Part 5 18/10/13

Singular Soergel bimodules
(in the sense of singular weights, i.e. those with non-trivial stabilizers).

In rank 1, we saw that complicated relations among B_s become simple consequences of biadjunction when we interpret $B_s = R \otimes_{R^s} R(1)$ as a composition of simpler 1-morphisms, namely $\mathrm{ind} = {}_R R \otimes_{R^s} -$ and $\mathrm{res}\ {}_{R^s} R \otimes_R -$.
Singular Soergel bimodules are (R^I, R^J)-bimodules for general finitary subsets $I, J \subset S$. These are used to categorify not the Hecke algebra but the Schur algebroid.

Say $I \subset S$ is finitary (notation: $I \overset{f}{\subset} S$) if $W_I^{\langle I \rangle}$ is finite.
Fix such $I \overset{f}{\subset} S$.

Define $\quad w_I =$ largest element of W_I,

$$\pi(I) = v^{-l(w_I)} \sum_{x \in W_I} v^{2(l(w_I) - l(x))}$$

Poincaré polynomial normalized so that $\overline{\pi(I)} = \pi(I)$.

$$\underline{H}_I = \underline{H}_{w_I} = \sum_{x \in W_I} v^{l(w_I) - l(x)} H_x .$$

Note that $(\underline{H}_I)^2 = \pi(I)\, \underline{H}_I$.

Let $\mathcal{H}_I$ be the Hecke algebra of (W_I, I).

The Schur algebroid $S\mathcal{H}$ is the category with:

1) objects $I \overset{f}{\subset} S$
2) $\mathrm{Hom}(J, I) = \underline{H}_I \mathcal{H} \cap \mathcal{H} \underline{H}_J =: {}^I\mathcal{H}^J$, which has $\mathbb{Z}[v^{\pm 1}]$ basis $\{\underline{H}_p \mid p$ max. length in its W_I-W_J double coset$\}$.

composition ${}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \longrightarrow {}^I\mathcal{H}^K$

$$(f, g) \longmapsto f *_J g := \frac{1}{\pi(J)} fg$$

(makes sense because $f = f'\underline{H}_J$, $g = \underline{H}_J g'$
so $fg = \pi(J) f' \underline{H}_J g'$)

Equivalently we could consider $S\mathcal{H}$ as an idempotented ring.

Example: 1) $\mathrm{End}(\emptyset) = \mathcal{H}$
2) $\mathrm{Hom}(\emptyset, I) = {}^I\mathcal{H}^\emptyset = \underline{H}_I \mathcal{H}$ is a right $\mathcal{H}$-module
isom. to $\mathrm{triv} \otimes_{\mathcal{H}_I} \mathcal{H}$.

Remark: If W is the Weyl gp of $G = G(\mathbb{F}_q)$ then
$$S\mathcal{H} = \mathrm{End}\Big(\bigoplus_{I \subset S} \mathrm{Ind}_{P_I}^G (\mathrm{triv}) \Big).$$

Example: 3). If $\mathfrak{g}$ is a complex semisimple Lie algebra, $\underline{h}$ a Cartan subalgebra, $\Phi^+ \subset \Phi$, (W, S) the Weyl group, P the weight lattice, $\hat{Z} = P/\mathbb{Z}\Phi$.

The category $\mathrm{Rep}_f\, \mathfrak{g}$ is $\hat{Z}$-graded. For $z, z' \in \hat{Z}$, let

$${}_z\mathrm{Rep}_{z'} = \{ V \in \mathrm{Rep}_f\, \mathfrak{g} \mid V \otimes (\mathrm{Rep}_f\, \mathfrak{g})_{z'} \subset (\mathrm{Rep}_f\, \mathfrak{g})_z \}.$$

Of course this is just $(\mathrm{Rep}_f\, \mathfrak{g})_{z-z'}$ but we want to identify these reps with the resulting functors $V \otimes -$.

Then we form a 2-category $R_{\mathfrak{g}}$ where the objects are elements of $\hat{Z}$ and $\mathrm{Hom}(z', z) = {}_z\mathrm{Rep}_{z'}$.

Let $W_{aff} = W \ltimes \mathbb{Z}\Phi$, a Coxeter gp with $S_{aff} = S \cup \{s_0\}$.

$\hat{Z}$ acts faithfully on (W_{aff}, S_{aff}) via diagram automorphisms.

e.g. W type A_{n-1}: $\hat{Z} = \mathbb{Z}/n\mathbb{Z}$ acting by rotation.

We then have the <u>Satake isomorphism</u>

$$[R_{\mathfrak{g}}] \otimes_{\mathbb{Z}} \mathbb{Z}[v^{\pm 1}] \cong \mathrm{End}_{S\mathcal{H}_{aff}}(\text{objects of the form } z(S) \text{ for } z \in \hat{Z})$$

with $[{}_z\mathrm{Rep}_{z'}]$ corresponding to ${}^{z(S)}\mathcal{H}_{aff}{}^{z'(S)}$.

e.g. $[{}_0\mathrm{Rep}_0] = [\mathrm{Rep}(G_{ad})]$ corresponds to ${}^S\mathcal{H}_{aff}{}^S = \underline{H}_{w_0} \mathcal{H}_{aff} \underline{H}_{w_0}$

This isomorphism sends classes of simple reps to KL basis elts.

(assume $\underline{h}$ is nice, e.g. faithful & symmetrizable)

Basic fact: if $I \subset J \overset{f}{\subset} S$, then $R^J \subset R^I$ is a Frobenius extension with trace $\partial_I^J : R^I \to R^J$ defined by

$$\partial_I^J = \partial_{s_1} \cdots \partial_{s_\ell} \quad \text{where } s_1 \cdots s_\ell \text{ is a reduced expression for } w_J w_I$$

Questions: 1) is there an explicit construction of dual bases for ∂_I^J, or even for ∂_ϕ^S with W finite?

(In type A one gets such a basis using Schubert polynomials.)

2) generators and relations for SSBim (singular Soergel bimodules, defined in an analogous way to SBim)?