Proof of hard Lefschetz: Last time we saw the importance of the statement:

$$HR(x,s)_\zeta := \text{the operator of multiplication by } L_\zeta,\ 0 \leq \zeta \in \mathbb{R} \text{ on } \overline{B_x B_s} \text{ satisfies the hard Lefschetz thm.}$$

Last time we assumed $xs > x$. However:

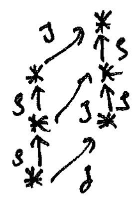
Thm: Suppose ~~HR~~ $xs < x$. Then $HR(x,s)_\zeta$ holds $\Leftrightarrow$ $\zeta > 0$ and ρ satisfies hL on $\overline{B_x}$.

(Sketch) Proof: It is not difficult to decompose $B_x B_s \cong B_x(1) \oplus B_x(-1)$.

If $\zeta = 0$ then L_ζ preserves this decomposition:

$\rho \circlearrowright \overline{B_x} =$  $\rightsquigarrow \rho \circlearrowright \overline{B_x B_s}$  $\rightsquigarrow$ hL fails.

If $\zeta \neq 0$ then L_ζ looks like a tensor product action:

$L_\zeta \circlearrowright \overline{B_x B_s}$  $\rightsquigarrow$ hL holds. $\square$

So we have reduced everything to:

Thm: If $xs > x$ and $\zeta \geq 0$ then $L_\zeta \circlearrowright \overline{B_x B_s}$ satisfies hard Lefschetz.

The proofs for $\zeta > 0$ and $\zeta \ngtr 0$ are different but follows a similar pattern. They are the subject of today's lecture.

Braid groups and Rouquier complexes

$C^b(\mathrm{SBim})$ complexes of Soergel bimodules.

Let $K^b(\mathrm{SBim})$ denote the homotopy category of ~~Rouquier complexes~~ Soergel bimodules.
Monoidal category under tensor product of complexes.
Let B_W denote the braid group assoc. to W, i.e.

$$B_W = \langle \sigma_s, s \in S \mid \underbrace{\sigma_s \sigma_t \cdots}_{m_{st}\text{- terms}} = \underbrace{\sigma_t \sigma_s \cdots}_{} \rangle .$$

Consider the complexes

$$F_s: \quad \longrightarrow \underset{-1}{0} \longrightarrow \underset{0}{B_s} \xrightarrow{\ f\ } \underset{1}{R(1)} \longrightarrow$$

$$F_s^{-1} \cong E_s: \quad \longrightarrow R(-1) \xrightarrow{\ b\ } B_s \longrightarrow 0 \longrightarrow$$

We have:

$$F_s E_s: \quad \longrightarrow \underset{-1}{B_s(-1)} \longrightarrow \underset{0}{B_s B_s \oplus R} \longrightarrow \underset{1}{B_s(1)} \longrightarrow$$

$$\cong$$

$$\longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow$$

<u>Exercise:</u> If $m_{st} = 3$ then

$$F_s F_t F_s \cong 0 \to \underset{0}{B_{sts}} \longrightarrow \underset{1}{B_{st}(1) \oplus B_{ts}(1)} \longrightarrow \underset{2}{B_s(2) \oplus B_t(2)} \longrightarrow \underset{3}{R} \longrightarrow 0 \cong F_t F_s F_t .$$

<u>Thm:</u> $F_s F_t \cdots \cong F_t F_s \cdots$ so one gets a homomorphism of groups

$$F : B_W \longrightarrow \text{isomorphism classes of invertible complexes in } K^b(\mathrm{SBim}).$$

$$\qquad\qquad\qquad /\!/ \qquad \mathrm{Pic}(K^b(\mathrm{SBim}))$$

Rmk: 1) Rouquier: in fact one can upgrade F to a monoidal functor "strict categorification"

$$F : \Omega B_W \longrightarrow K^b(SBim)$$

$\uparrow$

monoidal cat assoc. to B_W:

objects : B_W

$\mathrm{Hom}(x,y) = \begin{cases} id & x \neq y \\ \phi & \text{otherwise} \end{cases}$

$x \otimes y = xy.$

Hence F_σ is defined up to unique isomorphism.

2) Conjecture (Rouquier): $F_\sigma \cong R \Leftrightarrow \sigma = id$ (faithful).

Known in finite type A (Khovanov-Seidel), ADE (Brav-Thomas).

~~Diagonal miracle (~~

Minimal complexes: A complex $X \in C^b(SBim)$ is minimal if it contains not contractible direct summands.

Exercise: Suppose $d : X^i \to X^{i+1}$ is of the form

$$M \oplus B \xrightarrow{d} M' \oplus B' \quad \text{with} \quad d = \begin{pmatrix} \alpha & \beta \\ \gamma & iso \end{pmatrix}$$

Then one can choose new isomorphisms $X^i \cong M \oplus B$, $X^{i+1} \cong M' \oplus B'$ so that d has the form $\begin{pmatrix} \alpha' & 0 \\ 0 & iso \end{pmatrix}$. Hence a complex is minimal if it "contains no isomorphisms".

Lemma: Given $X \in C^b(SBim)$, there exists $X_{min} \overset{\oplus}{\subset} X$ minimal s.t. $X_{min} \cong X$ in $K^b(SBim)$

Moreover, given two minimal subcomplexes $X_{min}, X'_{min} \subset X$ the induced map $X_{min} \to X'_{min}$ is an isomorphism of complexes.

<u>Example:</u> $s - t - u$:

$$\begin{array}{lcccccccc} & & 0 & & 1 & & 2 & & 3 \\ F_{sut} & : & B_{sut} & \longrightarrow & B_{su}(1) \oplus B_{st}(1) \oplus B_{ut}(1) & \longrightarrow & B_s(2) \oplus B_t(2) \oplus B_u(2) & \longrightarrow & R(3) \end{array}$$

$$\begin{array}{lcccccccc} F_t F_{sut} & : & B_{tsut} & \longrightarrow & B_{tsu}(1) \oplus B_{sts}(1) \oplus B_t(1) \oplus \cancel{B_t(1)} \oplus B_{tut}(1) & \longrightarrow & B_{ts}(2) \oplus \cancel{B_t(1)} \oplus \cancel{B_t(3)} \oplus B_{tu}(2) & \longrightarrow & B_t(3) \\ & & & \searrow & \oplus\, B_{sut}(1) & \longrightarrow & B_{tsu}(2) \oplus B_{st}(2) \oplus B_{ut}(2) & \longrightarrow & B_s(3) \oplus \cancel{B_t(3)} \oplus B_u(3) \oplus R(\ldots) \end{array}$$

$$\begin{array}{lcccccccccc} F_{tsut} & : & B_{tsut} & \longrightarrow & B_{tsu}(1) \oplus B_{sts}(1) \oplus B_t(1) \oplus B_{tut}(1) \oplus B_{sut} & \longrightarrow & B_{ts}(2) \oplus B_{tu}(2) \oplus B_{su}(2) \oplus B_{st}(2) \oplus B_{ut}(2) & \longrightarrow & (B_t \oplus B_s \oplus B_u)(3) & \longrightarrow & B_{id}(4) \\ \uparrow & & & & & & & & & & \\ \text{minimal} & & & & & & & & & & \end{array}$$

Assume $S(\leq w)$:

<u>Thm</u> (Diagonal Miracle, Libedinsky-W): If F_w denotes a minimal complex for a positive lift of $w \in W$ to B_w

and $F_w : {}^0F \to {}^1F \to \ldots$ then ${}^iF \cong \left(\bigoplus B_x^{\oplus m_x}\right)(i)$.

Proof involves a close analysis of the standard/costandard filtrations on Soergel bimodules.

(There are geometric reasons to expect this for Weyl groups).

<u>Rmk:</u> This allows one to formally deduce summands of F_x from KL combinatorics.

Recall: $\rho \in \mathfrak{h}^*$ s.t. $\langle w\rho, \alpha_s^\vee\rangle > 0 \iff sw > w$. $(*)$

~~Let $\underline{w} = s_1 \dots s_m$ be a rex.~~

~~Recall: $\rho c_{id} = \rho \cdot 1\otimes 1 = 1\otimes 1 \cdot (s\rho) + \langle \rho, \alpha_s^\vee \rangle 1 \otimes 1$.~~

Recall: $B_s = R \otimes_{R^s} R(1)$, $\quad c_{id} = 1\otimes 1$, $\quad c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B_s$.

$$f \cdot c_{id} = c_{id} \cdot s(f) + \partial_s(f) c_s .$$

(Enough to check for $f = 1, \alpha_s$ here obvious).

Hence $\quad \rho \cdot c_{id} = \langle \rho, \alpha_s^\vee \rangle c_s + c_{id} \cdot s\rho$.

<u>Observation 1:</u>

If $\underline{w} = s_1 \dots s_m$ is a rex then, in $BS(\underline{w})$ we have:

$$\rho \cdot 1\otimes 1 \otimes \dots \otimes 1 = \sum_{i=1}^{m} \langle s_{i-1} \dots s_1 \rho, \alpha_{s_i}^\vee \rangle \, c_{id} \dots c_{s_i} \dots c_{id} + 1 \otimes \dots \otimes 1 \cdot w(\rho).$$

<u>Rmk:</u> If $\underline{w}$ is reduced then $(*)$ implies that all the coefficients

$$\langle s_{i-1} \dots s_1 \rho, \alpha_{s_i}^\vee \rangle > 0 \quad !$$

<u>Observation 2:</u> multiplication by c_s in B_s can be expressed as the composition: $B_s \longrightarrow R(1) \longrightarrow B_s(2)$

Let φ_i denote the map $BS(\underline{w}) \longrightarrow BS(s_1 \dots \hat{s}_i \dots s_m)(1)$ obtained by tensoring the multiplication map $B_{s_i} \to R(1)$ by identity maps.

$$\varphi : BS(\underline{w}) \xrightarrow{\sum_{i=1}^{m} \langle s_{i-1} \dots s_1 \rho, \alpha_{s_i}^\vee \rangle \varphi_i} \bigoplus_{i=1}^{m} BS(s_1 \dots \hat{s}_i \dots s_m)(1)$$

<u>Lemma:</u> if $\bar{\chi}$ denotes the induced map

$$\bar{\chi} : \overline{BS(\underline{w})} \longrightarrow \bigoplus_{i=1}^{m} \overline{BS(s_1 \dots \hat{s}_i \dots s_m)}(1)$$

then $\langle \bar{\chi} b, \bar{\chi} b' \rangle = \langle b, \rho b' \rangle \quad \forall\, b, b' \in \overline{BS(\underline{w})}$.

<u>Remark:</u> One can also introduce $\iota : \bigoplus \overline{BS(s_1, \dots, \hat{s}_i, \dots, s_m)} \to \overline{BS(\underline{w})}$ so that one has a "weak Lefschetz situation"

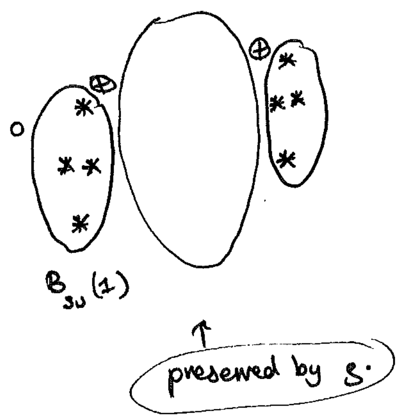
degree 1 maps

$$\overline{BS(\underline{w})} \underset{\bar{\chi}}{\overset{\bar{\iota}}{\rightleftarrows}} \bigoplus \overline{BS(s_1, \dots, \hat{s}_i, \dots, s_m)} \qquad \text{with } \bar{\iota} \circ \bar{\chi} = \rho \cdot - \,.$$

Main problem: the modules $\overline{BS(\underline{w})}$ generally do not satisfy hard Lefschetz.

Eg: $\overset{s}{\bullet} - \overset{t}{\circ} - \overset{u}{\bullet}$ $\qquad B_s B_u B_t B_s B_u \cong B_{sutsu} \oplus B_{su}(1) \oplus B_{su}(-1)$.

$$\Longrightarrow \quad \overline{B_s B_u B_t B_s B_u} = $$



This is where Rouquier complexes come to the rescue.

Assume $S(\leq x)$, $HR(\leq x)$, ... $\qquad \underline{x} = s_1 \dots s_m$

$$\begin{array}{ccc} \bar{\chi} : \overline{BS(\underline{x})} & \xrightarrow{\bar{\chi}} & \bigoplus_{i=1}^{m} \overline{BS(s_1, \dots, \hat{s}_i, \dots, s_m)}(1) \\ \cup & & \cup \\ \overline{B_x} & \longrightarrow & \bigoplus_{y<x} \overline{B}_y^{\oplus \mu(y,x)}(1) \end{array}$$

these modules satisfy hard Lefschetz.

At the end of the day one gets the following situation

$$\rightsquigarrow \quad \overline{B_x B_s} \xrightarrow{\bar{\chi}} \bigoplus_{y<x} \overline{B_y B_s}^{\oplus \mu(y,x)} \quad (1)$$

s.t. (case $J>0$)

1) $\bar{\chi}$ commutes with L_J.

2) $\bar{\chi}$ is injective in degrees $\leq \ell(x)$ (follows from exactness properties of Rouquier complexes)

3) $\langle b, L_J b' \rangle = \langle \bar{\chi} b, \bar{\chi} b' \rangle \qquad \forall b, b' \in \overline{B_x B_s}$

4) $\bigoplus_{y<x} \overline{B_y B_s}^{\oplus \mu(y,x)}$ satisfies (HR).

Now, if $0 \neq b \in (\overline{B_x B_s})^{-m}$ satisfies $L_J^m b = 0$ then $\bar{\chi}(L_J^m b) = L_J^m \bar{\chi}(b) = 0$

$\Rightarrow 0 \neq \bar{\chi}(b) \in \bigoplus_{y<x} \overline{B_y B_s}^{\oplus \mu(y,x)}$ is primitive.

$\Rightarrow \quad 0 \neq \langle \bar{\chi}(b), \bar{\chi}(b) \rangle = \langle b, L_J b \rangle$ a contradiction.

The case $J=0$ is a little more subtle, but the same basic idea applies.