Part 11 19/11/13

Rouquier complexes

Let $C^b(SBim)$ be the additive category of bounded complexes of SBim, $K^b(SBim)$ its homotopy category. These are both monoidal categories under tensor product.

Homological degree will be written on the left superscript:

a complex looks like $\cdots \to {}^iF \xrightarrow{d_i} {}^{i+1}F \to \cdots$ ${}^iF \in SBim$, d_i a degree-0 map.

Consider the complexes:

$$F_s: \quad \cdots \to \underset{-1}{0} \to \underset{0}{B_s} \xrightarrow[\text{(multiplication)}]{} \underset{1}{R(1)} \to \underset{2}{0} \to \cdots$$

$$E_s: \quad \cdots \to 0 \to R(-1) \to B_s \to 0 \to \cdots$$
$$1 \mapsto \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$$

Let $\mathcal{B}_W$ be the braid group $\langle \sigma_s, s \in S \mid \underbrace{\sigma_s \sigma_t \cdots}_{m_{st}\text{ terms}} = \underbrace{\sigma_t \sigma_s \cdots}_{m_{st}\text{ terms}} \rangle$

Thm One has a homomorphism

$$\mathcal{F}: \mathcal{B}_W \longrightarrow \operatorname{Pic}(K^b(SBim)) = \left\{ \begin{array}{l} \text{invertible objects in the} \\ \text{monoidal category } K^b(SBim) \end{array} \right\}$$

given by $\sigma_s \longmapsto F_s$.

Exercise (i)

$$F_s E_s \cong \cdots \to 0 \to \underset{-1}{B_s(-1)} \to \underset{0}{\begin{array}{c} B_s B_s \\ \oplus \\ R \end{array}} \longrightarrow \underset{1}{B_s(1)} \to 0 \to \cdots$$

$$\cong \cdots \to 0 \to B_s(-1) \to \begin{array}{c} B_s(1) \\ \oplus \\ R \\ \oplus \\ B_s(-1) \end{array} \longrightarrow B_s(1) \to 0 \to \cdots$$

$$\cong \qquad R$$

i.e. F_s is invertible with inverse E_s.

(ii) If $m_{st} = 3$:

$$F_s F_t F_s \cong \underset{0}{B_{sts}} \to \underset{1}{(B_{st} \oplus B_{ts})(1)} \to \underset{2}{(B_s \oplus B_t)(2)} \to \underset{3}{B_{id}(3)} \cong F_t F_s F_t.$$

Higher braid relations proved by Rouquier, 'Higher categorification and braid groups'.

Remark: Rouquier showed that $\mathcal{F}$ can be made into a strict categorification, i.e. a tensor functor $\Omega \mathcal{B}_W \longrightarrow K^b(SBim)$. Hence $\mathcal{F}(\sigma)$ is well defined up to unique isomorphism.

A complex $F \in C^b(\mathrm{SBim})$ is minimal if it contains no contractible direct summands.

<u>Exercise</u> Consider a complex of the form

$$\begin{array}{ccc} B & & B' \\ \oplus & \xrightarrow{d_i} & \oplus \\ M & & M' \\ \shortparallel & & \shortparallel \\ {}^{i}F & & {}^{i+1}F \end{array} \qquad \text{where } d_i = \begin{pmatrix} \mathrm{iso} & \alpha \\ \beta & \gamma \end{pmatrix}, \quad \mathrm{iso}: B \xrightarrow{\sim} B'$$

Show that one can choose new isoms ${}^{i}F \cong B \oplus M$, ${}^{i+1}F \cong B' \oplus M'$ so that d_i has the form $\begin{pmatrix} \mathrm{iso} & 0 \\ 0 & \alpha' \end{pmatrix}$. Hence "matrices for minimal complexes contain no isomorphisms".

Equivalently, a complex is minimal if all differentials belong to the radical of the category SBim (subcat. generated by the radicals of the endomorphism rings).

<u>Lemma</u> For $F \in C^b(\mathrm{SBim})$, there exists a minimal summand $F_{min} \subset F$ such that $F_{min} \to F$ is an iso in $K^b(\mathrm{SBim})$. For two such minimal summands F_{min}, F_{min}', the induced map $F_{min} \to F_{min}'$ is already an iso in $C^b(\mathrm{SBim})$.

Given $x \in W$, choose a rex $\underline{x} = s_1 \cdots s_m$.
Set F_x = minimal summand of $F_{s_1} \cdots F_{s_m}$ (independent of the rex).

<u>Example</u> Type A_3, $s - t - u$.
Want to calculate F_{tsut}. (corresponding to first singular Schubert variety)

$$\begin{array}{rcccccccc} & & 0 & & 1 & & 2 & & 3 \\ F_{sut} = F_s F_u F_t = & & B_{sut} & \longrightarrow & (B_{su} \oplus B_{st} \oplus B_{ut})(1) & \longrightarrow & (B_s \oplus B_t \oplus B_u)(2) & \to & B_{id}(3) \\ & & 0 & & 1 & & 2 & & 3 \end{array}$$

$$\begin{array}{rccccc} F_t F_{sut} = & B_{tsut} & \to & (B_{tsu} \oplus B_{sts} \oplus B_t \oplus B_{tut} \oplus \cancel{B_t})(1) & \to & B_{ts}(2) \oplus \cancel{B_t(1)} \oplus \cancel{B_t(3)} \oplus B_{tu}(2) \to \cancel{B_{id}(3)} \\ & & & \oplus & & \\ \oplus & & & B_{sut}(1) & \to & B_{su}(2) \oplus B_{st}(2) \oplus B_{ut}(2) \to \\ & & & & & B_s(3) \oplus B_t(3) \oplus B_u(3) \to B_{id}(4) \end{array}$$

$$= B_{tsut} \longrightarrow \begin{pmatrix} B_{tsu} \oplus B_{sts} \oplus B_{sut} \\ \oplus B_{tut} \oplus B_t \end{pmatrix}(1) \longrightarrow \begin{pmatrix} B_{ts} \oplus B_{tu} \oplus B_{su} \\ \oplus B_{st} \oplus B_{ut} \end{pmatrix}(2) \longrightarrow (B_s \oplus B_t \oplus B_u)(3) \to B_{id}(4)$$

Notice that the homological degree equals the shift in all terms.

Diagonal miracle (Libedinsky – W.)

Assume $S(\le x)$. Then ${}^iF_x \cong \bigoplus_{z\le x} B_z^{\oplus m_z}(i)$ for some $m_z \in \mathbb{N}$, and ${}^0F_x = B_x$.

So one can determine ${}^iF_x$ by Kazhdan-Lusztig combinatorics.

This can be expressed as saying that F_x, (and similarly F_x^{-1}) are in the heart of a suitable t-structure. Geometrically this corresponds to the fact that $j_! \underline{\mathbb{Q}}_{BwB/B}$ and $j_* \underline{\mathbb{Q}}_{BwB/B}$ are perverse.

Recall the assumption that $\langle w\rho, \alpha_s^\vee \rangle > 0 \iff sw > w$.

Let $c_{id} = 1\otimes 1$, $c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B_s$.

We have $f c_{id} = c_{id}\, s(f) + \partial_s(f)\, c_s$.

If $\deg(f) = 2$, $f c_{id} = c_{id}\, s(f) + \langle f, \alpha_s^\vee \rangle c_s$.

For an expression, $\underline{w} = s_1 \cdots s_m$,

$$\rho \cdot \underbrace{1\otimes \cdots \otimes 1}_{\in B_{\underline{w}}} = \sum_{i=1}^m \langle s_{i-1}\cdots s_1(\rho), \alpha_{s_i}^\vee \rangle\, c_{id} \cdots c_{s_i} \cdots c_{id}$$

$$+ (1\otimes \cdots \otimes 1)\, s_m \cdots s_1(\rho).$$

If $\underline{w}$ is reduced, all the coefficients $\langle s_{i-1}\cdots s_1(\rho), \alpha_{s_i}^\vee \rangle$ are >0.

Multiplication by c_s on B_s can be factored

$$B_s \xrightarrow{\ \text{⫯}\ } R(1) \xrightarrow{\ \text{⫰}\ } B_s(2)$$

If $\chi_i : B_{\underline{w}} \to B_{\underline{w}_{\hat{i}}}(1)$ denotes the tensor product of $B_{s_i} \to R(1)$ with identities on other factors,

we set $B_{\underline{w}} \xrightarrow{\chi} \bigoplus_{i=1}^m B_{\underline{w}_{\hat{i}}}(1)$

$$\chi = \bigoplus_{i=1}^m \langle s_{i-1}\cdots s_1(\rho), \alpha_{s_i}^\vee \rangle \chi_i .$$

<u>Lemma</u> The induced map $\overline{\chi} : \overline{B}_{\underline{w}} \to \bigoplus_{i=1}^m \overline{B}_{\underline{w}_{\hat{i}}}(1)$ satisfies $\langle b, \rho b' \rangle = \langle \overline{\chi}(b), \overline{\chi}(b') \rangle \quad \forall\, b, b' \in \overline{B}_{\underline{w}}$.

Compare the weak Lefschetz situation! However, neither $\overline{B}_{\underline{w}}$ nor $\bigoplus \overline{B}_{\underline{w}_{\hat{i}}}(1)$ satisfies hard Lefschetz. We need to strip away the summands other than $\overline{B}_w$.

<u>Example</u> Back to A_3, consider the other singular Schubert variety:

$$\overline{B_sB_uB_tB_sB_u} \cong \overline{B_{sutsu}} \oplus \overline{B_{su}(1)} \oplus \overline{B_{su}(-1)}.$$

This is a ρ-stable decomposition, and mult$^{\text{n}}$ by ρ from degree -1 to 1 is not an isom on $\overline{B_{su}(1)}$ (centred on deg. -1) or $\overline{B_{su}(-1)}$ (centred on deg. 1).

We're assuming $S(\leq x)$, $HR(\leq x)$, ...

We have $B_{\underline{x}} \xrightarrow{\chi} \bigoplus_{i=1}^{m} B_{\underline{x}_i}(1)$. Key observation: χ is (up to <u>positive</u> scalar) the first differential on $F_{s_1}\cdots F_{s_m}$ where $\underline{x}=s_1\cdots s_m$.
since $\langle s_1\cdots s_i(\rho), \alpha_i^\vee\rangle > 0$.

By the diagonal miracle, we have an induced map

$$B_x \xrightarrow{\chi} \bigoplus_{y<x} B_y^{\oplus\mu(y,x)}(1),$$

compatible with forms in the sense that the above Lemma holds.

Now tensor with F_s on the right and apply $\overline{}$:

$$\overline{B_xB_s} \xrightarrow{\chi'} \overline{B_x}(1) \oplus \bigoplus_{y<x} \overline{B_yB_s}^{\oplus\mu(y,x)}(1) \longrightarrow \cdots$$

Note the following four <u>properties</u> ("weak Lefschetz substitute").

1) χ' intertwines L_ζ on $\overline{B_xB_s}$ with $L' = (\text{mult by } \rho) \oplus \bigoplus_{y<x} (L_\zeta \text{ on } \overline{B_yB_s})$.

2) χ' is injective in degrees $\leq \ell(x)$.

This follows from the fact that $F_{xs} \cong R(-\ell(xs))$ as right R-modules.

3) $\langle b, L_\zeta(b')\rangle = \langle \chi'(b), \chi'(b')\rangle$ for all $b, b' \in \overline{B_xB_s}$.

(where the forms are induced from the ambient Bott-Samelson bimodules).

4) $\overline{B_x}(1) \oplus \bigoplus_{y<x} \overline{B_yB_s}^{\oplus\mu(y,x)}(1)$ satisfies (HR) if $\zeta > 0$, by the inductive hypothesis.

Here one needs:

<u>Prop</u> If $\overline{B_z}$ satisfies (HR) for $\zeta=0$, with $zs<z$, then $\overline{B_zB_s}$ satisfies (HR) for $\zeta>0$.

(Easy from $B_zB_s = B_z(1)\oplus B_z(-1)$.)

Now take $\zeta>0$ and nonzero $b \in (\overline{B_xB_s})^{-m}$ with $m \geq 0$. Because $\overline{B_xB_s}$ has symmetric Betti numbers, it's enough to show that $L_\zeta^m b \neq 0$.

Assume for a contradiction that $L_\zeta^m b = 0$.

Then $0 = \chi'(L_\zeta^m b) \overset{\text{property (1)}}{=} (L')^m \chi'(b)$, so $\chi'(b)$ is primitive (using property (2)).

Property (4) implies that $\langle (L')^{m-1}\chi'(b), \chi'(b)\rangle \neq 0$

$\parallel$ by property (3).

$\langle L_\zeta^m b, b\rangle$

But this doesn't give the $\zeta = 0$ case we actually care about.

To handle the $\zeta = 0$ case, we need a modification of the argument.

$$F_x : \quad B_x \longrightarrow {}^1B(1) \longrightarrow {}^2B(2) \longrightarrow \cdots$$

Write ${}^1B = B_\uparrow \oplus B_\downarrow$ where

$${}^1B = \bigoplus_{z \in W} B_z^{\oplus m_z}, \quad B_\uparrow = \bigoplus_{\substack{z \in W \\ zs > z}} B_z^{\oplus m_z}, \quad B_\downarrow = \bigoplus_{\substack{z \in W \\ zs < z}} B_z^{\oplus m_z}$$

$$F_x F_s : \quad B_x B_s \xrightarrow{\chi'} B_x(1) \oplus B_\uparrow B_s(1) \oplus B_\downarrow B_s(1) \longrightarrow \cdots$$

$\overline{B_x(1)} \oplus \overline{B_\uparrow B_s(1)}$ satisfy (HR) but $\overline{B_\downarrow B_s}(1) \cong \overline{B_\downarrow} \oplus \overline{B_\downarrow}(2)$ does not.

By the easier half of the diagonal miracle (independent of knowing $S(xs)$), $B_\downarrow(2)$ must cancel with something in ${}^2F_xF_s$.
So the projection onto

$$B_x B_s \xrightarrow{\chi''} B_x(1) \oplus B_\uparrow B_s(1) \oplus B_\downarrow$$

is an isometry for Lefschetz forms, and χ'' is still injective in degrees $\leq \ell(x)$.
If the component of $\chi''(b)$ in the $B_\downarrow$ summand is nonzero we've already done, so we can consider only b such that this component is zero.
After this reduction, the argument is the same.

Now the induction is closed and the proof of Soergel's conjecture is complete.