Part 10 15/11/13

Outline of the proof of Soergel's conjecture

Recall that we write $\underline{H}_x = \sum_{y \le x} h_{y,x} H_y$ for $h_{y,x} \in v\mathbb{Z}[v]$.
We write $\mu(y,x)$ for the coefficient of v in $h_{y,x}$, which is 0 unless $y < x$ and $\ell(x) - \ell(y)$ is odd.

If $xs > x$, the Kazhdan-Lusztig multiplication formula says

$$\underline{H}_x \underline{H}_s = \underline{H}_{xs} + \sum_{\substack{y<x \\ ys<y}} \mu(y,x) \underline{H}_y .$$

For $x \in W$, we'll write $S(x)$ for the statement $ch(B_x) = \underline{H}_x$, $S(\le x)$ for $S(y)\ \forall y \le x$ etc.

Assuming $S(\le x)$, $S(xs)$ (for s s.t. $xs > x$) is equivalent to the statement that

$$B_x B_s \cong B_{xs} \oplus \bigoplus_{\substack{y<x \\ ys<y}} B_y^{\oplus \mu(y,x)} .$$

The multiplicity of B_y as a summand inside $B_x B_s$ is equal to the rank of the pairing

$$\mathrm{Hom}(B_y, B_x B_s) \times \mathrm{Hom}(B_x B_s, B_y) \longrightarrow \mathrm{End}(B_y)$$

By assumption $S(y)$, $\mathrm{End}(B_y) \cong \mathbb{R}$.

By Soergel's hom formula, the graded rank of $\mathrm{Hom}^\bullet(B_y, B_x B_s)$ (and the same for $\mathrm{Hom}^\bullet(B_x B_s, B_y)$) equals

$$(ch(B_y), ch(B_x B_s)) = (\underline{H}_y, \underline{H}_x \underline{H}_s) \in \begin{Bmatrix} \mu(y,x) & \text{if } ys<y \\ 1 & \text{if } ys = x \\ 0 & \text{otherwise} \end{Bmatrix} + v\mathbb{Z}[v].$$

by assumption $S(\le x)$

So assuming $S(\le x)$, $S(xs)$ holds if and only if, for all $y < x$, the pairings $\mathrm{Hom}(B_y, B_x B_s) \times \mathrm{Hom}(B_x B_s, B_y) \longrightarrow \mathbb{R}$ (*) are nondegenerate.

But B_y, B_x, $B_x B_s$ carry non-degenerate invariant intersection forms. Hence we have a (canonical, once these forms are fixed) identification $\mathrm{Hom}(B_y, B_x B_s) = \mathrm{Hom}(B_x B_s, B_y)$, so (*) becomes a form $(-,-)_y^{x,s}$ on $\mathrm{Hom}(B_y, B_x B_s)$, the "local intersection form".

So assuming $S(\leq x)$, $S(xs)$ holds if and only if $(-,-)^{x,s}_y$ is nondegenerate $\forall y < x$.

<u>Embedding theorem</u> ~~The~~ Consider the map

$$\iota : \operatorname{Hom}(B_y, B_x B_s) \longrightarrow \overline{B_x B_s}$$
$$\varphi \longmapsto \varphi(\underbrace{1 \otimes \cdots \otimes 1}_{\text{lowest-degree element of } B_y})$$

1) ι is injective
2) $\operatorname{Im}(\iota) \subset P_\rho^{-\ell(y)}$ (recall that mult^n by ρ is a Lefschetz operator on $\overline{B_x B_s}$)

3) ι is an isometry up to a positive scalar.

Hence the Hodge-Riemann bilinear relation for $\overline{B_x B_s}$, $\langle -,- \rangle_{\mathbb{R}}$, mult^n by ρ, implies $S(xs)$. This packages all local forms $(-,-)^{x,s}_y$ into one global Lefschetz form on $\overline{B_x B_s}$.
However, Hodge-Riemann for $\overline{B_x B_s}$ seems difficult to attack directly.

Following the strategy of de Cataldo-Migliorini, we deform $L =$ mult^n by ρ on $\overline{B_x B_s}$: for $\zeta \in \mathbb{R}$, let $L_\zeta = L + \mathrm{id}_{B_x} \otimes$ (mult by $\rho\zeta$ on B_s)

<u>Theorem</u> If $(\overline{B_x}, \langle -,- \rangle_{\mathbb{R}}$, mult^n by $\rho)$ satisfy (HR) then $(\overline{B_x B_s}, \langle -,- \rangle_{\mathbb{R}}, L_\zeta)$ satisfies (HR) for $\zeta >> 0$.

Idea of proof: the action of L_ζ tends to the tensor product $\mathfrak{sl}_2$-action on $\overline{B_x} \otimes \mathbb{R}^2$ as $\zeta \longrightarrow \infty$.
The Limit Lemma then implies that if we know (hL) for all L_ζ, (HR) holds for all L_ζ and in particular for $L_0 = L$.

The other aspect of de Cataldo-Migliorini's strategy is the implication $\begin{pmatrix}\text{HR in} \\ \text{dim. } n-1\end{pmatrix} \implies \begin{pmatrix}\text{hL in} \\ \text{dim } n\end{pmatrix}$.

Geometric setting: $X \subset \mathbb{P}^N$, $d_X = \dim_{\mathbb{C}} X$, X_H hyperplane section, $r : X_H \hookrightarrow X$

$$\begin{tikzcd}
H^{d_X+2}(X) & \\
H^{d_X+1}(X) & H^{d_X}(X_H) \arrow[lu, "r_*"'] \\
H^{d_X}(X) & H^{d_X-1}(X_H) \arrow[lu, "r_*"'] \\
H^{d_X-1}(X) \arrow[ru, "r^*"] & H^{d_X-2}(X_H) \\
H^{d_X-2}(X) \arrow[ru, "r^*"] &
\end{tikzcd}$$

$r_* r^* = L$

(We'll give the argument assuming Hodge purity.)

Weak Lefschetz says that

$r^* : H^i(X) \longrightarrow H^i(X_H)$ is an iso for $i < d_X - 1$
injective for $i = d_X - 1$

$r_* : H^i(X_H) \longrightarrow H^{i+2}(X)$ is an iso for $i > d_X - 1$
surjective for $i = d_X - 1$

Hence $L^i : H^{d_X - i}(X) \longrightarrow H^{d_X + i}(X)$ is an isom if $i > 1$,
by ~~[illegible]~~ the inductive assumption that (hL) (a consequence of (HR))
holds for X_H.

To prove $L : H^{d_X - 1}(X) \xrightarrow{\sim} H^{d_X + 1}(X)$,
assume for a contradiction that $\exists$ nonzero $v \in H^{d_X - 1}(X)$ with $L(v) = 0$,
then $L(r^* v) = r^* L(v) = 0$, so $r^* v$ is primitive,
so (HR) for X_H says $0 \neq \langle r^* v, r^* v \rangle = \langle v, r_* r^* v \rangle = \langle v, L(v) \rangle$,
a contradiction.

We want to imitate this using complexes of Soergel bimodules.