}(\underbrace{|||||}_{m})$ which becomes the projector to V_m under the equivalence $\langle TL \rangle_{\oplus, K} \xrightarrow{\sim} \mathrm{Rep}\, \mathfrak{sl}_2(\mathbb{C})$.

<u>Eg:</u> $JW_1 = |$

$JW_2 = || + \frac{1}{2} \underset{\cap}{\cup}$

$JW_3 = ||| + \frac{1}{3} \cdots + \frac{1}{3} \cdots + \frac{2}{3} | \underset{\cap}{\cup} + \frac{2}{3} \underset{\cap}{\cup} |$.

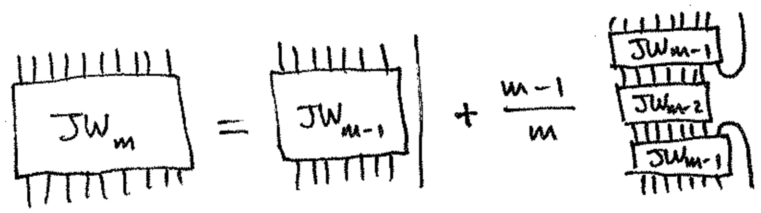
<u>Exercise:</u> 1) ~~the space of maps~~

the subspace of $\operatorname{Hom}_{TL}(\bullet \cdots \bullet)$ "killed by all caps"

(i.e. $\{ f \in \operatorname{Hom}_{TL}(\underbrace{\bullet \cdots \bullet}_{m}) \mid \text{cap}_{i,i+1} \circ f = 0 \ \forall\, 1 \leq i, i+1 \leq m \}$)

is one-dimensional, and contains JW_m (if it exists).

2) One has the recursive formula:

(*)  , $m \geq 2$.

3) Why can (*) be seen as a categorification of

$$V_{m-1} \otimes V \cong V_m \oplus V_{m-2} \ ? \qquad m \geq 2.$$

Now to Soergel bimodules:

$W \curvearrowright \mathfrak{h} \;/\; \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$R = S(\mathfrak{h}^*)$ graded with $\deg \mathfrak{h}^* = 2$.

$W \curvearrowright R$, $\quad R^s$ invariants.

$B_s := R \otimes_{R^s} R(1)$

$\mathbb{S}Bim$ = full $(\otimes, \oplus, (m))$ Karoubian subcat of R-Bim generated by $\{B_s \mid s \in S\}$.

The presentation will follow a similar pattern:

$BSBim$ = full subcat of R-Bim with objects

$BS(\underline{w}) = B_{s_1} \dots B_{s_m}$ where $\underline{w} = s_1 \dots s_m$.

By def^n: additive Karoubian envelope of $BSBim \cong SBim$.

We will present $BSBim$ by generators and relations.

~~Rank 1: Fix $s \in S$.~~

Difference: Hom spaces in R-Bim are graded left and right R-modules. Hence we will have "polynomials everywhere", and all hom spaces will be graded.

Rank 1: Fix $s \in S$: we define a diagrammatic category Diag_s:

objects: isotopy classes of finitely many dots on a line.

morphisms: R-linear combinations of isotopy classes of "Soergel graphs with polynomials": generators

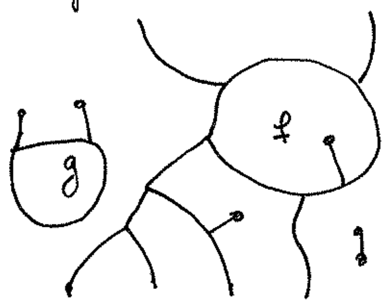
$\rightarrow$ polynomials in regions  $\deg = \deg f$

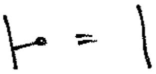
$\rightarrow$ trivalent vertices  $\deg = -1$

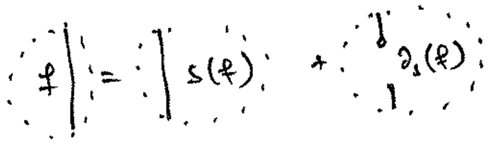
$\rightarrow$ dots  $\deg = +1$.

degree of a diagram = sum of degrees of components.

Eg:



modulo relations: "associativity"  "∘(co)unit" 

 $= \alpha_s$  $= \partial_s(f)$ 

Last time I explained how consideration of the Frobenius extension

$$R^s \underset{\partial_s}{\overset{i}{\rightleftarrows}} R$$

implies that we have a functor

$$r : \mathrm{Diag}_s \longrightarrow \langle B_s \rangle_{\otimes} \subset R\text{-Bim}$$

sending:

 $\longmapsto$ $\begin{array}{c} B_s = R \otimes_{R^s} R(1) \\ \uparrow \\ B_sB_s = R \otimes_{R^s} R \otimes_{R^s} R(2) \end{array}$ $\begin{array}{c} f \otimes \partial_s(g)h = f\partial_s(g) \otimes h. \\ \uparrow \\ f \otimes g \otimes h \end{array}$

 $\longmapsto$ $\begin{array}{c} R \\ \uparrow \\ B_s \end{array}$ $\begin{array}{c} fg \\ \uparrow \\ f \otimes g \end{array}$

 $\longmapsto$ $\begin{array}{c} f \otimes 1 \otimes g \\ \uparrow \\ f \otimes g \end{array}$ $\begin{array}{c} B_sB_s \\ \uparrow \\ B_s \end{array}$

 $\longmapsto$ $\begin{array}{c} B_s \\ \uparrow \\ R \end{array}$ $\begin{array}{c} \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \\ \uparrow \\ 1 \end{array}$

and that all the relations above are satisfied.

<u>Thm</u> (Libedinsky, Elias-Khovanov) r is an equivalence, hence $SBim_s \cong \langle Diag_s \rangle_{\oplus, Kar}$.

both did much more

<u>Idea of proof:</u>

<u>Step 0:</u> As in $\mathfrak{sl}_2$-case, use adjunctions to ~~compare~~ reduce to comparing

$$\mathrm{Hom}_{Digg}(\emptyset, \underbrace{\bullet\bullet\cdots\bullet}_{m \text{ dots}}) \quad \text{and} \quad \mathrm{Hom}_{SBim}(R, B_s^{\otimes m}).$$

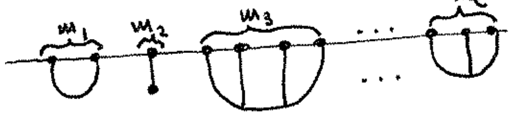
<u>Step 1:</u> $\mathrm{Hom}_{SBim}(R, B_s)$ is a free R-module generated by $1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

Last time: $B_s B_s \cong B_s(1) \oplus B_s(-1)$.

Hence: $\mathrm{Hom}_{SBim}(R, B_s^{\otimes m})$ is free as a left-R-module of rank $v(v+v^{-1})^{m-1}$.

<u>Step 2:</u> (Good exercise to get used to the diagrammatics)

Given a composition $\underline{m} = (m = m_1 + \ldots + m_\ell)$ consider

$LL_{\underline{m}} =$  . $\deg = \sum_{i=1}^{\ell} (2 - m_i) = 2\ell - m.$

~~Claim~~ <u>Exercise:</u> $\mathrm{Hom}_{Diag_s}(\emptyset, \bullet\bullet\cdots\bullet)$ is spanned (~~over~~ as a left R-module) by $\{LL_{\underline{m}}\}_{\underline{m} = (m_1 + \ldots + m_\ell = m)}$.

<u>Step 3:</u> Using localization (next time) it is not difficult to see that the images of $\{LL_{\underline{m}}\}$ are independent over R. Hence $\{LL_{\underline{m}}\}$ are a basis for $\mathrm{Hom}(\emptyset, \underbrace{\bullet\bullet\bullet\bullet\bullet}_{m \text{ dots}})$ over R.

<u>Step 4:</u> $\underline{m} = (m = m_1 + \ldots + m_\ell) \longleftrightarrow$ ~~pos~~ subset $X \subset \{1, \ldots, m-1\}$.

$$\deg LL_{\underline{m}} = 2\ell - m = 2(|X|+1) - m = 2|X| + 2 - m = 2|X| + 1 - (m-1).$$

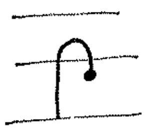
Hence $\sum_{\underline{m}} v^{\deg LL_{\underline{m}}} = \sum_{X \subset \{1,\ldots,m-1\}} v^{2|X|+1-(m-1)} = v(v+v^{-1})^{m-1}$.

Hence r induces an iso $\mathrm{Hom}(\emptyset, \bullet\cdots\bullet) \xrightarrow{\sim} \mathrm{Hom}(R, B_s^{\otimes m})$.

Singular Soergel bimodules:

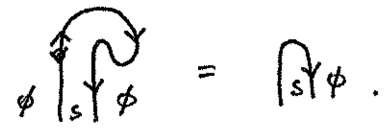
We have already seen that "complicated" relations among rank 1 Soergel bimodules become "simple" when we interpret B_s as a composition.

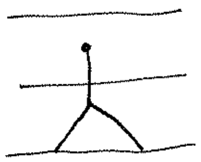
Eg:



$$B_s \to B_sB_s \to R$$

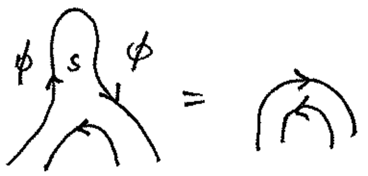
$\Leftarrow$





$$B_sB_s \to B_s \to R$$

$\Leftarrow$



We will see more evidence for this in rank 2. To do this we must introduce "singular Soergel bimodules".

Schur algebroid: (W, S) Coxeter system, $\mathcal{H}$ its Hecke algebra

$I \subset S$ finitary (notation $I \overset{f}{\subset} S$) if W_I finite.

Given $I \overset{f}{\subset} S$ set:

$w_I :=$ longest elt. in W_I

$$\pi(I) := v^{-\ell(w_I)} \sum_{x \in W_I} v^{2(\ell(w_I) - \ell(x))} H_x \qquad \text{"Poincaré polynomial"}$$

$$\underline{H}_I := \underline{H}_{w_I} = \sum_{w \in W_I} v^{\ell(w_I) - \ell(w)} H_w . \qquad \underline{H}_I^2 = \pi(I)\underline{H}_I .$$

$\mathcal{H}_I =$ Hecke algebra of $\mathbb{E}(W_I, I)$.

The Schur algebroid is the category with:

1) objects $I \overset{f}{\subset} S$;
2) $\mathrm{Hom}(I, J) = \underline{H}_I \mathcal{H} \cap \mathcal{H} \underline{H}_J$;
3) composition $\mathrm{Hom}(I, J) \times \mathrm{Hom}(J, K) \to \mathrm{Hom}(I, K)$

$$(f, g) \longmapsto \frac{1}{\pi(J)} fg =: f \underset{J}{*} g .$$

(makes sense because $f = f'\underline{H}_J$, $g = \underline{H}_J g'$, $fg = f'\underline{H}_J^2 g' = \pi(J) f' \underline{H}_J g$.

Equivalently one can view the Schur algebroid as a "ring with several objects"

$$1 = \sum_{I \subseteq S} e_I .$$

In general, Schur algebroid $= \mathrm{End}\Big(\bigoplus_{I \subseteq S} \cancel{\mathrm{End}}\big(\mathrm{triv} \otimes_{\mathcal{H}_I} \mathcal{H}\big)\Big)$.

Hence in type A we recover the Schur algebra.

Don't forget: $\mathrm{End}(\emptyset) = \mathcal{H}$
$\mathrm{Hom}(\emptyset, I) = \mathcal{H} \otimes_{\mathcal{H}_I} \mathrm{triv}_q$
as a left $\mathrm{End}(\emptyset) = \mathcal{H}$ module.

$\boxed{\text{LATER: } \mathrm{End}(\emptyset) = \mathrm{SBim},\ \mathrm{Hom}(\emptyset, I) \subset (R \otimes R^I)\text{-Mod}.}$

Consider the two category with;

1) objects consisting of finitary subsets $I \subset S$;
2) 1-morphisms given by graded (R^I, R^J)-bimodules, horizontal composition given by tensor product;
3) 2-morphisms bimodule homomorphisms.

SSBim "singular Soergel bimodules" is defined to be the full ~~additive~~ $(\oplus, (m))$ Karoubian 2-cat generated by the induction and restriction bimodules $R^I \subset (R^I, R^J)$-Bim $\forall J \supset I$ finitary.

Hence the indecomposable ~~objects~~ 1-morphisms in SSBim are the direct summands of bimodules of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{m-1}}} R^{I_m} \in (R^{J_0}, R^{J_m})\text{-Bim}$$

for finitary subsets $J_0 \supset I_1 \subset J_2 \supset I_2 \subset \ldots \supset I_m \subset J_m$.

Fact: If $\mathfrak{g}$ is reasonable (eg. if $\mathfrak{g}$ ~~co~~ is "symmetrizable")

then each extension $R^J \subset R^I$ for $J \supset I$ is a Frobenius extension with trace $\partial_I^J := \partial_{s_1} \cdots \partial_{s_m}$ with $s_1 \ldots s_m$ a reduced expression for $w_J w_I$.

Hence we have well-defined morphisms