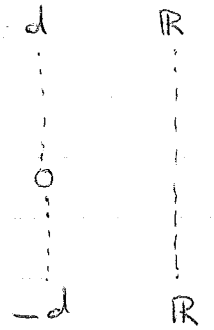
Part 9 12/11/13

Theme of the second part: positivity / Soergel's conjecture.
Want to axiomatize aspects of Hodge theory for complex varieties: "Lefschetz linear algebra".

Let M be an oriented $2d$-dimensional real compact manifold.
Put a grading shift on cohomology:
let $H^i = H^{d+i}(M;\mathbb{R})$ (de Rham or singular cohomology).



We have a non-degenerate intersection form
$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$.
If we assume that $H^{odd}(M;\mathbb{R}) = 0$, this form is symmetric.

Axiomatize: let $H = \bigoplus_i H^i$ be a fin. dim. graded $\mathbb{R}$-vector space with a symmetric nondegenerate form $\langle -,- \rangle : H \times H \to \mathbb{R}$ which is graded in the sense that $\langle H_i, H_j \rangle = 0$ unless $i = -j$.
A <u>Lefschetz operator</u> is a linear map $L : H^* \to H^{*+2}$ such that $\langle Lh, h' \rangle = \langle h, Lh' \rangle \quad \forall h, h' \in H$.

<u>Example</u> In the above setting of the manifold M, multiplication by any element of $H^2(M;\mathbb{R})$ is a Lefschetz operator.

We say that L satisfies the <u>hard Lefschetz (hL) theorem</u> if $L^i : H^{-i} \to H^i$ is an isomorphism for all $i \geq 0$.
<u>Example</u> M a d-dim smooth proj. variety / $\mathbb{C}$ with vanishing odd cohomology, $L = c_1$(ample line bundle) "generic hyperplane section".
<u>Exercise</u> L satisfies (hL) if & only if L is the action of e in some $\mathfrak{sl}_2(\mathbb{R})$-action on H for which $h.x = ix$ $\forall x \in H^i$.
If this is the case, the action of f is uniquely determined.

Suppose L satisfies (hL).

For $i \geq 0$, define $P_L^{-i} = \ker(L^{i+1}|_{H^{-i}})$ "primitive vectors".

Then we have the isotypic decomposition as $\mathfrak{sl}_2(\mathbb{R})$-module:

$$H = \bigoplus_{i \geq 0} \underbrace{\bigoplus_{0 \leq j \leq i} L^j P_L^{-i}}_{\substack{\text{sum of copies of the} \\ \text{irreducible } \mathfrak{sl}_2(\mathbb{R})\text{-rep w/ highest weight } i.}}$$

<u>Example</u> $M = Gr(2,4)$ over $\mathbb{C}$.

$$\begin{tikzcd}[row sep=small]
4 & & \mathbb{R} & & & \mathbb{R} & & \\
2 & & \mathbb{R} \arrow[u] & & & \mathbb{R} \arrow[u] & & \\
0 & \mathbb{R} \arrow[ur] & \oplus & \mathbb{R} \arrow[ul] & = & \mathbb{R} \arrow[u] & \oplus & \mathbb{R}. \\
-2 & & \mathbb{R} \arrow[ul] \arrow[ur] & & & \mathbb{R} \arrow[u] & & \\
-4 & & \mathbb{R} \arrow[u] & & & \mathbb{R} \arrow[u] & &
\end{tikzcd}$$

Define the <u>Lefschetz form</u> on H^{-i}:

$$(\alpha, \beta)_L^{-i} = \langle \alpha, L^i \beta \rangle \qquad \left(\begin{array}{c}\text{a nondegen. symmetric form} \\ \text{if \& only if (hL) holds.}\end{array}\right)$$

<u>Exercise</u>: (a) $(\alpha, \beta)_L^{-i} = (L\alpha, L\beta)_L^{-i+2}$ for $i \geq 2$.

(b) $H^{-i} = P_L^{-i} \oplus L(P_L^{-i-2}) \oplus L^2(P_L^{-i-4}) \oplus \cdots$ is an orthogonal decompn for the Lefschetz form.

<u>Hodge-Riemann bilinear relations:</u>

Assume either $H^{\text{even}} = 0$ or $H^{\text{odd}} = 0$.

Let $H^{\min}$ be the graded piece in the smallest nonzero degree.

Say that $H, \langle -,- \rangle, L$ satisfies (HR) if:

$\forall\, 0 \leq 2i \leq -\min$, the restriction of $(-,-)_L^{\min+2i}$ to $P_L^{\min+2i}$ is $(-1)^i$-definite. (In particular, this entails (hL).)

<u>Example</u> Again, this holds for M smooth projective etc.

Note that if $H^{\text{odd}} = 0$, the assumption is equivalent to saying that the restriction of $\langle -,- \rangle$ to $\underbrace{\widetilde{P}_{L,i} = L^{-i-\frac{\min}{2}} P_L^{\min+2i}}_{\subset H^0}$ is $(-1)^i$-definite.

Assume (hL) and (HR). Using the Exercise,
the signature of $(-,-)_L^{m-n+2i}$ (i.e. no. of positive evals − no. of negative evals)
$(-1)^i \dim P_L^{m-n+2i} + (-1)^{i+1} \dim P_L^{m-n+2i-2} + \cdots$,
which only depends on the Betti numbers $\dim H^i$.

Strategy of de Cataldo-Migliorini for establishing these properties:

- Assume (hL) and (HR) in dimension n.
  for us, Coxeter length
- By weak Lefschetz, this implies (hL) in dimension $n+1$.
- A limiting argument from known cases establishes (HR) in dim. $n+1$.

<u>Limit lemma</u> Suppose we have a continuous family
$[0,\infty) \longrightarrow \mathrm{Hom}(H, H(2)) : \gamma \longmapsto L_\gamma$ of Lefschetz operators
such that all L_γ satisfy (hL). If one L_γ satisfies (HR)
then they all do.

<u>Proof</u> All the forms $(-,-)_{L_\gamma}^{-i}$ are symmetric nondegenerate,
so their signatures must all be the same. □

Fix a realization $\underline{h}$ over $\mathbb{R}$ of (W,S) ← arbitrary Coxeter system satisfying:
$\exists\, \rho \in \underline{h}^*$ such that $\langle w\rho, \alpha_s^\vee \rangle > 0 \iff sw > w \quad \forall w \in W, s \in S$.
Fix this ρ also.

<u>Lemma</u> Such a realization always exists.

<u>Proof</u> Take $\underline{h}, \{\alpha_s\}, \{\alpha_s^\vee\}$ such that $\{\alpha_s\} \subset \underline{h}^*$ and $\{\alpha_s^\vee\} \subset \underline{h}$
are linearly independent and $\langle \alpha_s, \alpha_t^\vee \rangle = -2\cos(\pi/m_{st})$.
Then the property is satisfied for any ρ in the Tits cone.
(Ref: Humphreys 5.13.). □

We then form SBim as before, have isom $\mathrm{ch}: [\mathrm{SBim}] \xrightarrow{\sim} \mathcal{H}$.
Goal (Soergel's conjecture): $\mathrm{ch}(B_x) = \underline{H}_x$.
The key is to establish (hL) and (HR) for left mult. by
ρ on $\overline{B}_x = B_x \otimes_R \mathbb{R}$. First we need to say how to
define the form $\langle -,- \rangle$.
(Note: in the finite Weyl case, $\overline{B}_x = IH^*(\overline{BxB/B})$.)

Invariant forms on Soergel bimodules

A form $\langle -,- \rangle : B \times B \longrightarrow R$ on an R-bimodule B is graded if $\langle B^i, B^j \rangle \subset R^{i+j}$ and invariant if

$$\langle br, b' \rangle = \langle b, b'r \rangle = \langle b, b' \rangle r$$
$$\langle rb, b' \rangle = \langle b, rb' \rangle \qquad \forall\, b, b' \in B,\ r \in R.$$

(Note the privileging of right over left.)

Assume B is free finite-rank as a right R-module.

<u>Exercise</u>, Define the dual $\mathbb{D}(B) = \mathrm{Hom}_{-R}(B, R)$, an R-bimodule via $(r.f)(b) = f(rb)$, right R-modules
$(f.r)(b) = f(br) \quad \forall\, r \in R,\ f \in \mathbb{D}(B),\ b \in B.$

(i) $\mathbb{D}(BB_s) \cong \mathbb{D}(B) B_s$
$\mathbb{D}(B_s B) \cong B_s \mathbb{D}(B)$

(ii) Hence indecomposable Soergel bimodules are self-dual: $\mathbb{D}(B_x) \cong B_x$

(iii) $\mathrm{Hom}(B, \mathbb{D}(B))^0 \cong \{$ invariant graded forms on $B \}$. (degree zero)

<u>Def</u> An invariant graded form on an R-bimodule B is <u>nondegenerate</u> if it corresponds to an isom $B \xrightarrow{\sim} \mathbb{D}(B)$.

(iv) $\langle -,- \rangle$ is nondegenerate if & only if it induces a nondegenerate $\mathbb{R}$-valued form on $\overline{B} = B \otimes_R \mathbb{R}$.

Recall Soergel's hom formula: graded rank of $\mathrm{Hom}(B, B')$
$$= (\mathrm{ch}(B), \mathrm{ch}(B')) \in \mathbb{Z}[v, v^{-1}].$$

Recall also that $(\underline{H}_x, \underline{H}_y) \in \delta_{xy} + v\mathbb{Z}[v]$.

So $\mathrm{ch}(B_x) = \underline{H}_x$ implies that the space of invariant graded forms on B_x is 1-dimensional and hence that any nonzero one is nondegenerate. We will fix the sign of such a form by the condition $\langle \rho^{\ell(x)} 1 \otimes \cdots \otimes 1,\ 1 \otimes \cdots \otimes 1 \rangle > 0$ and call it the <u>intersection form</u> on B_x (unique up to positive scalar).

To construct forms on B_x, we first want to construct forms on Bott-Samelson bimodules.

Consider $R \otimes_{R^s} R$. Let $c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, $c_{id} = 1 \otimes 1$.

We have relations $f c_s = c_s f$, $f c_{id} = c_{id}(sf) + \partial_s(f) c_s$ for all $f \in R$.

For any expression $\underline{w} = s_1 \cdots s_m$ and $\underline{\varepsilon} = \varepsilon_1 \cdots \varepsilon_m$ $(\varepsilon_i \in \{0,1\})$, let $c_{\underline{\varepsilon}} = c_{s_1^{\varepsilon_1}} c_{s_2^{\varepsilon_2}} \cdots c_{s_m^{\varepsilon_m}} \in B_{\underline{w}} = B_{s_1} \cdots B_{s_m}$.

<u>Exercise</u> $\{ c_{\underline{\varepsilon}} \mid \underline{\varepsilon} \in \{0,1\}^m \}$ is a basis for $B_{\underline{w}}$ as a left, or right, R-module.

Define $\mathrm{Tr} : B_{\underline{w}} \longrightarrow R$ by $\mathrm{Tr}(x) = x_{11\cdots 1}$ where $x = \sum c_{\underline{\varepsilon}} x_{\underline{\varepsilon}}$. (i.e. the coefficient of $c_{s_1} \cdots c_{s_m}$, regarding $B_{\underline{w}}$ as a right R-module)

Let the <u>intersection form</u> on $B_{\underline{w}}$ be
$$\langle f, g \rangle = \mathrm{Tr}(fg)$$
(clearly symmetric, invariant and graded).
product in $B_{\underline{w}}$, which is after all a ring, being $R \otimes_{R^{s_1}} R \otimes \cdots \otimes_{R^{s_m}} R\,(m)$.

<u>Exercise</u> $\langle -,- \rangle$ is a nondegenerate form on $B_{\underline{w}}$, hence induces a nondegenerate form on $\overline{B}_{\underline{w}}$.

(Note: in the finite Weyl case, $\overline{B}_{\underline{w}} = H^*$(Bott-Samelson variety labelled by $\underline{w}$) and this is the intersection form on it; we then have $\overline{B}_w$ sitting inside it by the Decomposition Theorem.) if $\underline{w}$ is a rex for w,

Now fix x and a rex $\underline{x}$. Recall that B_x is a direct summand of $B_{\underline{x}}$.

<u>Lemma</u> (Fiebig) The restriction of $\langle -,- \rangle$ on $B_{\underline{x}}$ to B_x is nondegenerate.

This follows from:

<u>Exercise</u> Let $B = V_1 \otimes B_1 \oplus \cdots \oplus V_m \otimes B_m$ be the decomp^{n} of an object into indecomposables in a k-linear Krull-Remak-Schmidt category, V_i being the vector spaces giving the multiplicities. Then $\phi : B \to B$ is an isomorphism if & only if its components $\phi_i : V_i \otimes B_i \to V_i \otimes B_i$ are isomorphisms.